Lyapunov Functionals for Volterra Integro-Differential Equations

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Abstract of the Thesis

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The aim of this thesis is to study the qualitative behavior of a specific non-linear Volterra integro-differential equation with finite delays by using Lyapunov's second method. The non-linear Volterra integro-differential equation is:

$$x'(t) = b(t)x(t - r_1) - \int_{t-r_2}^{t} a(t, s)g(x(s))ds,$$

where r_1, r_2 are positive constants representing 2 finite delays, $t \geq 0$ and

$$a: \ [0, \infty) \times [-\tau, \infty) \to \mathbb{R}, \qquad \text{and} \qquad b: \ [0, \infty) \to \mathbb{R}$$

are two continuous functions.

In the first part, we study the qualitative behavior of the constant delay equation which is a specific case of the given integro-differential equation where $r_1 \neq 0$ and $r_2 = 0$. In the second part, we study the qualitative behavior of the integro-differential equation with one finite delay which is another specific case of the given integro-differential equation where $r_1 = 0$ and $r_2 \neq 0$. Three main steps are to be applied to each case separately. The first step is to construct a suitable, positive definite and non-decreasing, Lyapunov functional that yields the exponential stability of the zero solution of the given integro-differential equation. The second step is to derive inequalities and assumptions that guarantee the exponential stability of the zero solution of the given integro-differential equation. Finally, the third step is to derive inequalities and assumptions that guarantee the instability of the zero solution of the given integro-differential equation. Our theoretical results are extensions of many results found in the study of qualitative behavior of the zero solution of integro-differential equations with finite delay.

To Najib, Emma and Anna.

Contents

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Chapter 1

Introduction

The theory of linear and non-linear Volterra integro-differential equations has been developing rapidly in the last three decades.

In this chapter, we will start by introducing Professor Vito Volterra (3 May 1860 - 11 October 1940) and stating some examples about his research on linear and non-linear integro-differential equations. By an "integro-differential equation", we simply mean an equation that involves both integrals and derivatives of an unknown function.

Volterra was an Italian mathematician and physicist, known for his contributions to mathematical biology and integral equations. He is one of the founders of Functional Analysis. Volterra's work on elasticity was the origin of his theory of integro-differential equations: he found that for certain substances, the electric or magnetic polarization depends not only on the electromagnetic field at that moment, but also on the history of the electromagnetic state of the matter at all previous instants.

These physical facts are modeled by "integro-differential equations". Also, Volterra assumed "linear heredity", *i. e.*, that is the strain is a linear functional of the stress. In this case, the fundamental equations are systems of linear integro-differential equations, and he proved that the strain in a definite interval of time can be determined, given the forces in the body and the stress and strain on its surface for this time-interval. For more information on Professor Volterra's biography, we refer the reader to the Biography section in the book written by Volterra in 1930, a new edition was published in 2005 where his biography written by Sir Edmund Whittaker was added [1].

The theory of linear and non-linear Volterra integro-differential equations and systems and their solutions play an important role in many real-world phenomena in sciences and engineering such as atomic energy, control theory, economy, engineering techniques, fluid mechanics, biology, physics, medicine and many others [2–4]. For example, in biological applications, the population dynamics, and genetics are modeled by a system of integro-differential equations [5], the Initial-Value Problem (IVP) for a nonlinear system of integro-differential equations was used to model the competition between tumor cells and the immune system [6]. While in engineering, two systems of specific inhomogeneous integro-differential equations were studied in order to examine the noise term phenomenon [7].

In Physics, integro-differential equations model many situations such as in circuit analysis. For example, by Kirchhoff's second law, the net voltage drop across a closed loop equals the voltage impressed $E(t)$. An RLC circuit therefore is governed by the following equation

$$L\frac{d}{dt}I(t) + RI(t) + \frac{1}{C}\int_0^t I(\tau)d\tau = E(t)$$

where $I(t)$ is the electric current as a function of the time t, R is the resistance, L the inductance, and C the capacitance [3].

Few kinds of Volterra integro-differential equations and systems can be solved explicitly. Hence, we need to find analytic methods to study the qualitative behavior (stability, boundedness, asymptotic stability, etc.) of solutions without finding them. The study of qualitative behaviors of solutions of Volterra integro-differential equations plays an important role in engineering and sciences, and during the last fifty years, many results and methods have been obtained. These methods and techniques include the second method of Lyapunov, fixed point theory, perturbation theory, continuation methods and many others. For more details, we refer the reader to [8–12].

Further, non-linear Volterra integro-differential equations can be with or without delay. Many research works studied the qualitative behavior of solutions to Volterra integro-differential equation without delay. For example, in [13], Becker investigated the asymptotic behavior of solutions of the scalar linear homogeneous Volterra integro-differential equation

$$x'(t) = -a(t)x(t) + \int_0^t b(t,s)x(s)ds, \tag{1.0.1}$$

for $t \geq 0$, where a and b are real-valued functions that are continuous on the respective domains $[0, \infty)$ and $\Omega := \{(t,s) : 0 \leq s \leq t < \infty\}$. Becker employed the Lyapunov functional technique in order to study the qualitative behavior of the zero solution of the Volterra integro-differential equation (1.0.1) and found that this zero solution is:

- stable if for every $t > 0$ and every $t_0 \geq 0$, there exists a $\delta = \delta(t, t_0) > 0$ such that $\phi \in C[0, t_0]$ with $|\phi(t_0)| < \delta$ implies that $|x(t, t_0, \phi)| < \epsilon$ for all $t \geq t_0$, where $C[0, t_0]$ denotes the set of all continuous real-valued functions on $[0, t_0]$,

- globally asymptotically stable (asymptotically stable in the large) if it is stable and if every solution of (1.0.1) approaches zero as $t \to \infty$.

Furthermore, in [14], Burton transformed a large problem into several smaller ones and concluded that the solution of the large problem is some combination of the solutions of the small problems. He considered, under suitable continuity and smoothness assumptions, the following scalar equation:

$$x'(t) = A(t)f(x(t)) + \int_0^t \Big[B(t,s)g(x(s)) - C(t,s)h(x(s)) - D(t,s)r(x(s))\Big]ds \quad (1.0.2)$$

where $A(t), B(t), C(t)$ and $D(t)$ are continuous functions for $0 \le s \le t < \infty$ and $f(x(t)), g(x(s)), h(x(s))$ and $r(x(s))$ are continuous on $(-\infty, \infty)$. Then he looked at simplified equations

$$x'(t) = A(t)f(x(t)) + \int_0^t B(t,s)g(x(s))ds \quad (1.0.3)$$

$$x'(t) = -\int_0^t C(t,s)h(x(s))ds \quad (1.0.4)$$

$$x'(t) = -\int_0^t D(t,s)r(x(s))ds \quad (1.0.5)$$

Burton investigation is to construct Lyapunov functionals for (1.0.3), (1.0.4) and (1.0.5) and combine them to make a Lyapunov functional for (1.0.2) in order to study its qualitative behavior. In addition, in Burton et al. [15], a Lyapunov theory was developed that primarily seems to apply to Volterra integro-differential equations without delay. In what follows, we will use Lyapunov functionals which are (most of the time) non-increasing or strictly decreasing along solutions.

For more information on the construction of Lyapunov functionals and the qualitative behavior of the Volterra integro-differential equations without delay we refer the reader to [9, 11, 16–21].

Theoretically, the Lyapunov's second method is very attractive. However, the situation becomes more difficult when the integro-differential equation is with delay. In the literature, there are few papers on the qualitative behavior of Volterra integro-differential equations with delay. We refer the reader to the recent papers of Adivar and Raffoul [22], Graef and Tunc [23], Raffoul [24], Raffoul and Unal [25], Tunc [20, 26].

In this thesis, we consider the scalar non-linear Volterra integro- differential equation with finite delays

$$x'(t) = b(t)x(t - r_1) - \int_{t-r_2}^{t} a(t,s)g(x(s))ds \tag{1.0.6}$$

where r_1, r_2 are positive constants representing 2 finite delays, $t \geq 0$ and

$$a: \ [0,\infty) \times [-\tau,\infty) \to \mathbb{R}, \qquad \text{and} \qquad b: \ [0,\infty) \to \mathbb{R}$$

are again two continuous functions.

We note that the aim of this thesis is to use Lyapunov's second method to study the qualitative behavior of (1.0.6) where the key requirement is to find a positive definite functional that is non-decreasing along solutions. In fact, Driver [27] proved the following:

Theorem 1.0.1. *If there exists a functional $V(t,\phi(.))$, defined whenever $t \geq t_0 \geq 0$ and ϕ belongs to the Banach space of continuous functions $C([0,t],\mathbb{R}^n)$, such that*

i. $V(t,0) \equiv 0$, V is continuous in t and locally Lipschitz in ϕ,

ii. $V(t,\phi(.)) \geq W(|\phi(t)|)$, $W : [0,\infty) \to [0,\infty)$ is a continuous function with $W(0) = 0$, $W(r) > 0$ if $r > 0$, and W is strictly increasing (positive definiteness), and

iii. $V'(t,\phi(.)) \leq 0$,

then the zero solution of (1.0.6) will be stable, and

$$V(t,\phi(.)) = V(t,\phi(s) : 0 \leq s \leq t)$$

is called a Lyapunov functional for (1.0.6).

Therefore, in this thesis, Lyapunov functionals are employed to obtain sufficient conditions that guarantee the exponential stability of the zero solution of (1.0.6). We will consider the qualitative behavior of two different cases of specific integro-differential equations that may yield to the qualitative behavior of (1.0.6):

In Chapter 2, we consider a first particular case of the scalar nonlinear Volterra integro-differential equation (1.0.6) where $r_1 \neq 0$ and $r_2 = 0$. While in Chapter 3, we consider a second particular case of the scalar nonlinear Volterra integro-differential equation (1.0.6) where $r_1 = 0$ and $r_2 \neq 0$.

Note that this research work has 3 main steps to be applied to each case respectively:

- To construct a suitable Lyapunov functional that yields results concerning the exponential stability of the zero solution of the given integro-differential equation.

- To define the inequalities and assumptions needed to guarantee the exponential stability of the zero solution of the given integro-differential equation.

- To define the inequalities and assumptions needed to guarantee the exponential instability of the zero solution of the given integro-differential equation.

As we have mentioned above, few papers on the qualitative behavior of Volterra integro-differential equations with delay are found in the literature. For example, in [28], Wang considered the constant delay equation

$$x'(t) = a(t)x(t) + b(t)x(t-h), \tag{1.0.7}$$

where $a(t), b(t)$ are positive continuous functions on $\mathbb{R}$, and $h > 0$ represents a positive finite delay. Wang used Lyapunov functionals and obtained inequalities from which exponential stability of the zero solution of (1.0.7) was deduced provided that

$$-\frac{1}{2h} \leq a(t) + b(t+h) \leq hb^2(t+h).$$

In Chapter 2, instead of using Equation (1.0.7), we will consider Equation (1.0.6) with $r_1 \neq 0$ and $r_2 = 0$ and use Lyapunov functionals to study the qualitative behavior of the constant delay equation

$$x'(t) = b(t)x(t - r_1), \tag{1.0.8}$$

where $b : \mathbb{R} \to \mathbb{R}$ is a continuous function and r_1 is a positive constant.

Also, in [29], Levin and Nohel investigated the behavior, as t goes to zero, of the solutions of

$$x'(t) = -\frac{1}{L}\int_{t-L}^{t} (L - (t-\tau))\, g(x(\tau))d\tau,$$

where $L > 0$ is a given constant and $g(x)$ is the restoring force of a given spring, which is not necessarily linear. Specifically, they assumed $g(x)$ to be locally Lipschitz (so that for each $0 \leq A < \infty$, there exists $K = K(A) < \infty$ such that $|g(x) - g(y)| \leq K|x - y|$ if $|x|$, $|y| \leq A$) and

$$xg(x) > 0 \quad (x \neq 0), \qquad G(x) = \int_0^x g(\xi)d\xi \to \infty \qquad (|x| \to \infty).$$

Then, they constructed a suitable Lyapunov functional and showed that the zero solution of

$$x'(t) = -\int_{t-L}^{t} a(t-\tau)g(x(\tau))d\tau \qquad (0 \leq t < \infty),$$

is globally asymptotically stable provided that $a(r)$ is a continuous function with $a(r) = 0$, $a(t) \geq 0$, $a'(t) \leq 0$ for $0 \leq t \leq r$.

In addition, in [24], Raffoul considered the non-linear Volterra integro-differential equation with a uniformly distributed finite delay (which has been also used by Levin and Nohel [29]):

$$x'(t) = -\int_{t-r}^{t} a(t,s)g(x(s))ds, \tag{1.0.9}$$

where $r > 0$ is a constant and $a : [0,\infty) \times [0,\infty) \to (-\infty,\infty)$; $g(x)$ is continuous in x.

Raffoul used Lyapunov functionals to get enough conditions that guarantee the exponential stability of the zero solution of (1.0.9) and studied the stability and instability of the zero solution of (1.0.9).

Then, in [22], Adivar and Raffoul used Lyapunov functionals to obtain sufficient conditions that ensure exponential stability of the nonlinear Volterra integro-differential equation

$$x'(t) = p(t)x(t) - \int_{t-\tau}^{t} q(t,s)x(s)ds, \tag{1.0.10}$$

where the constant τ is positive, and $q(t,s)$ is a continuous function such that $q : [0,\infty) \times [-\tau,\infty) \to \mathbb{R}$ and $p : [0,\infty) \to \mathbb{R}$. The authors used Lyapunov functionals and deduced the exponential stability of the zero solution of (1.0.10).

In Chapter 3, instead of Equation (1.0.10), we study the qualitative behavior of the scalar nonlinear Volterra integro-differential equation

$$x'(t) = b(t)x(t-r_1) - \int_{t-r_2}^{t} a(t,s)g(x(s))ds, \tag{1.0.11}$$

with $r_1 = 0$ and $r_2 \neq 0$; *i. e.*, the scalar equation

$$x'(t) = b(t)x(t) - \int_{t-r_2}^{t} a(t,s)g(x(s))ds, \tag{1.0.12}$$

where r_2 is a positive constant representing a finite delay, $t \geq 0$, and $a: [0,\infty) \times [\tau,\infty) \to \mathbb{R}$, and $b: [0,\infty) \to \mathbb{R}$ and the function $g(x)$ is continuous in x.

Finally, we mention that our results will be different from those obtained in the literature (see [22, 26, 28]). In fact, Volterra integro-differential equations discussed and the assumptions and inequalities to be established in our investigations are different from those in the above-mentioned papers. This thesis contributes to the topic for the literature, and may be useful for further researches studying the stability and instability of the non-zero solution of nonlinear Volterra integro-differential equation with different positive delays.

In this thesis, the notation x_t means that $x_t(s) = x(t+s)$, $s \in [\tau, 0]$ as long as $x(t+s)$ is defined. Thus, x_t is a function which maps an interval $[\tau, 0]$ into $\mathbb{R}$. One can say that $x(t) \equiv x(\cdot, t_0, \psi)$ is a solution of (1.0.6) if $x(t)$ satisfies (1.0.6) for $t \geq t_0$ and $x_t = x(t+s) = \psi(s)$, $s \in [\tau, 0]$.

Chapter 2

Qualitative Behavior of the Constant Delay Equation

In this chapter, we consider the scalar non-linear Volterra integro-differential equation with finite delays:

$$x'(t) = b(t)x(t - r_1) - \int_{t-r_2}^{t} a(t,s)g(x(s))ds$$

where r_1, r_2 are positive constants representing 2 finite delays, $t \geq 0$ and

$$a: \ [0,\infty) \times [-\tau,\infty) \to \mathbb{R}, \qquad \text{and} \qquad b: \ [0,\infty) \to \mathbb{R}$$

with $r_1 \neq 0$ and $r_2 = 0$. This yields to the following equation

$$x'(t) = b(t)x(t - r_1) - \int_{t}^{t} a_1(t,s)g(x(s))ds \tag{2.0.1}$$

Hence, we get the constant delay equation

$$x'(t) = b(t)x(t - r_1). \tag{2.0.2}$$

The goal is to study the qualitative behavior of its solutions.

2.1 Construction of Lyapunov Functionals

Lemma 2.1.1. *Assume that r_1 is a positive constant and*

$$-\frac{1}{2r_1} \leq b(t+r_1) \leq -r_1 b^2(t+r_1). \tag{2.1.1}$$

Let $x(t) = x(t, t_0, \phi)$ be a solution of Equation (2.0.1) defined on $[t_0, \infty)$. Then for $t \geq t_0$, if

$$V(t) = \left[x(t) + \int_{t-r_1}^{t} b(s+r_1)x(s)ds\right]^2 + \int_{-r_1}^{0}\int_{t+s}^{t} b^2(z+r_1)x^2(z)dzds \tag{2.1.2}$$

then

$$V'(t) \leq b(t+r_1)V(t) \tag{2.1.3}$$

where $b(t+r_1) \leq 0$. Therefore $V'(t) \leq 0$.

Proof: Let us calculate $V'(t)$:

$$\begin{aligned}
V'(t) =& 2\left[x(t) + \int_{t-r_1}^{t} b(s+r_1)x(s)ds\right] b(t+r_1)x(t) + r_1 b^2(t+r_1)x^2(t) \\
&- \int_{t-r_1}^{t} b^2(s+r_1)x^2(s)ds \\
=& \Big[b(t+r_1)\Big] \left[x^2(t) + 2x(t)\int_{t-r_1}^{t} b(s+r_1)x(s)ds + \int_{t-r_1}^{t} b(s+r_1)x(s)ds\right] \\
&+ \left[b(t+r_1) + r_1 b^2(t+r_1)\right] x^2(t) - b(t+r_1)\left(\int_{t-r_1}^{t} b(s+r_1)x(s)ds\right)^2 \\
&- \int_{t-r_1}^{t} b^2(s+r_1)x^2(s)ds \\
=& b(t+r_1)\left\{\left[x(t) + \int_{t-r_1}^{t} b(s+r_1)x(s)ds\right]^2 + \int_{-r_1}^{0}\int_{t+s}^{t} b^2(z+r_1)x^2(z)dzds\right\}
\end{aligned}$$

$$+\left[b(t+r_1)+r_1 b^2(t+r_1)\right]x^2(t)$$
$$-b(t+r_1)\left[\left(\int_{t-r_1}^{t} b(s+r_1)x(s)ds\right)^2+\int_{-r_1}^{0}\int_{t+s}^{t} b^2(z+r_1)x^2(z)dzds\right]$$
$$-\int_{t-r_1}^{t} b^2(s+r_1)x^2(s)ds \tag{2.1.4}$$

In what follows, we perform some calculations in order to simplify Equation (2.1.4). Using Hölder's inequality, we get:

$$\left(\int_{t-r_1}^{t} b(s+r_1)x(s)ds\right)^2 \leq r_1\int_{t-r_1}^{t} b^2(s+r_1)x^2(s)ds \tag{2.1.5}$$

and we easily observe that

$$\int_{-r_1}^{0}\int_{t+s}^{t} b^2(z+r_1)x^2(z)dzds \leq r_1\int_{t-r_1}^{t} b^2(s+r_1)x^2(s)ds \tag{2.1.6}$$

So

$$\left(\int_{t-r_1}^{t} b(s+r_1)x(s)ds\right)^2+\int_{-r_1}^{0}\int_{t+s}^{t} b^2(z+r_1)x^2(z)dzds \leq 2r_1\int_{t-r_1}^{t} b^2(s+r_1)x^2(s)ds \tag{2.1.7}$$

Finally, we easily find

$$-\int_{t-r_1}^{t} b^2(s+r_1)x^2(s)ds \leq -\frac{1}{2r_1}\left[\left(\int_{t-r_1}^{t} b(s+r_1)x(s)ds\right)^2+\int_{-r_1}^{0}\int_{t+s}^{t} b^2(z+r_1)x^2(z)dzds\right] \tag{2.1.8}$$

Then, from (2.1.4), we get

$$V'(t) \leq b(t+r_1)V(t)+\left[b(t+r_1)+r_1 b^2(t+r_1)\right]x^2(t)$$
$$-b(t+r_1)\left[\left(\int_{t-r_1}^{t} b(s+r_1)x(s)ds\right)^2+\int_{-r_1}^{0}\int_{t+s}^{t} b^2(z+r_1)x^2(z)dzds\right]$$

$$
\begin{aligned}
&- \int_{t-r_1}^{t} b^2(s+r_1)x^2(s)ds \\
&\leq b(t+r_1)V(t) + \left[b(t+r_1) + r_1 b^2(t+r_1)\right] x^2(t) \\
&- \left(b(t+r_1) + \frac{1}{2r_1}\right) \left[\left(\int_{t-r_1}^{t} b(s+r_1)x(s)ds\right)^2 + \int_{-r_1}^{0}\int_{t+s}^{t} b^2(z+r_1)x^2(z)dzds\right]
\end{aligned}
$$

By invoking equations (2.1.5)-(2.1.8) into Equation (2.1.4), and using (2.1.1), we get

$$V'(t) \leq b(t+r_1)V(t), \tag{2.1.9}$$

this completes the proof. ■

Theorem 2.1.2. *Assume the hypothesis of Lemma 2.1.1 holds then any solution $x(t) = x(t, t_0, \phi)$ of Equation (2.0.2) satisfies the exponential inequality*

$$|x(t)| \leq \sqrt{6V(t_0)}e^{\left(\frac{1}{2}\int_{t_0}^{t-r_1/2} b(s+r_1)ds\right)} \qquad \textit{for} \quad t \geq t_0 + \frac{r_1}{2}. \tag{2.1.10}$$

Proof: By changing the order of integration, we have

$$
\begin{aligned}
\int_{-r_1}^{0}\int_{t+s}^{s} b^2(z+r_1)x^2(z)dzds &= \int_{t-r_1}^{t}\int_{-r_1}^{z-t} b^2(z+r_1)x^2(z)dsdz \\
&= \int_{t-r_1}^{t} b^2(z-r_1)x^2(z)(z-t+r_1)dz \qquad (2.1.11)
\end{aligned}
$$

Now, if $t - r_1/2 \leq z \leq t$, then

$$\frac{r_1}{2} \leq (z - t + r_1) \leq r_1 \tag{2.1.12}$$

Equation (2.1.11) yields

$$\int_{-r_1}^{0}\int_{t+s}^{t} b^2(z+r_1)x^2(z)dzds$$

$$= \int_{t-r_1}^{t} \int_{-r_1}^{z-t} b^2(z+r_1)x^2(z)(z-t+r_1)dz$$
$$= \int_{t-r_1}^{t-r_1/2} b^2(z+r_1)x^2(z)(z-t+r_1)dz + \int_{t-r_1/2}^{t} b^2(z+r_1)x^2(z)(z-t+r_1)dz$$
$$\geq \int_{t-r_1/2}^{t} b^2(z+r_1)x^2(z)(z-t+r_1)dz.$$

Then using Equation (2.1.12), we get

$$\int_{-r_1}^{0} \int_{t+s}^{t} b^2(z+r_1)x^2(z)dzds \geq \frac{r_1}{2} \int_{t-r_1/2}^{t} b^2(z+r_1)x^2(z)dz \tag{2.1.13}$$

Let $V(t)$ be given by Equation (2.1.2), then

$$\begin{aligned} V(t) &\geq \int_{-r_1}^{0} \int_{t+s}^{t} b^2(z+r_1)x^2(z)dzds \\ &\geq \frac{r_1}{2} \int_{t-r_1/2}^{t} b^2(s+r_1)x^2(s)ds \end{aligned} \tag{2.1.14}$$

Consequently,

$$\begin{aligned} V(t-\frac{r_1}{2}) &\geq \frac{r_1}{2} \int_{t-r_1/2-r_1/2}^{t-r_1/2} b^2(s+r_1)x^2(s)ds \\ &\geq \frac{r_1}{2} \int_{t-r_1}^{t-r_1/2} b^2(s+r_1)x^2(s)ds \end{aligned} \tag{2.1.15}$$

Therefore,

$$\begin{aligned} & V(t) + V(t-\frac{r_1}{2}) \\ &= \left[x(t) + \int_{t-r_1}^{t} b(s+r_1)x(s)ds\right]^2 + \int_{-r_1}^{0} \int_{t+r_1}^{t} b^2(z+r_1)x^2(z)dzds + V(t-\frac{r_1}{2}) \\ &\geq \left[x(t) + \int_{t-r_1}^{t} b(s+r_1)x(s)ds\right]^2 + \frac{r_1}{2} \int_{t-r_1/2}^{t} b^2(s+r_1)x^2(s)ds \\ &\quad + \frac{r_1}{2} \int_{t-r_1}^{t-r_1/2} b^2(s+r_1)x^2(s)ds \end{aligned}$$

$$\geq \left[x(t) + \int_{t-r_1}^{t} b(s+r_1)x(s)ds\right]^2 + \frac{r_1}{2}\int_{t-r_1}^{t} b^2(s+r_1)x^2(s)ds \tag{2.1.16}$$

Using Hölder's inequality, we get

$$\left(\frac{1}{2}\int_{t-r_1}^{t} b(s+r_1)x(s)ds\right)^2 \leq \frac{r_1}{2}\int_{t-r-1}^{t} b^2(s+r_1)x^2(s)ds$$

Then, Inequality (2.1.16) becomes

$$\begin{aligned}
V(t) + V(t-\frac{r_1}{2}) \geq & \left[x(t) + \int_{t-r_1}^{t} b(s+r_1)x(s)ds\right]^2 + \frac{1}{2}\left[\int_{t-r_1}^{t} b(s+r_1)x(s)ds\right]^2 \\
= & x^2(t) + 2x(t)\int_{t-r_1}^{t} b(s+r_1)x(s)ds + \left(\int_{t-r_1}^{t} b(s+r_1)x(s)ds\right)^2 \\
& + \frac{1}{2}\left(\int_{t-r_1}^{t} b(s+r_1)x(s)ds\right)^2 \\
= & x^2(t) + 2x(t)\int_{t-r_1}^{t} b(s+r_1)x(s)ds + \frac{3}{2}\left(\int_{t-r_1}^{t} b(s+r_1)x(s)ds\right)^2 \\
= & x^2(t) + 2x(t)\int_{t-r_1}^{t} b(s+r_1)x(s)ds + \left(\sqrt{\frac{3}{2}}\int_{t-r_1}^{t} b(s+r_1)x(s)ds\right)^2 \\
= & \frac{1}{3}x^2(t) + \frac{2}{3}x^2(t) + \left(2\sqrt{\frac{2}{3}}\sqrt{\frac{3}{2}}\right)x(t)\int_{t-r_1}^{t} b(s+r_1)x(s)ds \\
& + \frac{2}{3}\left(\int_{t-r_1}^{t} b(s+r_1)x(s)ds\right)^2 \\
= & \frac{1}{3}x^2(t) + \left[\sqrt{\frac{2}{3}}x(t) + \sqrt{\frac{3}{2}}\int_{t-r_1}^{t} b(s+r_1)x(s)ds\right]^2 \\
\geq & \frac{1}{3}x^2(t) \quad \text{for} \quad t \geq t_0 + \frac{r_1}{2}
\end{aligned} \tag{2.1.17}$$

Thus, Inequality (2.1.17) shows that:

$$\frac{1}{3}x^2(t) \leq V(t) + V(t-\frac{r_1}{2}) \leq 2V(t-\frac{r_1}{2}) \tag{2.1.18}$$

Integrating Equation (2.1.9) from t_0 to t, we get the following inequality:

$$V(t) \leq V(t_0) e^{\int_{t_0}^{t} b(s+r_1)ds} \quad \text{for} \quad t \geq t_0. \tag{2.1.19}$$

As a consequence, we get

$$V\left(t - \frac{r_1}{2}\right) \leq V(t_0) e^{\int_{t_0}^{t-r_1/2} b(s+r_1)ds} \tag{2.1.20}$$

and using Inequality (2.1.18) we get $x^2(t) \leq 6V\left(t - \frac{r_1}{2}\right)$, and hence

$$|x(t)| \leq \sqrt{6V\left(t - \frac{r_1}{2}\right)} \leq \sqrt{6V(t_0)} e^{(1/2)\int_{t_0}^{t-r_1/2} b(s+r_1)ds} \quad \text{for} \quad t \geq t_0 + \frac{r_1}{2},$$

which completes the proof. ■

2.2 Stability of the Solution

If $-1/2r_1 \leq b(t+r_1) \leq -r_1 b^2(t+r_1)$, then Inequality (2.1.10) clearly implies that the zero solution of the integro-differential equation given by (2.0.2) is uniformly stable. In addition, if $\int^{\infty} b^2(s+r_1)ds \longrightarrow \infty$, the zero solution of Equation (2.0.2) is uniformly asymptotically stable. This solution is exponentially stable if $\int_{t_0}^{t-r_1/2} b(s+r_1)ds \geq \gamma(t-t_0)$ for all $t \geq t_0 + r_1/2$ and $\gamma > 0$ constant.

Example 1: Let

$$x'(t) = -\frac{1}{2}x(t - \frac{1}{3}) \tag{2.2.1}$$

In this equation, $r_1 = 1/3$, $b(t) = -1/2$. Therefore,

$$-\frac{1}{2r_1} \leq b(t+r_1) \leq -r_1 b^2(t+r_1)$$

By Theorem 2.1.2, the solution of Equation (2.2.1) for $t \geq t_0 + \frac{2}{3}$, satisfies the following:

$$|x(t)| \leq \sqrt{6V(t_0)}e^{1/2\int_{t_0}^{t-2/3} b(s)ds}$$
$$|x(t)| \leq \sqrt{6V(t_0)}e^{-1/4(t-2/3-t_0)}$$

2.3 A Criterion for Instability

Next, we will discuss the instability of the zero solution of the integro-differential equation (2.0.2). We will start with the following lemma.

Lemma 2.3.1. *Suppose $b(t+r_1) \geq Kb^2(t+r_1)$ for some constant $K > r_1$. If*

$$V(t) = \left[x(t) + \int_{t-r_1}^{t} b(s+r_1)x(s)ds\right]^2 - K\int_{t-r_1}^{t} b^2(s+r_1)x^2(s)ds, \qquad (2.3.1)$$

then along the solution of (2.0.2) we have

$$V'(t) \geq b(t+r_1)V(t). \qquad (2.3.2)$$

Proof: Let $x(t) = x(t, t_0, \phi)$ be a solution of Equation (2.0.2) and define

$$V(t) = \left[x(t) + \int_{t-r_1}^{t} b(s+r_1)x(s)ds\right]^2 - K\int_{t-r_1}^{t} b^2(s+r_1)x^2(s)ds,$$

then we have

$$\begin{aligned} V'(t) &= 2\left[x(t) + \int_{t-r_1}^{t} b(s+r_1)x(s)ds\right] b(t+r_1)x(t) \\ &\quad - Kb^2(t+r_1)x^2(t) + Kb^2(t)x^2(t-r_1) \\ &= b(t+r_1)\left[x^2(t) + 2x(t)\int_{t-r_1}^{t} b(s+r_1)x(s)ds + \left(\int_{t-r_1}^{t} b(s+r_1)x(s)ds\right)^2\right] \end{aligned}$$

$$
\begin{aligned}
&+ b(t+r_1)x^2(t) - b(t+r_1)\left(\int_{t-r_1}^{t} b(s+r_1)x(s)ds\right)^2 \\
&- Kb^2(t+r_1)x^2(t) + Kb^2(t)x^2(t-r_1) \\
= {} & b(t+r_1)\left\{\left[x(t) + \int_{t-r_1}^{t} b(s+r_1)x(s)ds\right]^2 - K\int_{t-r_1}^{t} b^2(s+r_1)x^2(s)ds\right\} \\
&+ b(t+r_1)\left[K\int_{t-r_1}^{t} b^2(s+r_1)x^2(s)ds - \left(\int_{t-r_1}^{t} b(s+r_1)x(s)ds\right)^2\right] \\
&+ \left[b(t+r_1) - Kb^2(t+r_1)\right]x^2(t) + Kb^2(t)x^2(t-r_1) \\
\geq {} & b(t+r_1)V(t),
\end{aligned}
$$

which completes the proof. ■

Theorem 2.3.2. *Suppose the hypothesis of Lemma 2.3.1 holds and* $x(t) = x(t, t_0, \phi)$ *is a solution of Equation (2.0.2), then the zero solution of (2.0.2) is unstable provided that*

$$
\int^{\infty} b^2(s+r_1)ds \longrightarrow \infty,
$$

where

$$
x(t) \geq \sqrt{\frac{K-r_1}{K}V(t_0)e^{\frac{1}{2}\int_{t_0}^{t} b(s+r_1)ds}}
$$

Proof: Since, by Hölder's inequality

$$
\left(\int_{t+r_1}^{t} b(s+r_1)x^2(s)ds\right)^2 \leq r_1\int_{t-r_1}^{t} b^2(s+r_1)x^2(s)ds
$$

then

$$
r_1\int_{t-r_1}^{t} b^2(s+r_1)x^2(s)ds - \left(\int_{t+r_1}^{t} b(s+r_1)x^2(s)ds\right)^2 \geq 0 \qquad (2.3.3)
$$

So by integrating Equation (2.3.3), we get

$$V(t) \geq V(t_0)e^{\int_{t_0}^{t} b(s+r_1)ds} \tag{2.3.4}$$

It is clear that for any $\gamma > 0$, $\left(\frac{\sqrt{r_1}}{\sqrt{\gamma}}a - \frac{\sqrt{\gamma}}{\sqrt{r_1}}b\right)^2 \geq 0$.

Then, we have

$$2ab \leq \frac{r1}{\gamma}a^2 + \frac{\gamma}{r_1}b^2 \tag{2.3.5}$$

Let $\gamma = K - r_1$, then we have

$$\begin{aligned}
V(t) =& x^2(t) + 2x(t)\int_{t-r_1}^{t} b(s+r_1)x(s)ds + \left[\int_{t-r_1}^{t} b(s+r_1)x(s)ds\right]^2 \\
&- K\int_{t-r_1}^{t} b^2(s+r_1)x^2(s)ds \\
\leq& x^2(t) + \frac{r_1}{\gamma}x^2(t) + \frac{\gamma}{r_1}\left[\int_{t-r_1}^{t} b(s+r_1)x(s)ds\right]^2 + \left[\int_{t-r_1}^{t} b(s+r_1)x(s)ds\right]^2 \\
&- (r_1+\gamma)\int_{t-r_1}^{t} b^2(s+r_1)x^2(s)ds \\
\leq& x^2(t) + \frac{r_1}{\gamma}x^2(t) + (\gamma + r_1 - K)\int_{t-r_1}^{t} b^2(s+r_1)x^2(s)ds \\
\leq& \frac{\gamma + r_1}{\gamma}x^2(t) \\
\leq& \frac{K}{K - r_1}x^2(t)
\end{aligned}$$

Using Inequality (2.3.4), we get

$$\begin{aligned}
x^2(t) &\geq \frac{K - r_1}{K}V(t) = \frac{K - r_1}{K}V(t_0)\ e^{\int_{t_0}^{t} b(s+r_1)ds} \\
|x(t)| &\geq \sqrt{\frac{K - r_1}{K}V(t_0)}\ e^{\frac{1}{2}\int_{t_0}^{t} b(s+r_1)ds};
\end{aligned}$$

this completes the proof ■

Chapter 3

Qualitative Behavior of Integro-Differential Equation with One Finite Delay

In this chapter, we consider the scalar non-linear Volterra integro-differential equation with finite delays

$$x'(t) = b(t)x(t - r_1) - \int_{t-r_2}^{t} a(t, s)g(x(s))ds$$

where r_1, r_2 are positive constants representing 2 finite delays, $t \geq 0$ and

$$a : \ [0, \infty) \times [-\tau, \infty) \to \mathbb{R}, \qquad \text{and} \qquad b : \ [0, \infty) \to \mathbb{R}$$

are continuous functions with $r_1 = 0$ and $r_2 \neq 0$.

This yields to the following Integro-differential equation with one finite delay

$$x'(t) = b(t)x(t) - \int_{t-r_2}^{t} a(t, s)g(x(s))ds, \tag{3.0.1}$$

where $t \geq 0$ and $r_2 > 0$, $a : [0, \infty) \times [-r_2, \infty) \longrightarrow (-\infty, \infty)$, $b : [0, \infty) \longrightarrow \mathbb{R}$, and the function $g(x)$ is continuous in x.

In the next section, we will study the qualitative behavior of the solutions of the previous equation.

3.1 Construction of Lyapunov Functional

We will start by constructing a Lyapunov functional $V(t, x) := V(t)$ and show that for some non-positive function $P(t)$ we have

$$V'(t) \leq -P(t, x)V(t),$$

under suitable conditions, along the solution of (3.0.1).

In order to put Equation (3.0.1) in a form to get a suitable Lyapunov functional, we define

$$A(t, s) = \int_{t-s}^{r_2} a(u + s, s)du, \tag{3.1.1}$$

for $t \in [0, \infty)$ and $s \in [-r_2, \infty)$.

It is clear that

$$A(t, t - r_2) \equiv 0 \qquad \text{for } t > 0,$$

and

$$\begin{aligned} A(t, t) &= \int_{t-t}^{r_2} a(u + t, t)du \\ &= \int_{0}^{r_2} a(u + t, t)du. \end{aligned} \tag{3.1.2}$$

To get our main results, we assume there exists a positive constant Δ such that

$$|g(x)| \leq \Delta|x|, \tag{3.1.3}$$

$$\text{and} \qquad xg(x) \leq x^2 g(x). \tag{3.1.4}$$

Obviously Conditions (3.1.3) and (3.1.4) imply that

$$g(0) = 0.$$

In addition, we assume that

$$A(t,s)q(t,s) \geq 0, \tag{3.1.5}$$

where $q(t,s) = \frac{\partial A}{\partial s}$ for all $t \in [0,\infty)$ and all $s \in [t-r_2, t]$. And finally, we let

$$A^2(t)\left(t - \frac{(\alpha-1)r_2}{\alpha}, z\right) \geq A^2(t,z), \tag{3.1.6}$$

for $1 \leq \alpha \leq 2$ and $t \in [0,\infty)$ and all $z \in \left[\frac{t-r_2}{\alpha}, \frac{t-(\alpha-1)r_2}{\alpha}\right]$.

As a consequence of Equations (3.1.1) and (3.1.5) and for all $t \in [0,\infty)$ and all $s \in [t-r_2, t]$, we have

$$\begin{aligned}
&\int_{-r_2}^{0}\int_{t+s}^{t} A(t,z)\frac{\partial A(t,z)}{\partial t} g^2(x(z))dzds = -\int_{t-r_2}^{t}\int_{-r_2}^{z-t} A(t,z)q(t,z)g^2(x(z))dsdz \\
&= -\int_{t-r_2}^{t} (z-t-r_2)A(t,z)q(t,z)g^2(x(z))dz \\
&\leq 0.
\end{aligned} \tag{3.1.7}$$

Inequality (3.1.7) plays a very crucial role in the proof of the next lemma.

Writing Equation (3.0.1) in the form

$$\begin{aligned}
x'(t) &= b(t)x(t) - A(t,t)g(x(t)) + \frac{d}{dt}\int_{t-r_2}^{t} A(t,s)g(x(s))ds, \\
x'(t) &= b(t)x(t) + Q(t,x) + \frac{d}{dt}\int_{t-r_2}^{t} A(t,s)g(x(s))ds,
\end{aligned} \tag{3.1.8}$$

where

$$Q(t,x) = -A(t,t)g(x(t)). \tag{3.1.9}$$

Lemma 3.1.1. *Let $x'(t)$ be defined as in Equation (3.1.8) where $Q(t,x)$ is defined as in Equation (3.1.9). Let*

$$P(t,x) = b(t) + Q(t,x). \tag{3.1.10}$$

Suppose that Equations (3.1.3) - (3.1.8) hold and $0 < r_2 \le 1/2$ with

$$\frac{2r_2 - 1}{2r_2} \le P(t,x) \le -\left[(r_2+1)\Delta^2 + 1\right] A^2(t,t). \tag{3.1.11}$$

If

$$V(t) = \left[x(t) - \int_{t-r_2}^{t} A(t,s)g(x(s))ds\right]^2 + \int_{-r_2}^{0}\int_{t+s}^{t} A^2(t,z)g^2(x(z))dzds, \tag{3.1.12}$$

then along the solution of (3.0.1) we have

$$V'(t) \le P(t,x)V(t). \tag{3.1.13}$$

Proof: Let $x(t) = x(t, t_0, \phi)$ be a solution of (3.0.1) and define $V(t)$ by Equation (3.1.12). Let us calculate the time derivative of the functional $V(t)$ along the solution $x(t)$ of (3.0.1).

Due to Condition (3.1.11), it is clear that $P(t,x) < 0$ for all $t \ge 0$.

$$\begin{aligned} V'(t) =& 2\left[x(t) - \int_{t-r_2}^{t} A(t,s)g(x(s))ds\right]\left[b(t)x(t) - A(t,t)g(x(t))\right] + r_2A^2(t,t)g^2(x(t)) \\ & - \int_{-r_2}^{0} A^2(t,t+s)g^2(x(t+s))ds + \int_{-r_2}^{0}\int_{t+s}^{t} 2A(t,z)\frac{\partial A(t,z)}{\partial t}g^2(x(z))dzds \\ =& 2\left[x(t) - \int_{t-r_2}^{t} A(t,s)g(x(s))ds\right]\left[b(t)x(t)\right] \end{aligned}$$

$$
\begin{aligned}
&+2\left[x(t)-\int_{t-r_2}^{t} A(t,s)g(x(s))ds\right]\Big[-A(t,t)g(x(t))\Big]+r_2A^2(t,t)g^2(x(t))\\
&-\int_{-r_2}^{0} A^2(t,t+s)g^2(x(t+s))ds+\int_{-r_2}^{0}\int_{t+s}^{t} 2A(t,z)\frac{\partial A(t,z)}{\partial t}g^2(x(z))dzds \quad \text{by (3.1.8)}\\
\leq&2x(t)\left[b(t)x(t)\right]-2\left[b(t)x(t)\right]\int_{t-r_2}^{t} A(t,s)g(x(s))ds-2x(t)A(t,t)g(x(t))\\
&+2A(t,t)g(x(t))\int_{t-r_2}^{t} A(t,s)g(x(s))ds+r_2A^2(t,t)g^2(x(t))\\
&-\int_{-r_2}^{0} A^2(t,t+s)g^2(x(t+s))ds+\int_{-r_2}^{0}\int_{t+s}^{t} 2A(t,z)\frac{\partial A(t,z)}{\partial t}g^2(x(z))dzds.
\end{aligned}
$$

By using Equation (3.1.3), we get

$$
\begin{aligned}
V'(t)\leq&2x(t)\left[b(t)x(t)\right]-2\left[b(t)x(t)\right]\int_{t-r_2}^{t} A(t,s)g(x(s))ds-x(t)A(t,t)g(x(t))\\
&-x(t)A(t,t)g(x(t))+2A(t,t)g(x(t))\int_{t-r_2}^{t} A(t,s)g(x(s))ds+r_2\Delta^2A^2(t,t)x^2(t)\\
&-\int_{-r_2}^{0} A^2(t,t+s)g^2(x(t+s))ds\\
\leq&x^2(t)b(t)+b(t)\Bigg\{\Bigg[x^2(t)-2x(t)\int_{t-r_2}^{t} A(t,s)g(x(s))ds\\
&+\left(\int_{t-r_2}^{t} A(t,s)g(x(s))ds\right)^2\Bigg]+\int_{-r_2}^{0}\int_{t+s}^{t} A^2(t,z)g^2(x(z))dzds\Bigg\}\\
&-b(t)\left(\int_{t-r_2}^{t} A(t,s)g(x(s))ds\right)^2-b(t)\int_{-r_2}^{0}\int_{t+s}^{t} A^2(t,z)g^2(x(z))dzds\\
&-A(t,t)g(x)x^2(t)-A(t,t)g(x)x^2(t)+2A(t,t)g(x)x(t)\int_{t-r_2}^{t} A(t,s)g(x(s))ds\\
&-2A(t,t)g(x)x(t)\int_{t-r_2}^{t} A(t,s)g(x(s))ds+r\Delta^2A^2(t,t)x^2(t)\\
&+2A(t,t)g(x(t))\int_{t-r_2}^{t} A(t,s)g(x(s))ds-\int_{-r_2}^{0} A^2(t,t+s)g^2(x(t+s))ds
\end{aligned}
$$

$$
\begin{aligned}
\leq&x^2b(t)+b(t)V(t)-b(t)\left[\int_{t-r_2}^{t} A(t,s)g(x(s))ds\right]^2\\
&-b(t)\int_{-r_2}^{0}\int_{t+s}^{t} A^2(t,z)g^2(x(z))dzds+\Delta^2A^2(t,t)x^2(t)
\end{aligned}
$$

$$- \int_{-r_2}^{0} A^2(t,t+s)g^2(x(t+s))ds - A(t,t)g(x)x^2(t)$$
$$- 2A(t,t)g(x)x(t)\int_{t-r_2}^{t} A(t,s)g(x(s))ds + 2A(t,t)g(x(t))\int_{t-r_2}^{t} A(t,s)g(x(s))ds$$

$$=b(t)V(t) - A(t,t)g(x)V(t) + x^2(t)b(t) - b(t)\left[\int_{t-r_2}^{t} A(t,s)g(x(s))ds\right]^2$$
$$- b(t)\int_{-r_2}^{0}\int_{t+s}^{t} A^2(t,z)g^2(x(z))dzds + A(t,t)g(x)\left[\int_{t-r_2}^{t} A(t,s)g(x(s))ds\right]^2$$
$$+ A(t,t)g(x)\int_{-r_2}^{0}\int_{t+s}^{t} A^2(t,z)g^2(x(z))dzds$$
$$- 2A(t,t)g(x)x(t)\int_{t-r_2}^{t} A(t,s)g(x(s))ds$$
$$+ 2A(t,t)g(x(t))\int_{t-r_2}^{t} A(t,s)g(x(s))ds + r\Delta^2 A^2(t,t)x^2(t)$$
$$- A(t,t)g(x(t))x^2(t) - \int_{-r_2}^{0} A^2(t,t+s)g^2(x(t+s))ds. \quad (3.1.14)$$

We can write some expressions in what to follow in order to simplify (3.1.14). First, let $u = t + s$, then

$$- \int_{-r_2}^{0} A^2(t,t+s)g^2(x(t+s))ds = - \int_{t-r_2}^{t} A^2(t,s)g^2(x(s))ds, \quad (3.1.15)$$

and, from Hölder's inequality and $2|ab| \le a^2 + b^2$, we get

$$\Big[- b(t) + A(t,t)g(x)\Big]\left(\int_{t-r_2}^{t} A(t,s)g(x(s))ds\right)^2$$
$$\le \left[-b(t) + A(t,t)g(x)\right] r_2 \int_{t-r_2}^{t} A^2(t,s)g^2(x(s))ds. \quad (3.1.16)$$

Finally, by changing the order of integration, we write

$$[-b(t) + A(t,t)g(x)]\int_{-r_2}^{0}\int_{t+s}^{t} A^2(t,z)g^2(x(z))dzds$$

$$\leq [-b(t) + A(t,t)g(x)]\, r_2 \int_{t-r_2}^{t} A^2(t,s)g^2(x(s))ds, \tag{3.1.17}$$

and

$$-2A(t,t)g(x(t)) \int_{t-r_2}^{t} A(t,s)g(x(s))ds \leq A^2(t,t)x^2(t) + r_2 \int_{t-r_2}^{t} A^2(t,s)g^2(x(s))ds. \tag{3.1.18}$$

Similarly,

$$2A(t,t)g(x(t)) \int_{t-r_2}^{t} A(t,s)g(x(s))ds \leq \Delta^2 A^2(t,t)x^2(t) + r_2 \int_{t-r_2}^{t} A^2(t,s)g^2(x(s))ds. \tag{3.1.19}$$

By invoking Equation (3.1.15) and substituting (3.1.15)-(3.1.19) into (3.1.14), we get

$$\begin{aligned}
V'(t) \leq & b(t)V(t) - A(t,t)g(x)V(t) \\
& + \left[A^2(t,t) + \Delta^2 A^2(t,t) + r_2\Delta^2 A^2(t,t) - A(t,t)g(x) + b(t)\right] x^2(t) \\
& + \left[2r_2 - b(t)r_2 + A(t,t)g(x)r_2 - 1 - b(t)r_2 + A(t,t)g(x)r_2\right] \int_{t-r_2}^{t} A^2(t,s)g^2(x(s))ds \\
\leq & b(t)V(t) + Q(t,x)V(t) + \left[\left((r_2+1)\Delta^2 + 1\right) A^2(t,t) + b(t) + Q(t,x)\right] x^2(t) \\
& \left[-2r_2(b(t) + Q(t,x)) - 1 + 2r_2)\right] \int_{t-r_2}^{t} A^2(t,s)g^2(x(s))ds \\
\leq & [b(t) + Q(t,x)]\, V(t) \qquad \text{by (3.1.11)} \\
\leq & P(t,x)V(t), \qquad (3.1.20)
\end{aligned}$$

this completes the proof. ■

Theorem 3.1.2. *Assume that the hypothesis of Lemma 3.1.1 and Condition (3.1.6) both hold, and let $1 < \alpha \leq 2$, then any solution $x(t) = x(t, t_0, \phi)$ of (3.0.1) satisfies* *the following exponential inequality*

$$|x(t)| \leq \sqrt{2\frac{1+\frac{\alpha-1}{\alpha}}{\frac{\alpha-1}{\alpha}}V(t_0)}\; e^{\frac{1}{2}\int_{t_0}^{t-\left(\frac{\alpha-1}{\alpha}\right)r_2}[b(s)-A(s,s)g(x(s))ds]}, \tag{3.1.21}$$

for $t \geq t_0 + \frac{\alpha-1}{\alpha}r_2$.

Proof: Let us change the order of integration of

$$\begin{aligned}\int_{-r_2}^{0}\int_{t+s}^{t} & A^2(t,z)g^2(x(z))dzds \\ &= \int_{t-r_2}^{t}\int_{-r_2}^{z-t} A^2(t,z)g^2(x(z))dsdz \\ &= \int_{t-r_2}^{t} A^2(t,z)g^2(x(z))(z-t+r_2)dz. \end{aligned} \tag{3.1.22}$$

For $1 < \alpha \leq 2$, and if $t - \frac{r_2}{\alpha} \leq z \leq t$, then

$$\left(\frac{\alpha-1}{\alpha}\right) r_2 \leq z - t + r_2 \leq r_2. \tag{3.1.23}$$

Then, Equation (3.1.22) gives

$$\begin{aligned}\int_{-r_2}^{0}\int_{t+s}^{t} & A^2(t,z)g^2(x(z))dzds \\ &= \int_{t-r_2}^{t} A^2(t,z)g^2(x(z))(z-t+r_2)dz \\ &= \int_{t-r_2/\alpha}^{t} A^2(t,z)g^2(x(z))(z-t+r_2)dz \qquad \text{by Chasles' rule} \\ &\quad + \int_{t-r_2}^{t-r_2/\alpha} A^2(t,z)g^2(x(z))(z-t+r_2)dz \\ &\geq \int_{t-r_2/\alpha}^{t} A^2(t,z)g^2(x(z))(z-t+r_2)dz \\ &\geq \left(\frac{\alpha-1}{\alpha}\right) r_2 \int_{t-r_2/\alpha}^{t} A^2(t,z)g^2(x(z))dz \qquad \text{by (3.1.23)}. \end{aligned} \tag{3.1.24}$$

Then, the function $V(t)$ in (3.1.12) can be written as

$$V(t) \geq \int_{-r_2}^{0} \int_{t+s}^{t} A^2(t,z) g^2(x(z)) dz ds$$
$$\geq \left(\frac{\alpha - 1}{\alpha} \right) r_2 \int_{t - r_2/\alpha}^{t} A^2(t,z) g^2(x(z)) dz. \tag{3.1.25}$$

This implies that for $1 < \alpha \leq 2$, we have $-r_2 + \frac{r_2}{\alpha} \geq -\frac{r_2}{\alpha}$ and hence using (3.1.6)

$$V\left(t - \frac{\alpha - 1}{\alpha} r_2\right) \geq (\alpha - 1) \frac{r_2}{\alpha} \int_{t-r_2}^{t - r_2 + r_2/\alpha} A^2(t,z) g^2(x(z)) dz$$
$$\geq \frac{\alpha - 1}{\alpha} r_2 \int_{t-r_2}^{t - r_2/\alpha} A^2(t,z) g^2(x(z)) dz. \tag{3.1.26}$$

Since $V'(t) \leq 0$ and for $t \geq t_0 + \left(\frac{\alpha-1}{\alpha}\right) r_2$ we have

$$0 \leq V(t) + V\left(t - \frac{\alpha - 1}{\alpha} r_2\right) \leq 2V\left(t - \frac{\alpha - 1}{\alpha} r_2\right). \tag{3.1.27}$$

From Inequalities (3.1.20), (3.1.25) and (3.1.26) we get

$$V(t) + V\left(t - \frac{\alpha - 1}{\alpha} r_2\right) \geq \left(x(t) - \int_{t-r_2}^{t} A(t,s) g(x(s)) ds \right)^2$$
$$+ \int_{-r_2}^{0} \int_{t+s}^{t} A^2(t,z) g^2(x(z)) dz ds$$
$$+ \frac{\alpha - 1}{\alpha} r_2 \int_{t-r_2}^{t - r_2/\alpha} A^2(t,z) g^2(x(z)) dz$$
$$\geq \left(x(t) - \int_{t-r_2}^{t} A(t,s) g(x(s)) ds \right)^2$$
$$+ \frac{\alpha - 1}{\alpha} r_2 \int_{t - r_2/\alpha}^{t} A^2(t,z) g^2(x(z)) dz$$
$$+ \frac{\alpha - 1}{\alpha} r_2 \int_{t-r_2}^{t - r_2/\alpha} A^2(t,z) g^2(x(z)) dz \qquad \text{by (3.1.24)}$$
$$= \left(x(t) - \int_{t-r_2}^{t} A(t,s) g(x(s)) ds \right)^2$$
$$+ \frac{\alpha - 1}{\alpha} r_2 \int_{t-r_2}^{t} A^2(t,z) g^2(x(z)) dz.$$

Using Hölder's inequality, we get

$$
\begin{aligned}
&V(t) + V\left(t - \frac{\alpha-1}{\alpha} r_2\right) \\
\geq &\left(x(t) - \int_{t-r_2}^{t} A(t,s)g(x(s))ds\right)^2 + \frac{\alpha-1}{\alpha}\left(\int_{t-r_2}^{t} A(t,s)g(x(s))ds\right)^2 \\
= &x^2(t) - \frac{1}{1+\frac{\alpha-1}{\alpha}}x^2(t) + \frac{1}{1+\frac{\alpha-1}{\alpha}}x^2(t) - 2x(t)\int_{t-r_2}^{t} A(t,s)g(x(s))ds \\
&+ \left(\frac{\alpha-1}{\alpha}+1\right)\left[\int_{t-r_2}^{t} A(t,s)g(x(s))ds\right]^2 \\
= &\frac{\frac{\alpha-1}{\alpha}}{1+\frac{\alpha-1}{\alpha}}x^2(t) \\
&+ \left(\frac{1}{1+\frac{\alpha-1}{\alpha}}x^2(t) - 2\frac{1}{\sqrt{1+\frac{\alpha-1}{\alpha}}}x(t)\sqrt{1+\frac{\alpha-1}{\alpha}}\int_{t-r_2}^{t} A(t,s)g(x(s))ds\right. \\
&\left. + \left(\frac{\alpha-1}{\alpha}+1\right)\left[\int_{t-r_2}^{t} A(t,s)g(x(s))ds\right]^2\right) \\
= &\frac{\frac{\alpha-1}{\alpha}}{1+\frac{\alpha-1}{\alpha}}x^2(t) + \left[\frac{1}{\sqrt{1+\frac{\alpha-1}{\alpha}}}x(t) - \sqrt{1+\frac{\alpha-1}{\alpha}}\int_{t-r_2}^{t} A(t,s)g(x(s))ds\right]^2,
\end{aligned}
$$

which yields to

$$
V(t) + V\left(t - \frac{\alpha-1}{\alpha} r_2\right) \geq \frac{\frac{\alpha-1}{\alpha}}{1+\frac{\alpha-1}{\alpha}}x^2(t). \tag{3.1.28}
$$

In this way, Inequality (3.1.28) shows that

$$
\begin{aligned}
\frac{\frac{\alpha-1}{\alpha}}{1+\frac{\alpha-1}{\alpha}}x^2(t) &\leq V(t) + V\left(t - \frac{\alpha-1}{\alpha} r_2\right) \\
&\leq 2V(t - \frac{\alpha-1}{\alpha} r_2).
\end{aligned} \tag{3.1.29}
$$

Integrating (3.1.13) from t_0 to t, we get

$$
V(t) \leq V(t_0)\, e^{\int_{t_0}^{t} P(s,x(s))ds}.
$$

Thus,

$$V(t-\frac{\alpha-1}{\alpha}r_2) \leq V(t_0)\ e^{\int_{t_0}^{t-\frac{\alpha-1}{\alpha}r_2} P(s,x(s))ds},$$

and from Inequality (3.1.28), we get

$$\frac{\frac{\alpha-1}{\alpha}}{1+\frac{\alpha-1}{\alpha}}x^2(t) \leq 2V(t-\frac{\alpha-1}{\alpha}r_2),$$

and hence

$$\begin{aligned}
\frac{\frac{\alpha-1}{\alpha}}{1+\frac{\alpha-1}{\alpha}}x^2(t) &\leq 2V(t_0)\ e^{\int_{t_0}^{t-\frac{\alpha-1}{\alpha}r_2} P(s,x(s))ds}; \\
x^2(t) &\leq 2\frac{1+\frac{\alpha-1}{\alpha}}{\frac{\alpha-1}{\alpha}}V(t_0)\ e^{\int_{t_0}^{t-\frac{\alpha-1}{\alpha}r_2} P(s,x(s))ds}; \\
|x(t)| &\leq \sqrt{2\frac{1+\frac{\alpha-1}{\alpha}}{\frac{\alpha-1}{\alpha}}V(t_0)}\ e^{(1/2)\int_{t_0}^{t-\frac{\alpha-1}{\alpha}r_2} P(s,x(s))ds}; \\
|x(t)| &\leq \sqrt{2\frac{1+\frac{\alpha-1}{\alpha}}{\frac{\alpha-1}{\alpha}}V(t_0)}\ e^{(1/2)\int_{t_0}^{t-\frac{\alpha-1}{\alpha}r_2}[b(s)-A(s,s)g(x(s))]ds}; \qquad (3.1.30)
\end{aligned}$$

this completes the proof. ■

3.2 Stability of the Solution

Inequality (3.1.21) shows that the zero solution of (3.0.1) is asymptotically stable provided that

$$\int^{\infty} P(s,x(s)) \longrightarrow \infty$$

Assuming that

$$b(t) - A(t,t)g(x(t)) \leq -\beta P(t,x) \leq -\beta,$$

and

$$\int_{t_0}^{t-\frac{\alpha-1}{\alpha}r_2} P(t,x) \leq -\beta(t-t_0),$$

for some positive constant β and all $t \geq t_0 + \frac{\alpha-1}{\alpha}r_2$, then Theorem 3.1.2 implies that the zero solution of (3.0.1) is exponentially stable as a consequence of Inequality (3.1.21).

3.3 A Criterion for Instability

Now, we will use a non-negative definite Lyapunov functional and obtain a criterion that can be easily used to check the instability of the zero solution of Equation (3.0.1). We begin with the following lemma:

Lemma 3.3.1. *Suppose Inequalities (3.1.3), (3.1.4) and (3.1.5) hold and that there exists a positive constant $\beta > r_2$ such that*

$$A^2(t,t)\ \left[\left(1+\Delta^2\right)+\Delta^2\beta\right] \leq P(t,x) \leq \frac{2r_2}{\beta - r_2}. \tag{3.3.1}$$

If

$$V(t) = \left(x(t) - \int_{t-r_2}^{t} A(t,s)g(x(s))ds\right)^2 - \beta \int_{t-r_2}^{t} A^2(t,z)g^2(x(z))dz. \tag{3.3.2}$$

Then along the solution of (3.0.1) we have

$$V'(t) \geq P(t,x)V(t). \tag{3.3.3}$$

Proof: We first note a consequence of (3.1.5),

$$\beta \int_{t-r_2}^{t} A(t,z)\frac{\partial A(t,z)}{\partial t}g^2(x(z))dz = \beta \int_{t-r_2}^{t} A(t,z)q(t,s)g^2(x(z))dz \geq 0. \tag{3.3.4}$$

Then, because of (3.3.1), it is clear that $P(x,t) > 0$ for all $t \geq 0$. Let $x(t) = x(t, t_0, \phi)$ be a solution of (3.0.1). Using Inequality (3.1.16) and calculating the time derivative of the functional $V(t)$ defined by (3.3.2) along the solution $x(t)$ of (3.0.1), and using (3.3.4)we get:

$$
\begin{aligned}
V'(t) =& 2\left(x(t) - \int_{t-r_2}^{t} A(t,s)g(x(s))ds\right)[b(t)x(t) - A(t,t)g(x(t))] \\
& - \beta A^2(t,t)g^2(x(t)) - \beta \int_{t-r_2}^{t} 2A(t,s)\frac{\partial A(t,z)}{\partial t} g^2(x(z))dz \\
\geq& 2\left(x(t) - \int_{t-r_2}^{t} A(t,s)g(x(s))ds\right)[b(t)x(t)] \\
& + 2\left(x(t) - \int_{t-r_2}^{t} A(t,s)g(x(s))ds\right)[A(t,t)g(x(t))] - \beta A^2(t,t)g^2(x(t)) \\
=& b(t)V(t) - A(t,t)g(x)V(t) + b(t)x^2(t) - x^2(t)A(t,t)g(x(t)) \\
& - 2A(t,t)g(x(t))x(t)\int_{t-r_2}^{t} A(t,s)g(x(s))ds \\
& + 2A(t,t)g(x(t))\int_{t-r_2}^{t} A(t,s)g(x(s))ds - \beta A^2(t,t)g^2(x(t)) \\
\geq& [b(t) - A(t,t)g(x)]\, V(t) + [-b(t) + A(t,t)g(x)]\left[\int_{t-r_2}^{t} A(t,s)g(x(s))ds\right]^2 \\
& + [b(t) - A(t,t)g(x)]\, \beta \int_{t-r_2}^{t} A^2(t,z)g^2(x(z))dz \\
& + 2A(t,t)g(x(t))\int_{t-r_2}^{t} A(t,s)g(x(s))ds - \beta A^2(t,t)|\Delta x(t)|^2 \\
& - x^2(t)A(t,t)g(x(t)) + b(t)x^2(t).
\end{aligned}
\tag{3.3.5}
$$

Using Inequalities (3.1.18), (3.1.19) and Hölder's inequality, we get

$$
A(t,t)g(x(t))\left[\int_{t-r_2}^{t} A(t,s)g(x(s))ds\right]^2 \geq r_2 A(t,t)g(x(t))\int_{t-r_2}^{t} A^2(t,s)g^2(x(s))ds. \tag{3.3.6}
$$

Then (3.3.5) simply becomes

$$
\begin{aligned}
V'(t) \geq & P(t,x)V(t) \\
& + \left[A^2(t,t) + \Delta^2 A^2(t,t) - \beta A^2(t,t)\Delta - A(t,t)g(x(t)) + b(t)\right] x^2(t) \\
& + [-r_2 b(t) + A(t,t)g(x(t))r_2 + b(t)\beta \\
& \quad -A(t,t)g(x)\beta + 2r_2] \int_{t-r_2}^{t} A^2(t,s)g^2(x(s))ds \\
\geq & P(t,x)V(t) + A^2(t,t)\left[\left(1+\Delta^2\right) - \beta\Delta^2\right] x^2(t) \\
& + [-r_2 P(t,x) + \beta P(t,x) + 2r_2] \int_{t-r_2}^{t} A^2(t,s)g^2(x(s))ds. \qquad (3.3.7)
\end{aligned}
$$

Using (3.3.1), we get

$$V'(t) \geq P(t,x)V(t), \qquad (3.3.8)$$

this completes the proof. ■

Theorem 3.3.2. *Assume that the conditions of Lemma 3.3.1 hold. Then the zero solution of (3.0.1) is unstable provided that*

$$\int_{t_0}^{\infty} P(s,x(s))ds = \infty, \qquad (3.3.9)$$

and

$$|x(t)| \geq \sqrt{\frac{\beta - r_2}{\beta} V(t_0)}\; e^{1/2 \int_{t_0}^{t} P(s,x(s))ds}. \qquad (3.3.10)$$

Proof: Integrating Equation (3.3.8) from t_0 to t gives

$$V(t) \geq V(t_0) e^{\int_{t_0}^{t} P(s,x(s))ds} \qquad (3.3.11)$$

The function $V(t)$ in Inequality (3.3.2) can be written as

$$V(t) = x^2(t) - 2x(t) \int_{t-r_2}^{t} A(t,s)g(x(s))ds$$

$$+\left[\int_{t-r_2}^{t} A(t,s)g(x(s))ds\right]^2 - \beta\int_{t-r_2}^{t} A^2(t,z)g^2(x(z))dz. \qquad (3.3.12)$$

Let

$$k = \beta - r_2, \qquad (3.3.13)$$

then from

$$\left(\frac{\sqrt{r_2}}{\sqrt{k}}a - \frac{\sqrt{k}}{\sqrt{r_2}}b\right)^2 \geq 0, \qquad (3.3.14)$$

we have

$$2ab \leq \frac{r_2}{k}a^2 + \frac{k}{r_2}b^2. \qquad (3.3.15)$$

Keeping (3.3.14) and (3.3.15) in mind, we get

$$\begin{aligned}-2x(t)\int_{t-r_2}^{t} A(t,s)g(x(s))ds &\leq 2|x(t)||\int_{t-r_2}^{t} A(t,s)g(x(s))ds| \\ &\leq \frac{r_2}{k}x^2(t) + \frac{k}{r_2}\left[\int_{t-r_2}^{t} A(t,s)g(x(s))ds\right]^2.\end{aligned}$$

Using Hölder's inequality, we get

$$\begin{aligned}-2x(t)\int_{t-r_2}^{t} A(t,s)g(x(s))ds &\leq \frac{r_2}{k}x^2(t) + \frac{k}{r_2}r_2\int_{t-r_2}^{t} A^2(t,s)g^2(x(s))ds \\ &\leq \frac{r_2}{k}x^2(t) + k\int_{t-r_2}^{t} A^2(t,s)g^2(x(s))ds. \qquad (3.3.16)\end{aligned}$$

Substituting (3.3.16) into (3.3.12) and using Hölder's inequality, we get

$$\begin{aligned}V(t) \leq& x^2(t) + \frac{r_2}{k}x^2(t) + \beta\int_{t-r_2}^{t} A^2(t,s)g^2(x(s))ds \\ &+ r_2\int_{t-r_2}^{t} A^2(t,s)g^2(x(s))ds - \beta\int_{t-r_2}^{t} A^2(t,z)g^2(x(z))dz.\end{aligned}$$

Using Equation (3.3.13)

$$\begin{aligned} V(t) \leq& x^2(t) + \frac{r_2}{k}x^2(t) + (k + r_2 - \beta)\int_{t-r_2}^{t} A^2(t,s)g^2(x(s))ds \\ =& \frac{k+r_2}{k}x^2(t) \\ =& \frac{\beta}{\beta - r_2}x^2(t). \end{aligned}$$

Using Equations (3.3.1) and (3.3.11), we get:

$$V(t) \leq \frac{\beta}{\beta - r_2}x^2(t),$$

then

$$\begin{aligned} |x(t)| &\geq \sqrt{\frac{\beta - r_2}{\beta}V(t)}; \\ |x(t)| &\geq \sqrt{\frac{\beta - r_2}{\beta}V(t_0)}\ e^{1/2\int_{t_0}^{t} P(s,x(s))ds}, \end{aligned}$$

this completes the proof. ■

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