# Introduction to Differential Geometry of Space Curves and Surfaces

A Thesis Presented

by

**Tatiana Nicolakis**

to

The Faculty of Natural and Applied Sciences

in Partial Fulfillment of the Requirements

for the Degree of

**Master of Science**

in

**Mathematics**

Notre Dame University-Louaize

Zouk Mosbeh, Lebanon

**September 2020**

**Notre Dame University-Louaize,**
**Zouk Mosbeh, Lebanon**

Department of Mathematics and Statistics

**Tatiana Nicolakis**

We, the thesis committee for the above candidate for the Master of Science degree, hereby recommend acceptance of this thesis.

**Roger Nakad – Thesis Advisor**
**Department of Mathematics and Statistics**

**George Eid – First Reader**
**Department of Mathematics and Statistics**

**Holem Saliba – Second Reader**
**Department of Mathematics and Statistics**

**Roger Nakad – Third Reader**
**Department of Mathematics and Statisctis**

This thesis is accepted by the Faculty of Natural and Applied Sciences.

**George Eid**
Dean of the Faculty of Natural and
Applied Sciences

Abstract of the Thesis

# Introduction to Differential Geometry of Space Curves and Surfaces


by

**Tatiana Nicolakis**

**Master of Science**

in

**Mathematics**

Notre Dame University-Louaize,

Zouk Mosbeh,
Lebanon


2020


This thesis is an introduction to some of the classical theory and results of Differential Geometry: The geometry of curves and surfaces lying (mostly) in 3-dimensional space.

One of the most important tools used to analyze a curve is the Frenet frame, a moving frame that provides a coordinate system at each point of the curve that is adapted to the curve near that point. Given a curve, one can define two quantities: its curvature and torsion. Both quantities are scalar fields and depend on

some parameter which parametrizes the curve that is in general the arclength of the curve.

The Fundamental Theorem of Space Curves states that every regular curve in three-dimensional space, with non-zero curvature, is completely determined by its curvature and torsion. It means that from just the curvature and torsion, the vector fields for the tangent, normal, and binormal vectors can be derived using the Frenet–Serret formulas. Then, integrating the tangent field yields the curve. In the first chapter of this thesis, we present the proof of the Fundamental Theorem of Space Curves using two approaches. The first proof is the traditional one used in almost all differential geometry references [1, 2]. The second approach is a new one established recently by H. Guerrrero in [3]. It is based on finding a solution of a nonlinear differential equation of second order. As applications of the second approach, general slants and helices are characterized.

The second chapter revolves around defining a parametrized surface in the plane and introducing its first and second fundamental forms. This will allow to define the notions of curvature: the Gaussian curvature and the Mean curvature. The Gaussian curvature describes the intrinsic geometry of the surface, whereas the Mean curvature describes how it bends in space. The Gaussian curvature of a cone is zero: This is why we can make a cone out of a flat piece of paper. The Gaussian curvature of a sphere is strictly positive: This is why planar maps of the earth's surface invariably distort distances. The Gauss-Codazzi equations (also called the Gauss–Codazzi–Mainardi equations) are fundamental equations which link together the induced scalar product on $\mathbb{R}^3$ and the second fundamental form of a surface. The first equation, often called the Gauss equation was discovered by Carl Friedrich Gauss. It states that the Gauss curvature of the surface, at any given point, is encoded by the second fundamental form. The second equation, called the Codazzi equation or Codazzi-Mainardi equation, discovered by Gaspare Mainardi (1856) and Delfino Codazzi (1868–1869) states that the covariant derivative of the second fundamental form is fully symmetric. It turns out that the Gauss-Codazzi equations are sufficient to prove the existence of a surface in $\mathbb{R}^3$. This is called the Fundamental Theorem of Surfaces and it is proved in Chapter 3. In fact, consider a symmetric, positive definite matrix field of order two and a symmetric matrix field of order two that satisfy together the Gauss-Codazzi equations in a connected and simply connected open subset of $\mathbb{R}^2$. If the matrix fields are respectively of class $C^2$ and $C^1$, the fundamental theorem of surface theory asserts that there exists a surface immersed in the three-dimensional Euclidean space with these fields as its first and second fundamental forms.

To my family.

# Contents

# Acknowledgements

Many thanks go first and foremost to my advisor Dr. Roger Nakad for his continuous motivation and support throughout this entire journey. I also thank him for all the effort he has put in order to make this experience memorable.

Also, I would like to thank Dr. George Eid and all my instructors for their guidance and patience during the entire period of my masters studies.

Finally, I can never forget to thank my family and all my close friends for always being there and supporting me through every step of the way. I could never have done it without them.

# Chapter 1

# Parametrized Curves in $\mathbb{R}^n$

In this first chapter, we introduce the concept of a parametrized curve, its parametrization by arclength, curvature and torsion. We then prove the Fundamental Theorem of Curves in two different methods. The first method is the traditional one used in almost all differential geometry references ([1, 2, 4, 5]). The second method is a new one recently established by H. Guerrero in [3]. It is based on finding a solution of a non linear differential equation of second order. As applications of this second method, general helices and slant helices are characterized.

## 1.1 Parametrized Curves

In this section, we define parametrized curves and give some common examples.

**Definition 1.1.1.** *Let* $[a, b]$ *be a closed interval in* $\mathbb{R}$*. A map*

$$\begin{aligned} F : [a, b] \subseteq \mathbb{R} &\longrightarrow \mathbb{R}^n \\ t &\longrightarrow F(t) = \Big(x_1(t), x_2(t), \cdots, x_n(t)\Big) \end{aligned}$$

*is smooth if all derivatives of its components* $x_j : [a, b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ $(1 \leq j \leq n)$ *exist and are continuous for all* $t \in (a, b)$*. In this case, we write* $F'(t) = \Big(x_1'(t), x_2'(t), \cdots, x_n'(t)\Big)$.

**Definition 1.1.2.** *A parametrized curve* $\alpha$ *in* $\mathbb{R}^n$ *is a smooth map, explicitly given by*

$$\begin{aligned} \alpha : [a, b] \subseteq \mathbb{R} &\longrightarrow \mathbb{R}^n \\ t &\longrightarrow \alpha(t) = \Big(x_1(t), x_2(t), \cdots, x_n(t)\Big). \end{aligned}$$

*It is called a parametrized regular curve in* $\mathbb{R}^n$ *if* $\alpha'(t) \neq 0$ *for all* $t \in (a, b)$.

**Example 1.1.3.**

- **Straight line***: A parametrization of the straight line in* $\mathbb{R}^2$ *passing through the points* $(1, 2)$ *and* $(2, -3)$ *can be written as*

$$\begin{aligned} \alpha : \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ t &\longrightarrow \alpha(t) = (t, -5t + 7). \end{aligned}$$

  *This curve is regular since* $\alpha'(t) = (1, -5) \neq (0, 0)$ *for all* $t \in \mathbb{R}$.

- **Ellipse**: *A parametrization of the ellipse of equation* $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ *can be written as:*

$$\begin{aligned} \alpha : [0, 2\pi] &\longrightarrow \mathbb{R}^2 \\ t &\longrightarrow \alpha(t) = (2\cos t, 3\sin t). \end{aligned}$$

  *The curve is regular since* $\alpha'(t) = (-2\sin t, 3\cos t) \neq (0,0)$ *for all* $t \in (0, 2\pi)$.

- **The graph of a function**: *If* $f : [a,b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ *is a smooth function, then a parametrization of its graph can be written as:*

$$\begin{aligned} \alpha : [a, b] &\longrightarrow \mathbb{R}^2 \\ x &\longrightarrow \alpha(x) = (x, f(x)). \end{aligned}$$

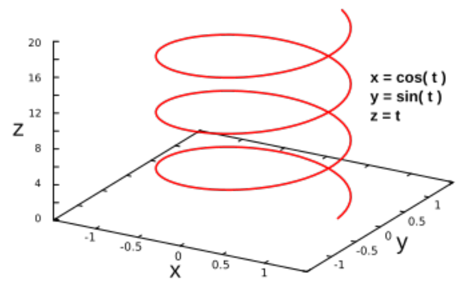
  *The curve is regular since* $\alpha'(x) = (1, f'(x)) \neq (0,0)$ *for all* $x \in (a,b)$.

**Remark** A parametrization of a curve is not unique, and regularity of a curve does not depend on its parametrization. For instance, $\alpha(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$ and $\beta(t) = (\cos 2t, \sin 2t)$ for $t \in [0, \pi]$ are two distinct parametrizations that represent the same curve which is the circle of equation $x^2 + y^2 = 1$ in $\mathbb{R}^2$. Considering $\alpha$ or $\beta$, we have that the curve is regular since $\alpha'(t) = (-\sin t, \cos t) \neq (0,0)$ for all $t \in (0, 2\pi)$ and $\beta'(t) = (-2\sin t, 2\cos t) \neq (0,0)$ for all $t \in (0, \pi)$.

**Example 1.1.4. Helix:** *The parametrized curve* $\theta : \mathbb{R} \longrightarrow \mathbb{R}^3$ *given by*

$$\theta(t) = (r\cos t, r\sin t, at),$$

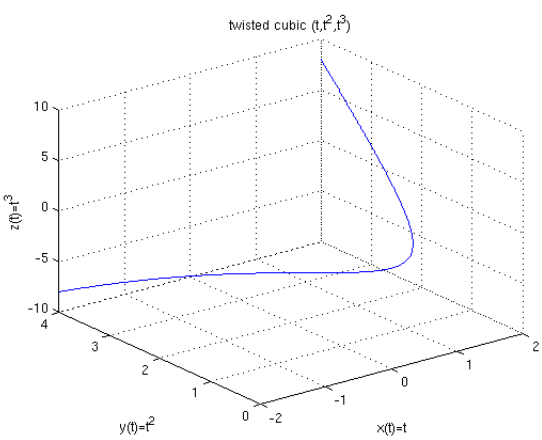
*with* $r > 0$ *and* $a \in \mathbb{R}^*$, *has its image the circle helix having radius* $r$ *and pitch* $a$. *The curve is regular since* $\theta'(t) = (-r\sin t, r\cos t, a) \neq (0,0,0)$ *for all* $t \in \mathbb{R}$.




**Example 1.1.5. Twisted cubic**: *The parametrized curve* $\theta : \mathbb{R} \longrightarrow \mathbb{R}^3$ *given by*

$$\theta(t) = (t, t^2, t^3)$$

*represents a twisted cubic.*




*The curve is regular because* $\theta'(t) = (1, 2t, 3t^2) \neq (0,0,0)$ *for all* $t \in \mathbb{R}$.

**Definition 1.1.6.** *For a parametrized curve* $\alpha : [a, b] \longrightarrow \mathbb{R}^n$, *the map*

$$\alpha' : (a, b) \longrightarrow \mathbb{R}^n$$

*is called the tangent of* $\alpha$. *From now on, it will be denoted by* $T$.

## 1.2 Arclength of Regular Curves

**Definition 1.2.1.** *The arclength of a regular parametrized curve*

$$\begin{aligned} \alpha : [a, b] &\longrightarrow \mathbb{R}^n \\ t &\longrightarrow \alpha(t) = \Big(x_1(t), x_2(t), ..., x_n(t)\Big) \end{aligned}$$

*from the point* $t_0 \in (a, b)$ *is given by*

$$s(t) = \int_{t_0}^{t} \|\alpha'(r)\| dr,$$

*for all* $t \in (a, b)$, *where* $\|\alpha'(t)\| = \sqrt{\Big((x_1'(t))^2 + (x_2'(t))^2 + .... + (x_n'(t)\Big)^2}$.

**Remark** Clearly, the arclength function $s(t)$ is one-to-one. In fact, if $s(t) = s(t')$, we have

$$\begin{aligned} & \int_{t_0}^{t} \|\alpha'(x)\| dx = \int_{t_0}^{t'} \|\alpha'(x)\| dx \\ \Longrightarrow & \int_{t}^{t'} \|\alpha'(x)\| dx = 0 \\ \Longrightarrow & \ t = t', \text{ since } \|\alpha'(t)\| > 0. \end{aligned}$$

**Definition 1.2.2.** *A parametrized regular curve* $\alpha : [a,b] \longrightarrow \mathbb{R}^n$ *is said to be parametrized by arclength if* $\|\alpha'(t)\| = 1$ *for all* $t \in (a,b)$*.*

**Remark** A regular curve can always be parametrized by arclength. In fact, assume $\alpha : [a,b] \longrightarrow \mathbb{R}^n$ is a parametrized regular curve with $s : (a,b) \longrightarrow \mathbb{R}^+$ being its arclength function. Denoting the inverse function of $s$ by $t = t(s)$, one can check that the map $\beta(s) = \alpha\big(t(s)\big)$ is a regular curve and it is parametrized by arclength, because

$$\begin{aligned}
\|\beta'(s)\| &= \|\alpha'\big(t(s)\big)t'(s)\| \\
&= |t'(s)|\|\alpha'\big(t(s)\big)\| \\
&= \frac{1}{|s'(t(s))|}\|\alpha'\big(t(s)\big)\| \\
&= \frac{1}{\|\alpha'\big(t(s)\big)\|}.\|\alpha'\big(t(s)\big)\| = 1.
\end{aligned}$$

**Example 1.2.3.**

- **Straight Line***: From Example 1.1.3, we have* $\alpha'(t) = (1,-5)$*. The arclength function starting from* $t_0 = 0$ *is given by*

  $$s(t) = \int_{t_0}^{t} \|\alpha'(x)\|dx = \int_{t_0}^{t} \sqrt{1^2 + (-5)^2}dx = \int_{t_0}^{t} \sqrt{1+25}dx = \sqrt{26}t.$$

  *So, the inverse function of* $s$ *is* $t(s) = \dfrac{s}{\sqrt{26}}$*. Then,*

  $$\beta(s) = \alpha\Big(t(s)\Big) = \alpha\Big(\frac{s}{\sqrt{26}}\Big) = \Big(\frac{s}{\sqrt{26}}, \frac{-5s}{\sqrt{26}} + 7\Big)$$

  *is the arclength parametrization of* $\alpha$*.*

- **Circle***: Let* $(C)$ *be the circle of center* $(1,-1)$ *and radius 2. A parametriza-*

*tion of this circle is given by:* $\alpha(t) = (1+2\cos t, -1+2\sin t)$ *for* $t \in [0, 2\pi]$. *We have* $\alpha'(t) = (-2\sin t, 2\cos t)$ *and the arclength function starting from* $t_0 = 0$ *is given by:*

$$s(t) = \int_0^t \sqrt{4\sin^2 x + 4\cos^2 x}dx = \int_0^t 2dx = 2t$$

*Then,* $t(s) = \dfrac{s}{2}$ *and the arclength parametrization of* $\alpha$ *is given by*

$$\beta(s) = \alpha\big(t(s)\big) = \alpha\Big(\frac{s}{2}\Big) = \left(1 + 2\cos\left(\frac{s}{2}\right), -1 + 2\sin\left(\frac{s}{2}\right)\right).$$

- **Helix**: *From Example 1.1.4, we have* $\alpha'(t) = (-r\sin t, r\cos t, a)$ *for all* $t \in \mathbb{R}$. *The arclength function starting from* $t_0 = 0$ *is then given by*

$$s(t) = \int_0^t \sqrt{r^2 + a^2}dx = t\sqrt{r^2 + a^2}$$

*Then,* $t(s) = \dfrac{s}{\sqrt{r^2 + a^2}}$ *and the arclength parametrization of* $\alpha$ *is given by*

$$\begin{aligned}\beta(s) = \alpha\big(t(s)\big) &= \alpha\Big(\frac{s}{\sqrt{r^2+a^2}}\Big) \\ &= \left(r\cos\left(\frac{s}{\sqrt{r^2+a^2}}\right), r\sin\left(\frac{s}{\sqrt{r^2+a^2}}\right), \frac{as}{\sqrt{r^2+a^2}}\right).\end{aligned}$$

- **Twisted Cubic**: *From Example 1.1.5, we have* $\theta'(t) = (1, 2t, 3t^2)$ *for all* $t \in \mathbb{R}$. *The arclength function starting from* $t_0 = 0$*:*

$$s(t) = \int_0^t \sqrt{1 + 4x^2 + 9x^4}dx.$$

*The arclength parametrization of $\theta$ is: $\beta(s) = \alpha(t(s))$ where $t = t(s)$ is the inverse function of $s(t)$.*

## 1.3 Curvature and Torsion

In this section, we define the curvature and torsion of a regular curve $\alpha$ parametrized by arclength.

**Definition 1.3.1.** *Let $\alpha : [a, b] \longrightarrow \mathbb{R}^n$ be a regular curve parametrized by arclength. For $s \in (a, b)$, the number $k(s)$ defined by*

$$k(s) = \|\alpha''(s)\| = \|T'(s)\|$$

*is called the curvature of $\alpha$ at $s$. In this case, the radius of curvature of $\alpha$ at the point $s$ is $r(s) = \dfrac{1}{k(s)}$ for $k(s) \neq 0$.*

**Remark** If $\alpha : [a, b] \longrightarrow \mathbb{R}^n$ is a regular curve parametrized by arclength, we say that $\alpha$ is biregular if its curvature $k(s) \neq 0$ for all $s \in (a, b)$.

**Example 1.3.2.** *Consider the following parametrization of a straight line in $\mathbb{R}^3$:*

$$\begin{aligned} \alpha : \mathbb{R} &\longrightarrow \mathbb{R}^3 \\ t &\longrightarrow \alpha(t) = (at + a_0, bt + b_0, ct + c_0). \end{aligned}$$

*A parametrization by arclength starting with $t_0 = 0$ is given by:*

$$\beta(s) = \left( \frac{as + a_0}{\sqrt{a^2 + b^2 + c^2}}, \frac{bs + b_0}{\sqrt{a^2 + b^2 + c^2}}, \frac{cs + c_0}{\sqrt{a^2 + b^2 + c^2}} \right).$$

*We calculate* $\beta'(s)$ *and get*

$$\beta'(s) = \left( \frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}} \right).$$

*Hence,* $\beta''(s) = (0,0,0)$ *and* $\|\beta''(s)\| = 0$. *This shows that the curvature of a straight line in* $\mathbb{R}^3$ *is* $0$.

**Example 1.3.3.** *Let* $(C)$ *be a circle of center* $(x_0, y_0)$ *and radius* $r > 0$. *A parametrization* $\alpha$ *of this circle is given by*

$$\begin{aligned} \alpha : [0.2\pi] &\longrightarrow \mathbb{R}^2 \\ t &\longrightarrow \alpha(t) = (x_0 + r\cos t, y_0 + r\sin t). \end{aligned}$$

*The arclength function starting from* $t_0 = 0$ *is given by*

$$\begin{aligned} s(t) &= \int_0^t \|\alpha'(x)\| dx \\ &= \int_0^t \sqrt{r^2 \sin^2 x + r^2 \cos^2 x} dx \\ &= \int_0^t r dx = rt. \end{aligned}$$

*Hence,* $t = \dfrac{s}{r}$ *and* $\beta(s) = \alpha(t(s)) = \left(x_0 + r\cos\left(\frac{s}{r}\right), y_0 + r\sin\left(\frac{s}{r}\right)\right)$. *One can calculate* $\beta'(s)$ *and* $\beta''(s)$ *to get*

$$\begin{aligned} \beta'(s) &= \left( -\sin\left(\frac{s}{r}\right), \cos\left(\frac{s}{r}\right) \right), \\ \beta''(s) &= \left( -\frac{1}{r}\cos\left(\frac{s}{r}\right), -\frac{1}{r}\sin\left(\frac{s}{r}\right) \right). \end{aligned}$$

*The curvature $k(s)$ is then given by*

$$\begin{aligned} k(s) &= \|\beta''(s)\| \\ &= \sqrt{\frac{1}{r^2}\cos^2\left(\frac{s}{r}\right) + \frac{1}{r^2}\sin^2\left(\frac{s}{r}\right)} \\ &= \frac{1}{r}, \end{aligned}$$

*which is a constant. So, $(C)$ has a constant curvature and a radius of curvature $r$.* **This is the reason why the reciprocal of the curvature is called the radius of curvature.**

**Example 1.3.4.** *From Example 1.2.3, the arclength parametrization of a helix is given by:*

$$\beta(s) = \left(r\cos\left(\frac{s}{\sqrt{r^2+a^2}}\right), r\sin\left(\frac{s}{\sqrt{r^2+a^2}}\right), \frac{as}{\sqrt{r^2+a^2}}\right).$$

*One can calculate $\beta'(s)$ and $\beta''(s)$ to get*

$$\begin{aligned} \beta'(s) &= \left(-\frac{r}{\sqrt{r^2+a^2}}\sin\left(\frac{s}{\sqrt{r^2+a^2}}\right), \frac{r}{\sqrt{r^2+a^2}}\cos\left(\frac{s}{\sqrt{r^2+a^2}}\right), \frac{a}{\sqrt{r^2+a^2}}\right), \\ \beta''(s) &= \left(-\frac{r}{r^2+a^2}\cos\left(\frac{s}{\sqrt{r^2+a^2}}\right), -\frac{r}{r^2+a^2}\sin\left(\frac{s}{\sqrt{r^2+a^2}}\right), 0\right). \end{aligned}$$

*The curvature $k(s)$ is then given by:*

$$\begin{aligned} k(s) &= \|\beta''(s)\| \\ &= \sqrt{\frac{r^2}{(r^2+a^2)^2}\cos^2\left(\frac{s}{\sqrt{r^2+a^2}}\right) + \frac{r^2}{(r^2+a^2)^2}\sin^2\left(\frac{s}{\sqrt{r^2+a^2}}\right)} = \frac{|r|}{r^2+a^2}. \end{aligned}$$

*Hence, a helix has a constant curvature of $\dfrac{|r|}{r^2+a^2}$.*

**Remark** Assume that $\alpha : [a,b] \longrightarrow \mathbb{R}^2$ is a curve in $\mathbb{R}^2$ parametrized by arclength. Since $\alpha'(t)$ is a unit vector, we can write

$$\alpha'(t) = \Big(\cos\theta(t), \sin\theta(t)\Big),$$

where $\theta$ is the angle between the positive $x$-axis and the tangent vector $\alpha'(t)$ (measured counterclockwise). Thus, we can say that $\theta(t)$ is the direction of the curve $\alpha$ at $\alpha(t)$. So, the curvature of $\alpha$ is also defined to be the instantaneous rate of change of $\theta$ with respect to the arclength. It means, $k = \dfrac{\partial\theta}{\partial s}$.

**Proposition 1.3.5.** *Let $\alpha$ be a regular curve parametrized by arclength and defined by:*

$$\begin{aligned} \alpha : [a,b] &\longrightarrow \mathbb{R}^2 \\ t &\longrightarrow \alpha(t) = \Big(x(t), y(t)\Big). \end{aligned}$$

*Then, its curvature is given by*

$$k = \frac{y''x' - x''y'}{\Big[(x')^2 + (y')^2\Big]^{\frac{3}{2}}}.$$

*Proof.* We know that $\alpha'(t) = (x'(t), y'(t))$. From the above remark, since $\alpha$ is parametrized by arclength, we can conclude that $\theta(t) = \tan^{-1}\Big(\dfrac{y'(t)}{x'(t)}\Big)$, and hence

$$\frac{\partial\theta}{\partial s} = \frac{\partial\theta}{\partial t}.\frac{\partial t}{\partial s}$$

$$
\begin{aligned}
&= \left(\frac{\frac{y''(t)x'(t)-x''(t)y'(t)}{(x'(t))^2}}{1+(\frac{y'(t)}{x'(t)})^2}\right).\frac{1}{\frac{\partial s}{\partial t}} \\
&= \frac{y''x'-x''y'}{x'^2+y'^2}.\frac{1}{s'(t)} \\
&= \frac{y''x'-x''y'}{x'^2+y'^2}.\frac{1}{\|\alpha'(t)\|} \\
&= \frac{y''x'-x''y'}{x'^2+y'^2}.\frac{1}{\sqrt{x'^2+y'^2}} \\
&= \frac{y''x'-x''y'}{(x'^2+y'^2)^{\frac{3}{2}}}.
\end{aligned}
$$

□

**Example 1.3.6.** *If we go back to Example 1.3.3, we can see that we can recalculate the curvature using Proposition 1.3.5. In fact, we have*

$$
\begin{aligned}
k &= \frac{y''x'-x''y'}{\left[(x')^2+(y')^2\right]^{\frac{3}{2}}} \\
&= \frac{\frac{1}{r}\sin(\frac{s}{r})\sin(\frac{s}{r})+\frac{1}{r}\cos(\frac{s}{r})\cos(\frac{s}{r})}{\left(\sin^2(\frac{s}{r})+\cos^2(\frac{s}{r})\right)^{\frac{3}{2}}} = \frac{\frac{1}{r}\sin^2(\frac{s}{r})+\frac{1}{r}\cos^2(\frac{s}{r})}{1^{\frac{3}{2}}} = \frac{1}{r}.
\end{aligned}
$$

**Theorem 1.3.7.** *Let $\alpha : [a,b] \longrightarrow \mathbb{R}^n$ be any regular parametrized curve. Then, the curvature $k : [a,b] \longrightarrow \mathbb{R}^+$ of $\alpha$ is given by:*

$$
k = \frac{\sqrt{\|\alpha'\|^2\|\alpha''\|^2 - |\langle\alpha'',\alpha'\rangle|^2}}{\|\alpha'\|^2},
$$

*where $|\langle .,. \rangle|$ is the Euclidean scalar product in $\mathbb{R}^n$.*

*Proof.* Only in this proof, we shall denote $\dfrac{\partial\alpha}{\partial s}$ by $\dot{\alpha}$ and $\dfrac{\partial\alpha}{\partial t}$ by $\alpha'$. Let $s : (a,b) \longrightarrow \mathbb{R}$ be the arclength of $\alpha$ measured starting from any point. We have that

$$\alpha'(t) = \frac{\partial \alpha}{\partial t}(t) = \frac{\partial \alpha}{\partial s}(s(t)).\frac{\partial s}{\partial t}(t) = \frac{\partial \alpha}{\partial s}(s(t)).\|\alpha'(t)\| = \dot{\alpha}(s(t)).\|\alpha'(t)\|.$$

Hence

$$\dot{\alpha}(s(t)) = \frac{\alpha'(t)}{\|\alpha'(t)\|}. \tag{1.3.1}$$

We derive now $\dot{\alpha}(s(t))$ with respect to $t$ and get

$$\frac{\partial \dot{\alpha}}{\partial t}(s(t)) = \frac{\partial \dot{\alpha}}{\partial s}(s(t)).\frac{\partial s}{\partial t}(t) = \frac{\partial \dot{\alpha}}{\partial s}(s(t)).\|\alpha'(t)\|.$$

Thus, we have

$$\frac{\partial \dot{\alpha}}{\partial s}(s(t)) = \frac{\partial \dot{\alpha}}{\partial t}(s(t)).\frac{1}{\|\alpha'(t)\|} = \frac{1}{\|\alpha'(t)\|}.\frac{\partial}{\partial t}\Big(\frac{\alpha'(t)}{\|\alpha'(t)\|}\Big). \tag{1.3.2}$$

But, we have that

$$\frac{\partial}{\partial t}\Big(\frac{\alpha'(t)}{\|\alpha'(t)\|}\Big) = \frac{\alpha''(t).\|\alpha'(t)\| - (\|\alpha'(t)\|)'.\alpha'(t)}{\|\alpha'(t)\|^2}. \tag{1.3.3}$$

and we also know that $\|\alpha'(t)\|^2 = \langle \alpha'(t), \alpha'(t)\rangle$. Deriving both sides of the last identity with respect to $t$, we get $2\|\alpha'(t)\|(\|\alpha'(t)\|)' = 2\langle \alpha'(t), \alpha''(t)\rangle$, which is

$$(\|\alpha'(t)\|)' = \frac{\langle \alpha'(t), \alpha''(t)\rangle}{\|\alpha'(t)\|}. \tag{1.3.4}$$

Hence, inserting (1.3.4) in (1.3.3), we obtain,

$$\begin{aligned}
\frac{\partial}{\partial t}\Big(\frac{\alpha'(t)}{\|\alpha'(t)\|}\Big) &= \frac{\alpha''(t)}{\|\alpha'(t)\|} - \frac{\alpha'(t)}{\|\alpha'(t)\|^2}.\frac{\langle \alpha'(t), \alpha''(t)\rangle}{\|\alpha'(t)\|} \\
&= \frac{1}{\|\alpha'(t)\|}\left(\alpha''(t) - \frac{\langle \alpha'(t), \alpha''(t)\rangle}{\|\alpha'(t)\|^2}.\alpha'(t)\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
k = \left\|\frac{\partial \dot{\alpha}}{\partial s}(s(t))\right\| &= \left\|\frac{1}{\|\alpha'(t))\|^2}\left(\alpha''(t) - \frac{\langle \alpha'(t), \alpha''(t)\rangle}{\|\alpha'(t)\|^2}.\alpha'(t)\right)\right\| \\
&= \frac{1}{\|\alpha'(t)\|^2}\left\|\alpha''(t) - \frac{\langle \alpha'(t), \alpha''(t)\rangle}{\|\alpha'(t)\|^2}.\alpha'(t)\right\|.
\end{aligned}$$

But we have,

$$\begin{aligned}
&\left\|\alpha''(t) - \frac{\langle \alpha'(t), \alpha''(t)\rangle}{\|\alpha'(t)\|^2}.\alpha'(t)\right\|^2 \\
= &\left\langle \alpha''(t) - \frac{\langle \alpha'(t), \alpha''(t)\rangle}{\|\alpha'(t)\|^2}.\alpha'(t), \alpha''(t) - \frac{\langle \alpha'(t), \alpha''(t)\rangle}{\|\alpha'(t)\|^2}.\alpha'(t)\right\rangle \\
= &\langle \alpha''(t), \alpha''(t)\rangle - \left\langle \alpha''(t), \frac{\langle \alpha'(t), \alpha''(t)\rangle}{\|\alpha'(t)\|^2}.\alpha'(t)\right\rangle - \left\langle \alpha''(t), \frac{\langle \alpha'(t), \alpha''(t)\rangle}{\|\alpha'(t)\|^2}.\alpha'(t)\right\rangle \\
&+\left\langle \frac{\langle \alpha'(t), \alpha''(t)\rangle}{\|\alpha'(t)\|^2}.\alpha'(t), \frac{\langle \alpha'(t), \alpha''(t)\rangle}{\|\alpha'(t)\|^2}.\alpha'(t)\right\rangle \\
= &\|\alpha''(t)\|^2 - \frac{\langle \alpha'(t), \alpha''(t)\rangle}{\|\alpha'(t)\|^2}\langle \alpha''(t), \alpha'(t)\rangle - \frac{\langle \alpha'(t), \alpha''(t)\rangle}{\|\alpha'(t)\|^2}\langle \alpha''(t), \alpha'(t)\rangle \\
&+\frac{|\langle \alpha'(t), \alpha''(t)\rangle|^2}{\|\alpha'(t)\|^4}\langle \alpha'(t), \alpha'(t)\rangle \\
= &\|\alpha''(t)\|^2 - 2\frac{|\langle \alpha'(t), \alpha''(t)\rangle|^2}{\|\alpha'(t)\|^2} + \frac{|\langle \alpha'(t), \alpha''(t)\rangle|^2}{\|\alpha'(t)\|^2}.\frac{|\langle \alpha'(t), \alpha''(t)\rangle|^2}{\|\alpha'(t)\|^2} \\
= &\|\alpha''(t)\|^2 - 2\frac{|\langle \alpha'(t), \alpha''(t)\rangle|^2}{\|\alpha'(t)\|^2} + \frac{|\langle \alpha'(t), \alpha''(t)\rangle|^2}{\|\alpha'(t)\|^2}.\frac{\|\alpha'(t)\|^2}{\|\alpha'(t)\|^2} \\
= &\|\alpha''(t)\|^2 - \frac{|\langle \alpha'(t), \alpha''(t)\rangle|^2}{\|\alpha'(t)\|^2}.
\end{aligned}$$

Finally, we can calculate $k$ and get

$$\begin{aligned} k &= \frac{1}{\|\alpha'(t)\|^2}\sqrt{\|\alpha''(t)\|^2 - \frac{|\langle\alpha'(t),\alpha''(t)\rangle|^2}{\|\alpha'(t)\|^2}} \\ &= \frac{1}{\|\alpha'(t)\|^2}\sqrt{\frac{\|\alpha''(t)\|^2.\|\alpha'(t)\|^2 - |\langle\alpha'(t),\alpha''(t)\rangle|^2}{\|\alpha'(t)\|^2}} \\ &= \frac{1}{\|\alpha'(t)\|^3}\sqrt{\|\alpha''(t)\|^2.\|\alpha'(t)\|^2 - |\langle\alpha'(t),\alpha''(t)\rangle|^2} \\ &= \frac{\sqrt{\|\alpha''(t)\|^2.\|\alpha'(t)\|^2 - |\langle\alpha'(t),\alpha''(t)\rangle|^2}}{\|\alpha'(t)\|^3}. \end{aligned}$$

□

**Example 1.3.8. (Helix)** *Its curvature is equal to* $\frac{|r|}{r^2+a^2}$. *Let's apply Theorem 1.3.7 to find the curvature again. We use the parametrization* $\alpha$ *given in Example 1.1.4. We have*

$$\alpha'(t) = (-r\sin t, r\cos t, a),$$

$$\alpha''(t) = (-r\cos t, -r\sin t, 0).$$

*Thus* $\|\alpha'(t)\|^2 = r^2 + a^2$ *and* $\|\alpha''(t)\|^2 = r^2$. *Moreover, one can calculate*

$$\langle\alpha'(t),\alpha''(t)\rangle = r^2\cos t\sin t - r^2\sin t\cos t = 0,$$

*so* $|\langle\alpha'(t),\alpha''(t)\rangle|^2 = 0$. *Since,* $\|\alpha'(t)\|^3 = (r^2+a^2)^{\frac{3}{2}}$, *we get:*

$$k = \frac{\sqrt{r^2(r^2+a^2) - 0}}{(r^2+a^2)^{\frac{3}{2}}} = \frac{|r|.(r^2+a^2)^{\frac{1}{2}}}{(r^2+a^2)^{\frac{3}{2}}} = \frac{|r|}{r^2+a^2},$$

*which is the desired result.*

**Definition 1.3.9.** *Let $\alpha(s)$ be any regular curve parametrized by arclength with $\|\alpha''(s)\| \neq 0$ for all $s \in (a, b)$. The unit vector $N(s) = k(s)^{-1}T'(s)$ is called the normal vector along the curve.*

**Remark** Note that: $N(s)$ and $T(s)$ are orthogonal for all $s \in (a, b)$, and $N(s)$ and $T'(s)$ are collinear or all $s \in (a, b)$. In fact, since $\alpha(s)$ is parametrized by arclength, then $\|\alpha'(s)\| = 1$ and so $\langle \alpha', \alpha' \rangle = 1$. Deriving both sides with respect to $s$, we get

$$\langle \alpha', \alpha'' \rangle = 0.$$

Since $T = \alpha'$ and $T' = \alpha''$, hence $\langle T, T' \rangle = 0$. As a result, we have that $N(s)$ and $T(s)$ are orthogonal.

**Definition 1.3.10.** *Let $\alpha(s)$ be any regular curve parametrized by arclength. Assume that $k(s) \neq 0$ for all $s$. The unit vector $B(s) = T(s) \times N(s)$ is called the binomial along the curve, where $\times$ denotes the cross product of vectors. The orthonormal system $\Big(T(s), N(s), B(s)\Big)$ is called the Frenet-Serret Frame.*

**Definition 1.3.11.** *Let $\alpha(s)$ be any regular curve parametrized by arclength. Assume that $k(s) \neq 0$ for all $s$. The scalar function $\tau(s) = -\langle B'(s), N(s) \rangle$ is called the torsion of the curve $\alpha$.*

**Theorem 1.3.12. (The Frenet Serret-Equations)** *Let $\alpha(s)$ be a regular curve parametrized by arclength with $k(s) \neq 0$. The following equations are satisfied:*

$$T'(s) = k(s)N(s) \quad \textit{and} \quad k(s) = \langle T'(s), N(s) \rangle, \tag{1.3.5}$$

$$N'(s) = -k(s)T(s) + \tau(s)B(s), \tag{1.3.6}$$

$$B'(s) = -\tau(s)N(s). \tag{1.3.7}$$

*Proof.* We know from Definition 1.3.9 that $N(s) = k^{-1}(s)T'(s)$, so we have $T'(s) = k(s)N(s)$. Moreover,

$$\langle T'(s), N(s)\rangle = \langle k(s)N(s), N(s)\rangle = k(s)\langle N(s), N(s)\rangle = k(s), \tag{1.3.8}$$

which proves (1.3.5). Since $T(s)$ and $N(s)$ are orthogonal, we have that $\langle T(s), N(s)\rangle = 0$. Deriving both sides with respect to $s$, we get

$$\begin{aligned} & \langle T'(s), N(s)\rangle + \langle T(s), N'(s)\rangle = 0 \\ \overset{(1.3.8)}{\Longrightarrow}\quad & k(s) + \langle T(s), N'(s)\rangle = 0 \\ \Longrightarrow\quad & -k(s) = \langle T(s), N'(s)\rangle. \end{aligned}$$

Since $B(s)$ and $N(s)$ are orthogonal, we also have $\langle B(s), N(s)\rangle = 0$. Deriving both sides with respect to $s$, we get

$$\begin{aligned} & \langle B'(s), N(s)\rangle + \langle B(s), N'(s)\rangle = 0 \\ \Longrightarrow\quad & -\tau(s) + \langle B(s), N'(s)\rangle = 0 \\ \Longrightarrow\quad & \tau(s) = \langle B(s), N'(s)\rangle. \end{aligned}$$

In the Frenet- Serret frame, each vector $v$ can be written as:

$$v = \langle v, B(s)\rangle B(s) + \langle v, N(s)\rangle N(s) + \langle v, T(s)\rangle T(s).$$

In particular and for $v = N'(s)$, we get

$$N'(s) = \langle N'(s), B(s)\rangle B(s) + \langle N'(s), N(s)\rangle N(s) + \langle N'(s), T(s)\rangle T(s).$$

Since $N(s)$ and $N'(s)$ are orthogonal, we obtain $N'(s) = \tau(s)B(s) - k(s)T(s)$, which proves (1.3.6). Now, deriving both sides with respect to $s$ of $\langle T(s), B(s)\rangle = 0$, we get

$$\langle T'(s), B(s)\rangle + \langle T(s), B'(s)\rangle = 0.$$

We have $\langle T'(s), B(s)\rangle = k(s)\langle N(s), B(s)\rangle = k(s).0 = 0$, hence $\langle T(s), B'(s)\rangle = 0$. The vector $B'(s)$ can be written as:

$$\begin{aligned} B'(s) &= \langle B'(s), B(s)\rangle B(s) + \langle B'(s), N(s)\rangle N(s) + \langle B'(s), T(s)\rangle T(s). \\ &= \langle B'(s), N(s)\rangle N(s) \\ &= -\tau(s)N(s), \end{aligned}$$

which proves (1.3.7). □

**Remark** The geometric meaning of curvature is the amount by which a curve deviates from being a straight line (curvature of a straight line is 0). The torsion measures the turnaround of the binormal vector. The larger the torsion is, the faster the binormal vector rotates around the axis given by the tangent vector.

**Example 1.3.13. (Helix)** *We recall the arclength parametrization of the helix:*

$$\beta(s) = \left(r\cos\left(\frac{s}{\sqrt{r^2+a^2}}\right), r\sin\left(\frac{s}{\sqrt{r^2+a^2}}\right), \frac{as}{\sqrt{r^2+a^2}}\right).$$

*One can easily calculate $T(s), T'(s), N(s), B(s)$ and $B'(s)$*

$$\begin{aligned}
T(s) &= \beta'(s) = \left(\frac{-r}{\sqrt{r^2+a^2}}\sin\left(\frac{s}{\sqrt{r^2+a^2}}\right), \frac{r}{\sqrt{r^2+a^2}}\cos\left(\frac{s}{\sqrt{r^2+a^2}}\right), \frac{a}{\sqrt{r^2+a^2}}\right),\\
T'(s) &= \beta''(s) = \left(\frac{-r}{r^2+a^2}\cos\left(\frac{s}{\sqrt{r^2+a^2}}\right), \frac{-r}{r^2+a^2}\sin\left(\frac{s}{\sqrt{r^2+a^2}}\right), 0\right),\\
N(s) &= \frac{1}{k(s)}T'(s) = \frac{r^2+a^2}{r}\left(\frac{-r}{r^2+a^2}\cos\left(\frac{s}{\sqrt{r^2+a^2}}\right), \frac{-r}{r^2+a^2}\sin\left(\frac{s}{\sqrt{r^2+a^2}}\right), 0\right)\\
&= \left(-\cos\left(\frac{s}{\sqrt{r^2+a^2}}\right), -\sin\left(\frac{s}{\sqrt{r^2+a^2}}\right), 0\right),\\
B(s) &= T(s)\times N(s) = \begin{vmatrix} i & j & k \\ \frac{-r}{\sqrt{r^+a^2}}\sin\left(\frac{s}{\sqrt{r^2+a^2}}\right) & \frac{r}{\sqrt{r^+a^2}}\cos\left(\frac{s}{\sqrt{r^2+a^2}}\right) & \frac{a}{\sqrt{r^+a^2}} \\ -\cos\left(\frac{s}{\sqrt{r^2+a^2}}\right) & -\sin\left(\frac{s}{\sqrt{r^2+a^2}}\right) & 0 \end{vmatrix}\\
&= \left(\frac{a}{\sqrt{r^+a^2}}\sin\left(\frac{s}{\sqrt{r^2+a^2}}\right), \frac{-a}{\sqrt{r^+a^2}}\cos\left(\frac{s}{\sqrt{r^2+a^2}}\right), \frac{r}{\sqrt{r^+a^2}}\right),\\
B'(s) &= \left(\frac{a}{r^+a^2}\cos\left(\frac{s}{\sqrt{r^2+a^2}}\right), \frac{a}{r^+a^2}\sin\left(\frac{s}{\sqrt{r^2+a^2}}\right), 0\right).
\end{aligned}$$

*We conclude that*

$$\tau(s) = \langle -B'(s), N(s)\rangle = \frac{-a}{r^+a^2}\cos^2\left(\frac{s}{\sqrt{r^2+a^2}}\right) - \frac{a}{r^+a^2}\sin^2\left(\frac{s}{\sqrt{r^2+a^2}}\right) = \frac{-a}{r^2+a^2}$$

**Example 1.3.14. (Circle)** *Let $(C)$ be a circle of center $(x_0, y_0)$ and radius $r$*

*whose arclength parametrization is given by:*

$$\alpha(s) = \left(x_0 + r\cos\left(\frac{s}{r}\right), y_0 + r\sin\left(\frac{s}{r}\right), 0\right).$$

*Then,*

$$\begin{aligned}
T(s) &= \alpha'(s) = \left(-\sin\left(\frac{s}{r}\right), \cos\left(\frac{s}{r}\right), 0\right),\\
T'(s) &= \alpha''(s) = \left(\frac{-1}{r}\cos\left(\frac{s}{r}\right), \frac{-1}{r}\sin\left(\frac{s}{r}\right), 0\right),\\
N(s) &= \frac{1}{k(s)}T'(s) = r\left(\frac{-1}{r}\cos\left(\frac{s}{r}\right), \frac{-1}{r}\sin\left(\frac{s}{r}\right), 0\right),\\
N(s) &= \left(-\cos\left(\frac{s}{r}\right), -\sin\left(\frac{s}{r}\right), 0\right),\\
B(s) &= T(s) \times N(s) = \begin{vmatrix} i & j & k \\ -sin(\frac{s}{r}) & cos(\frac{s}{r}) & 0 \\ -cos(\frac{s}{r}) & -sin(\frac{s}{r}) & 0 \end{vmatrix} = 0i + 0j + 1k,\\
B'(s) &= (0,0,0).
\end{aligned}$$

*Hence* $\tau(s) = \langle -B'(s), N(s)\rangle = 0.$

**Proposition 1.3.15.** *If the curvature of a curve $\alpha(t)$ not necessarily parametrized by arclength is non-zero, then the curvature and torsion are given by*

$$k(t) = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}, \tag{1.3.9}$$

$$\tau(t) = \frac{\langle \alpha' \times \alpha'', \alpha'''\rangle}{\|\alpha' \times \alpha''\|^2}. \tag{1.3.10}$$

*Proof.* In this proof, let $\alpha'$ denote $\dfrac{\partial \alpha}{\partial t}$. Then:

$$
\begin{aligned}
\alpha' &= \frac{\partial \alpha}{\partial s}.\frac{\partial s}{\partial t}, \\
\alpha'' &= \frac{\partial}{\partial t}\Big(\frac{\partial \alpha}{\partial s}\Big).\frac{\partial s}{\partial t} + \frac{\partial}{\partial t}\Big(\frac{\partial s}{\partial t}\Big).\frac{\partial \alpha}{\partial s} \\
&= \frac{\partial}{\partial s}.\frac{\partial s}{\partial t}\Big(\frac{\partial \alpha}{\partial s}\Big).\frac{\partial s}{\partial t} + \frac{\partial \alpha}{\partial s}.\frac{\partial^2 s}{\partial t^2} \\
&= \frac{\partial}{\partial s}\Big(\frac{\partial \alpha}{\partial s}\Big)(\frac{\partial s}{\partial t})^2 + \frac{\partial \alpha}{\partial s}.\frac{\partial^2 s}{\partial t^2} \\
&= \frac{\partial^2 \alpha}{\partial s^2}.\Big(\frac{\partial s}{\partial t}\Big)^2 + \frac{\partial \alpha}{\partial s}.\frac{\partial^2 s}{\partial t^2}.
\end{aligned}
$$

The unit tangent vector is $\dfrac{\alpha'}{\|\alpha'\|}$ or $\dfrac{\partial \alpha}{\partial s}$. So, using $\alpha' = \dfrac{\partial \alpha}{\partial s}.\dfrac{\partial s}{\partial t}$, we conclude that $\alpha' = \dfrac{\alpha'}{\|\alpha'\|}.\dfrac{\partial s}{\partial t}$ and so, $\|\alpha'\| = \dfrac{\partial s}{\partial t}$. Hence,

$$\alpha' = \frac{\partial \alpha}{\partial s}\|\alpha'\| \text{ and } \alpha'' = \frac{\partial^2 \alpha}{\partial s^2}\|\alpha'\|^2 + \frac{\partial \alpha}{\partial s}.\frac{\partial^2 s}{\partial t^2}.$$

Now, we calculate $\alpha' \times \alpha''$ and get

$$
\begin{aligned}
\alpha' \times \alpha'' &= \Big(\frac{\partial \alpha}{\partial s}\|\alpha'\|\Big) \times \Big(\frac{\partial^2 \alpha}{\partial s^2}\|\alpha'\|^2 + \frac{\partial \alpha}{\partial s}\frac{\partial^2 s}{\partial t^2}\Big) \\
&= \Big(\frac{\partial \alpha}{\partial s}\|\alpha'\| \times \frac{\partial^2 \alpha}{\partial s^2}\|\alpha'\|^2\Big) + \Big(\frac{\partial \alpha}{\partial s}\|\alpha'\| \times \frac{\partial \alpha}{\partial s}.\frac{\partial^2 s}{\partial t^2}\Big) \\
&= \|\alpha'\|^3\Big(\frac{\partial \alpha}{\partial s} \times \frac{\partial^2 \alpha}{\partial s^2}\Big).
\end{aligned}
$$

Using the definition of $k(s), \tau(s)$, and $N(s)$, we have:

$$\tau(s) = \frac{\partial \alpha}{\partial s} \text{ and } \tau'(s) = \frac{\partial^2 \alpha}{\partial s^2} = k(s)N(s).$$

Hence,

$$\begin{aligned}\alpha' \times \alpha'' &= \|\alpha'\|^3(T(s) \times k(s)N(s)) \\ &= \|\alpha'\|^3 k(s)(T(s) \times N(s)) = \|\alpha'\|^3 k(s)B(s).\end{aligned}$$

Thus, $\|\alpha' \times \alpha''\| = \|\alpha'\|^3 k(s)\|B(s)\| = \|\alpha'\|^3 k(s)$ and we conclude that

$$k(s) = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}.$$

As for the torsion, we recall that

$$\alpha' = \frac{\partial \alpha}{\partial t} = \frac{\partial \alpha}{\partial s}.\frac{\partial s}{\partial t},$$

where $s(t)$ is the arclength parameter. Hence,

$$\begin{aligned}\alpha'' &= \Big[\frac{\partial}{\partial t}\Big(\frac{\partial \alpha}{\partial s}\Big)\Big].\frac{\partial s}{\partial t} + \Big[\frac{\partial}{\partial t}\Big(\frac{\partial s}{\partial t}\Big)\Big]\Big(\frac{\partial \alpha}{\partial s}\Big) = \frac{\partial}{\partial s}.\frac{\partial s}{\partial t}\Big(\frac{\partial \alpha}{\partial s}\Big).\frac{\partial s}{\partial t} + \frac{\partial^2 s}{\partial t^2}.\frac{\partial \alpha}{\partial s} \\ &= \frac{\partial^2 \alpha}{\partial s^2}.\frac{\partial s}{\partial t}.\frac{\partial s}{\partial t} + \frac{\partial^2 s}{\partial t^2}.\frac{\partial \alpha}{\partial s} = \frac{\partial^2 \alpha}{\partial s^2}.\Big(\frac{\partial s}{\partial t}\Big)^2 + \frac{\partial^2 s}{\partial t^2}.\frac{\partial \alpha}{\partial s}\end{aligned}$$

The unit tangent vector is $\dfrac{\alpha'(t)}{\|\alpha'(t)\|}$ or $\dfrac{\partial \alpha}{\partial s} = \alpha'(s)$, so we get: $\alpha' = \dfrac{\alpha'}{\|\alpha'\|}.\dfrac{\partial s}{\partial t}$.

Hence, $\|\alpha'\| = \frac{\partial s}{\partial t}$ and so $\alpha' = \frac{\partial \alpha}{\partial s}\|\alpha'\|$. We now calculate

$$\begin{aligned}\alpha'' &= \frac{\partial^2 \alpha}{\partial s^2}\|\alpha'\|^2 + \frac{\partial^2 s}{\partial t^2}.\frac{\partial \alpha}{\partial s}, \\ \alpha' \times \alpha'' &= \Big(\frac{\partial \alpha}{\partial s}\|\alpha'\|\Big) \times \Big(\frac{\partial^2 \alpha}{\partial s^2}\|\alpha'\|^2 + \frac{\partial \alpha}{\partial s}.\frac{\partial^2 s}{\partial t^2}\Big), \\ \alpha \times \alpha' &= \|\alpha'\|^3\Big(\frac{\partial \alpha}{\partial s} \times \frac{\partial^2 \alpha}{\partial s^2}\Big) + \|\alpha'\|\Big(\frac{\partial \alpha}{\partial s} \times \frac{\partial \alpha}{\partial s}.\frac{\partial^2 s}{\partial t^2}\Big) = \|\alpha'\|^3\Big(\frac{\partial \alpha}{\partial s} \times \frac{\partial^2 \alpha}{\partial s^2}\Big),\end{aligned}$$

$$
\begin{aligned}
\alpha''' &= \frac{\partial}{\partial t}\Big[\frac{\partial^2\alpha}{\partial s^2}\Big]\Big(\frac{\partial s}{\partial t}\Big)^2 + \frac{\partial}{\partial t}\Big(\frac{\partial s}{\partial t}\Big)^2\frac{\partial^2\alpha}{\partial s^2} + \frac{\partial}{\partial t}\Big(\frac{\partial \alpha}{\partial s}\Big).\frac{\partial^2 s}{\partial t^2} + \frac{\partial}{\partial t}\Big(\frac{\partial^2 s}{\partial t^2}\Big)\Big(\frac{\partial \alpha}{\partial s}\Big)\\
&= \frac{\partial}{\partial s}.\Big(\frac{\partial s}{\partial t}.\frac{\partial^2\alpha}{\partial s^2}\Big)\Big(\frac{\partial s}{\partial t}\Big)^2 + \frac{\partial}{\partial s}\Big(\frac{\partial s}{\partial t}.\frac{\partial \alpha}{\partial s}\Big).\frac{\partial^2 s}{\partial t^2} + \frac{\partial^3 s}{\partial t^3}.\frac{\partial \alpha}{\partial s} + 2.\frac{\partial s}{\partial t}.\frac{\partial}{\partial t}\Big(\frac{\partial s}{\partial t}\Big).\frac{\partial^2\alpha}{\partial s^2}\\
&= \frac{\partial^3\alpha}{\partial s^3}\Big(\frac{\partial s}{\partial t}\Big)^3 + \frac{\partial^2\alpha}{\partial s^2}\Big(\frac{\partial s}{\partial t}\Big)\frac{\partial^2 s}{\partial t^2} + \frac{\partial^3 s}{\partial t^3}.\frac{\partial \alpha}{\partial s} + 2.\frac{\partial s}{\partial t}.\frac{\partial^2 s}{\partial t^2}.\frac{\partial^2\alpha}{\partial s^2}\\
&= \frac{\partial^3\alpha}{\partial s^3}\Big(\frac{\partial s}{\partial t}\Big)^3 + 3\Big(\frac{\partial^2\alpha}{\partial s^2}\Big)\Big(\frac{\partial s}{\partial t}\Big)\Big(\frac{\partial^2 s}{\partial t^2}\Big) + \frac{\partial^3 s}{\partial t^3}.\frac{\partial \alpha}{\partial s} = \frac{\partial^3\alpha}{\partial s^3}\|\alpha'\|^3\\
&\quad +3.\frac{\partial^2\alpha}{\partial s^2}.\frac{\partial^2 s}{\partial t^2}\|\alpha'\| + \frac{\partial \alpha}{\partial s}.\frac{\partial^3 s}{\partial t^3}.
\end{aligned}
$$

The scalar product $\langle \alpha' \times \alpha'', \alpha''' \rangle$ is then given by

$$
\begin{aligned}
&\langle \alpha' \times \alpha'', \alpha''' \rangle\\
&= \Big\langle \|\alpha'\|^3\Big(\frac{\partial \alpha}{\partial s} \times \frac{\partial^2\alpha}{\partial s^2}\Big), \frac{\partial^3\alpha}{\partial s^3}\|\alpha'\|^3 + 3.\frac{\partial^2\alpha}{\partial s^2}.\frac{\partial^2 s}{\partial t^2}\|\alpha'\| + \frac{\partial \alpha}{\partial s}.\frac{\partial^3 s}{\partial t^3}\Big\rangle\\
&= \|\alpha'\|^6\Big\langle \frac{\partial \alpha}{\partial s} \times \frac{\partial^2\alpha}{\partial s^2}, \frac{\partial^3\alpha}{\partial s^3}\Big\rangle + 3\|\alpha'\|^4\Big\langle \frac{\partial \alpha}{\partial s} \times \frac{\partial^2\alpha}{\partial s^2}\frac{\partial^2 s}{\partial t^2}\Big\rangle + \|\alpha'\|^3\Big\langle \frac{\partial \alpha}{\partial s} \times \frac{\partial^2\alpha}{\partial s^2}, \frac{\partial \alpha}{\partial s}.\frac{\partial^3 s}{\partial t^3}\Big\rangle\\
&= \|\alpha'\|^6\Big\langle \frac{\partial \alpha}{\partial s} \times \frac{\partial^2\alpha}{\partial s^2}, \frac{\partial^3\alpha}{\partial s^3}\Big\rangle
\end{aligned}
$$

Now, by definition, we have $\dfrac{d\alpha}{ds} = T(s)$ and $\dfrac{d^2\alpha}{ds^2} = T'(s) = k(s)N(s)$. Hence,

$$
\begin{aligned}
\Longrightarrow \langle \alpha' \times \alpha'', \alpha''' \rangle &= |\alpha'|^6\Big\langle T \times kN, \frac{\partial}{\partial s}(kN)\Big\rangle = k|\alpha'|^6\Big\langle T \times N, \frac{\partial k}{\partial s}N + \frac{\partial N}{\partial s}k\Big\rangle\\
&= k|\alpha'|^6\Big\langle B, \frac{\partial k}{\partial s}N\Big\rangle + k|\alpha'|^6\Big\langle B, \frac{\partial N}{\partial s}k\Big\rangle = k|\alpha'|^6\Big\langle B, kN'(s)\Big\rangle\\
&= k^2|\alpha'|^6\Big\langle B, -kT + \tau B\Big\rangle = k^2|\alpha'|^6\Big\langle B, \tau B\Big\rangle = k^2\tau|\alpha'|^6\\
&= \left(\frac{\|\alpha' \times \alpha''\|}{|\alpha'|^3}\right)^2 \tau|\alpha'|^6,
\end{aligned}
$$

which gives the desired result. □

**Definition 1.3.16.** *A map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is called a rigid motion if there ex-*

*ists an orthogonal matrix $A$ (a matrix with determinant $\pm 1$ and whose columns are orthonormal vectors) with $\det(A) = 1$ and a constant vector $b \in \mathbb{R}^n$ such that $f(X) = AX + b$ for all $X \in \mathbb{R}^n$.*

**Remark** The reason we require $det(A) = 1$ in Definition 1.3.16 is to preserve the orientation of $\mathbb{R}^n$ in rigid motions. Given a constant angle $\rho$, and $b \in \mathbb{R}^2$, $f(X) = R_\rho X + b$ is a rigid motion of $\mathbb{R}^2$ where $R_\rho$ is the rotation by an angle $\rho$ and $b$ is a translation. So, a rigid motion is a combination of a rotation and a translation.

**Proposition 1.3.17.** *Suppose $\alpha$ is a curve in $\mathbb{R}^3$ parametrized by arclength such that $\alpha'' \neq 0$ and $f(X) = AX + b$ is a rigid motion in $\mathbb{R}^3$. Then, the curve $\beta(s) = f(\alpha(s))$ has the same curvature and torsion as $\alpha$. Moreover, if $(T(s), N(s), B(s))$ is the Frenet-Serret frame along $\alpha$, then $(AT(s), AN(s), AB(s))$ is the Frenet-Serret frame along $\beta$.*

*Proof.* The curve $\beta$ is given by

$$\beta(s) = f(\alpha(s)) = A(\alpha(s)) + b,$$

and hence $\beta'(s) = A\alpha'(s)$. So, $\|\beta'(s)\| = \|A\alpha'(s)\| = \|\alpha'(s)\| = 1$. Therefore, $\beta$ is paramterized by arclength. We denote the curvature and the torsion of $\beta$ by $\overline{k}(s)$ and $\overline{\tau}(s)$. We have $\beta''(s) = A\alpha''(s)$ and so

$$\overline{k}(s) = \|\beta''(s)\| = \|A\alpha''(s)\| = \|\alpha''(s)\| = k(s).$$

Thus, $\beta(s)$ and $\alpha(s)$ have the same curvature. Now, let $(\overline{T}(s), \overline{N}(s), \overline{B}(s))$ be the Frenet-Serret frame of $\beta$. since $T(s) = \alpha'(s)$ and $\overline{T}(s) = \beta'(s)$, we obviously have $\overline{T}(s) = AT(s)$. Now,

$$\overline{N}(s) = \frac{1}{k(s)}\overline{T}'(s) = \frac{1}{k(s)}.AT'(s) = A.\frac{1}{k(s)}.T'(s) = A.N(s).$$

Also,

$$\overline{B}(s) = \overline{T}(s) \times \overline{N}(s) = AT(s) \times AN(s) = A(T(s) \times N(s)) = AB(s).$$

Finally, the torsion $\overline{\tau}(s)$ of $\beta(s)$ is given by

$$\begin{aligned}\overline{\tau}(s) &= -\langle \overline{B}'(s), \overline{N}(s)\rangle = -\langle AB'(s), AN(s)\rangle \\ &= -\langle B'(s), N(s)\rangle = \tau(s).\end{aligned}$$

Therefore, $\alpha(s)$ and $\beta(s)$ have the same torsion $\tau(s)$ and

$$\left(\overline{T}(s) = AT(s), \overline{N}(s) = AN(s), \overline{B}(s) = AB(s)\right)$$

is the Frenet-Serret frame for $\beta(s)$. □

## 1.4 The Fundamental Theorem of Curves in $\mathbb{R}^3$

In this section, we recall the Fundamental Theorem of Curves in $\mathbb{R}^3$ and give its traditional proof written in almost all Differential Geometry books and references [1, 2, 4, 5]. We start by stating the local existence and uniqueness Theorem of Ordinary Differential Equations, needed to prove the Fundamental

Theorem of Curves in $\mathbb{R}^3$.

**Theorem 1.4.1.** *[6]* **(The Local Existence and Uniqueness Theorem of Ordinary Differential Equations).** *Let $U \subset \mathbb{R}^n$ be any open set of $\mathbb{R}^n$ and $F : (a,b) \times U \longrightarrow \mathbb{R}^n$, a $C^1$ map, where $C^1$ is the class of all continuously differentiable functions. Fix $c_0 \in (a,b)$. Then, given any point $p_0 \in U$, there exists $\varepsilon > 0$ and a unique solution $\alpha : (c_0 - \varepsilon, c_0 + \varepsilon) \longrightarrow U$ of the following initial value problem:*

$$\begin{cases} \dfrac{\partial \alpha}{\partial t} = F(t, \alpha(t)), \\ \alpha(c_0) = p_0. \end{cases}$$

**Proposition 1.4.2.** *Let $M_{3\times 3}$ be the space of all square matrices of size 3. Suppose that the map $A : [a,b] \longrightarrow M_{3\times 3}$ is smooth, $A(t)$ is skew-symmetric for all $t \in [a,b]$ (i.e. $A(t)^T = -A(t)$), $c_0 \in (a,b)$, and $C$ is a 3×3 orthogonal matrix. If $g : [a,b] \longrightarrow M_{3\times 3}$ is a smooth solution to the following initial value problem*

$$\begin{cases} \dfrac{dg}{dt} = g(t).A(t), \\ g(c_0) = c, \end{cases}$$

*then, $g(t)$ is orthogonal for all $t \in [a,b]$.*

*Proof.* Let $y(t) = g(t)^T g(t)$. By the product chain rule, we have

$$\begin{aligned} y'(t) &= g'(t)^T g(t) + g(t)^T g'(t) = (g(t)A)^T.g(t) + g(t)^T (g(t)A) \\ &= A^T g(t)^T g(t) + g(t)^T g(t) A = A^T y(t) + y(t) A. \end{aligned}$$

Also, by the product rule and since $C$ is orthogonal, we have

$$y(c_0) = g(c_0)^T g(c_0) = C^T.C = \mathrm{I}_3,$$

where $\mathrm{I}_3$ denotes the identity matrix of size 3. So, $y(t)$ satisfies the following initial value problem:

$$\begin{cases} \dfrac{\partial \alpha}{\partial t} = A^T\alpha + \alpha A = F(t, \alpha(t)), \\ \alpha(c_0) = \mathrm{I}_3, \end{cases} \tag{1.4.1}$$

However, the function $z(t) = \mathrm{I}_3$ also satisfies (1.4.1) because $z(c_0) = \mathrm{I}_3$, $z'(t) = 0$ and

$$A^T z + zA = A^T.\mathrm{I}_3 + \mathrm{I}_3.A = A^T + A = 0,$$

since $A$ is skew-symmetric. Hence, by the existence and uniqueness theorem of ODE, Theorem 1.4.1, we have $y(t) = z(t)$. This implies that $g(t)^T g(t) = \mathrm{I}_3$. Therefore, $g(t)$ is orthogonal. □

**Proposition 1.4.3.** *Let* $p_0, q_0 \in \mathbb{R}^3$ *and* $\{u_1, u_2, u_3\}$ *and* $\{v_1, v_2, v_3\}$ *be orthonormal bases of* $\mathbb{R}^3$ *such that* $\det(u_1, u_1, u_3) = \det(v_1, v_2, v_3) = 1$. *Then, there exists a unique rigid motion* $f(x) = Ax + b$ *such that* $f(p_0) = q_0$ *and* $Au_i = v_i$ *for* $1 \leq i \leq 3$, *where* $A$ *is an orthogonal* $3 \times 3$ *matrix with* $\det(A) = 1$ *and* $b \in \mathbb{R}^3$.

*Proof.* Let $U = (u_1, u_2, u_3)$ and $V = (v_1, v_2, v_3)$ be the $3 \times 3$ matrices with $u_i$ and $v_i$ as their $i^{th}$ columns respectively ($1 \leq i \leq 3$). Since $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ are orthonormal, we have that $U$ and $V$ are orthogonal matrices. Consider the rigid motion $f(x) = Ax + b$ where $A$ is the matrix given by

$A = VU^{-1} = VU^T$ and $b$ the vector given by $b = q_0 - Ap_0$. Of course we have $Au_i = v_i$ for $1 \leq i \leq 3$, and $Ap_0 + b = q_0$. We still need to prove the uniqueness of the rigid motion. Let $g(x) = Bx + c$ be another rigid motion such that $Bu_i = v_i$ and $g(p_0) = q_0$. Then $BU = V$ and hence

$$B = VU^{-1} = VU^T = A,$$

and since $q_0 = Bp_0 + c$, we get $c = q_0 - Bp_0 = q_0 - Ap_0 = b$. Finally, we get that $Ax + b = Bx + c$, which means that $f(x) = g(x)$ and the rigid motion is unique. $\square$

Now, we are ready to state and prove the Fundamental Theorem of Curves in $\mathbb{R}^3$.

**Theorem 1.4.4. (Fundamental Theorem of Curves in $\mathbb{R}^3$)**
*Given two smooth functions $k, \tau : (a, b) \longrightarrow \mathbb{R}$ such that $k(t) > 0$.*

1. *Let $t_0 \in (a, b)$, $p_0 \in \mathbb{R}^3$, and $(u_1, u_2, u_3)$ be a fixed orthonormal basis of $\mathbb{R}^3$, then there exists $\delta > 0$ and a unique curve $\alpha : (t_0 - \delta, t_0 + \delta) \longrightarrow \mathbb{R}^3$ parametrized by arclength whose curvature and torsion are given respectively by $k$ and $\tau$, such that $\alpha(0) = p_0$ and $(u_1, u_2, u_3)$ is the Frenet-Serret frame of $\alpha$ at $t = t_0$.*

2. *Suppose $\alpha, \widetilde{\alpha} : [a, b] \longrightarrow \mathbb{R}^3$ are curves parametrized by arclength and $\alpha, \tilde{\alpha}$ have the same curvature function $k$ and torsion function $\tau$. Then, there exists a rigid motion $f$ so that $\tilde{\alpha} = f(\alpha)$.*

*Proof.* 1. Consider the function $F$ given by

$$\begin{aligned} F:(a,b)\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3 &\longrightarrow \mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3 \\ (t,X,Y,Z) &\longrightarrow F(t,X,Y,Z)=(k(t)Y,-k(t)X+\tau(t)Z,-\tau(t)Y). \end{aligned}$$

$F$ is clearly $C^1$. Thus, by the existence and uniqueness Theorem of ODE, Theorem 1.4.1, there exists $\delta>0$ and a unique solution say $g:(t_0-\delta,t_0+\delta)\longrightarrow\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3$ of the initial value problem

$$\begin{cases} g'(t)=F(t,g(t))=g(t)A(t), \\ g(t_0)=(u_1,u_2,u_3), \end{cases}$$

where

$$A(t)=\begin{pmatrix} 0 & -k(t) & 0 \\ k(t) & 0 & -\tau(t) \\ 0 & \tau(t) & 0 \end{pmatrix}.$$

Now, $\{u_1,u_2,u_3\}$ is an orthonormal basis of $\mathbb{R}^3$, so $g(t_0)=(u_1,u_2,u_3)$ is an orthogonal $3\times3$ matrix. Also, $A$ is smooth and since $A^T=-A$, we get that $A$ is skew-symmetric and $g$ is smooth. By Proposition 1.4.2, we have that $g(t)$ is orthogonal for all $t\in(t_0-\delta,t_0+\delta)$ and $e_1(t),e_2(t)$, and $e_3(t)$ are the columns of $g(t)$. Hence, $\{e_1(t),e_2(t),e_3(t)\}$ is an orthonormal basis of $\mathbb{R}^3$ for all $t\in(t_0-\delta,t_0+\delta)$. Define the curve $\alpha$ by

$$\alpha(t)=p_0+\int_{t_0}^{t}e_1(s)ds.$$

We have $\alpha'(t) = e_1(t)$ and $\|\alpha'(t)\| = \|e_1(t)\| = 1$. It means that $\alpha(t)$ is parametrized by arclength. Now, $g(t) = (e_1(t), e_2(t), e_3(t))$ and $g'(t) = g(t)A(t)$. Hence

$$\begin{aligned}(e_1'(t), e_2'(t), e_3'(t)) &= (e_1(t), e_2(t), e_3(t)) \begin{pmatrix} 0 & -k(t) & 0 \\ k(t) & 0 & -\tau(t) \\ 0 & \tau(t) & 0 \end{pmatrix} \\ &= (k(t)e_2(t), -k(t)e_1(t) + \tau(t)e_3(t), -\tau(t)e_2(t)).\end{aligned}$$

This gives that

$$\begin{cases} e_1'(t) = k(t)e_2(t), \\ e_2'(t) = -k(t)e_1(t) + \tau(t)e_3(t), \\ e_3'(t) = -\tau(t)e_2(t), \\ (e_1(t_0), e_2(t_0), e_3(t_0)) = (u_1, u_2, u_3). \end{cases}$$

Thus, $k$ and $\tau$ are respectively the curvature and the torsion of the curve $\alpha(t)$ and $\{e_1, e_2, e_3\}$ is the Frenet-Serret frame of $\alpha$, which proves the first statement of the theorem.

2. Consider $t_0 \in (a, b)$ fixed and $\{e_1(t), e_2(t), e_3(t)\}$ the Frenet-Serret frame of $\alpha$. Let $\{\tilde{e_1}(t), \tilde{e_1}(t), \tilde{e_3}(t)\}$ be the Frenet-Serret frame of $\tilde{\alpha}$. We have $\alpha(t_0) \in \mathbb{R}^3$ and $\tilde{\alpha}(t_0) \in \mathbb{R}^3$. Also, $\{e_1(t), e_2(t), e_3(t)\}$ and $\{\tilde{e_1}(t), \tilde{e_2}(t), \tilde{e_3}(t)\}$ are orthonormal bases. By Proposition 1.4.3, there exists a unique rigid motion $f(x) = Ax + b$ such that $f(\alpha(t_0)) = \tilde{\alpha}(t_0)$ and $Ae_i(t_0) = \tilde{e_i}(t_0)$, for $1 \leq i \leq 3$. Consider the curve $\beta = f \circ \alpha$. By Proposition 1.3.17,

$\beta$ and $\alpha$ has the same curvature $k$ and same torsion $\tau$ and the Frenet frame of $\beta$ is $\{Ae_1(t), Ae_2(t), Ae_3(t)\}$. Now, it is easy to check that

$$(\tilde{\alpha}, \tilde{e_1}, \tilde{e_2}, \tilde{e_3})' = (\tilde{\alpha}', \tilde{e_1}', \tilde{e_2}', \tilde{e_3}') = (\tilde{e_1}, k\tilde{e_2}, -k\tilde{e_1} + \tau\tilde{e_3}, -\tau\tilde{e_2}),$$

and

$$\begin{aligned}(\beta, Ae_1, Ae_2, Ae_3)' &= (\beta', (Ae_1)', (Ae_2)', (Ae_3)') \\ &= (Ae_1, kAe_2, -kAe_1 + \tau Ae_3, -\tau Ae_2).\end{aligned}$$

Moreover, at $t_0$, we have

$$\begin{aligned}\beta(t_0) &= f(\alpha(t_0)) = \tilde{\alpha}(t_0), \\ Ae_i(t_0) &= \tilde{e_i}(t_0) \ \ for \ \ 1 \leq i \leq 3.\end{aligned}$$

Thus, both $\{\tilde{\alpha}, \tilde{e_1}, \tilde{e_2}, \tilde{e_3}\}$ and $\{\beta, Ae_1, Ae_2, Ae_3\}$ satisfy the same differential equation given by

$$(X, y_1, y_2, y_3)' = (y_1, ky_2, -ky_1 + \tau y_3, -\tau y_2).$$

Therefore, by the existence and uniqueness Theorem of ODE, Theroem 1.4.1, we get that $\beta(t) = \tilde{\alpha}(t)$ for all $t \in (a, b)$, and $\beta(t) = f(\alpha(t))$. So, $f(\alpha(t)) = \tilde{\alpha}(t)$ which proves the second statement of the theorem.

□

## 1.5 A New Proof of the Fundamental Theorem of Curves in $\mathbb{R}^3$

In this section, we will give another proof of the Fundamental Theorem of Curves in $\mathbb{R}^3$. This new proof, established by H. F. Guerrero in [3], is based on finding solutions of a non-linear differential equation of second order. First, we restate the Fundamental Theorem of Curves in $\mathbb{R}^3$.

**Theorem 1.5.1. (The Fundamental Theorem of Curves in $\mathbb{R}^3$).** *Given a differentiable function $k(s) > 0$ and a continuous function $\tau(s)$ where $s \in (a,b)$, there exists a regular curve $\alpha$ parametrized by arclength $\alpha : J \longrightarrow \mathbb{R}^3$ such that $s$ is the arclength, $k(s)$ is its curvature, and $\tau(s)$ its torsion. Moreover, any other curve $\widetilde{\alpha}$ satisfying the same conditions, differs from $\alpha$ by a rigid motion. i.e, there exists an orthogonal matrix $A$ of size 3 with $\det(A) > 0$, and a vector $b$ such that $\tilde{\alpha} = A\alpha + b$.*

**Remark** We point out here that in Theorem 1.4.4, $\tau(s)$ was given differentiable. However, in Theorem 1.5.1, $\tau(s)$ is only continuous.

Before proving Theorem 1.5.1, we need to establish two lemmas.

**Lemma 1.5.2.** *Let $k : [c,d] \longrightarrow \mathbb{R}$ be a positive function of class $C^1$ and $\tau : [c,d] \longrightarrow \mathbb{R}$ a function of class $C^0$. The $2^{nd}$ order differential equation*

$$\frac{\partial}{\partial s}\Big(\frac{1}{k}\frac{\partial w}{\partial s}\Big) = -kw + \tau\sqrt{1 - w^2 - \Big(\frac{1}{k}\frac{\partial w}{\partial s}\Big)^2} \tag{1.5.1}$$

*with intial value*

$$\begin{cases} w(s_1) = w_1, \\ w'(s_1) = v_1, \end{cases}$$

*where*

$$\begin{cases} s_1 \in (c, d), \\ (w_1, v_1) \in \left\{(w, v) \in \mathbb{R}^2 / w^2 + \frac{v^2}{{k_0}^2} < 1\right\}, \\ k_0 = \min\{k(s), s \in [c, d]\}, \end{cases}$$

*has a unique solution* $w : J \subseteq (c, d) \to \mathbb{R}$ *on some open interval* $J \subseteq (c, d)$ *containing* $s_1$.

*Proof.* By the Chain rule, we get

$$\frac{\partial}{\partial s}\left(\frac{1}{k}\frac{\partial w}{\partial s}\right) = \frac{-k'}{k^2}w' + \frac{1}{k}w''.$$

Then, Equation 1.5.1 can be written as:

$$\frac{-k'}{k^2}w' + \frac{1}{k}w'' \quad = \quad -kw + \tau\sqrt{1 - w^2 - \frac{1}{k^2}w'^2}.$$

Thus, we have

$$w'' = \frac{k'}{k}w' - k^2 w + \tau\sqrt{k^2(1 - w^2) - w'^2}. \tag{1.5.2}$$

Letting $v = w'$, Equation (1.5.2) can be written as:

$$v' = \frac{k'}{k}v - k^2 w + \tau\sqrt{k^2(1-w^2) - v^2}.$$

Thus,

$$(w,v)' = (w',v') = \left(v, \frac{k'}{k}v - k^2 w + \tau\sqrt{k^2(1-w^2) - v^2}\right).$$

Consider the function $F$ defined by

$$\begin{aligned} F(s,w,v) &= (F_1(s,w,v), F_2(s,w,v)) \\ &= \left(v, \frac{k'(s)}{k(s)}v - k^2(s)w + \tau\sqrt{k^2(s)(1-w^2) - v^2}\right), \end{aligned}$$

for $(s,w,v) \in L = (c,d) \times \left\{(w,v) \in \mathbb{R}^2 / w^2 + \frac{v^2}{{k_0}^2} < 1\right\}$. First, $F$ is well defined since

$$\begin{aligned} w^2 + \frac{v^2}{{k_0}^2} < 1 &\implies 1 - w^2 - \frac{v^2}{{k_0}^2} > 0 \\ &\implies {k_0}^2(1-w^2) - v^2 > 0 \implies k^2(1-w^2) - v^2 > 0. \end{aligned}$$

Clearly, $F$ is continuous. Also, the partial derivatives of $F$ given by

$$\begin{aligned} \frac{\partial F_1}{\partial w} &= 0 \quad ; \quad \frac{\partial F_2}{\partial w} = -k^2 - \frac{\tau k^2 w}{\sqrt{k^2(1-w^2) - v^2}}, \\ \frac{\partial F_1}{\partial v} &= 0 \quad ; \quad \frac{\partial F_2}{\partial v} = \frac{k'}{k} - \frac{v\tau}{\sqrt{k^2(1-w^2) - v^2}}, \end{aligned}$$

are all continuous. Then, $F$ is continuously differentiable, so $F$ is $C^1$ with respect to $(w,v)$ in a neighborhood $D$ of $(s_1, w_1, v_1) \in L$. By the uniqueness and existence theorem of ODE, Theorem 1.4.1, the initial value problem

$$\begin{cases} (w,v)' = F(s,w,v) = \left(v, \frac{k'}{k}v - k^2 w + \tau\sqrt{k^2(1-w^2)-v^2}\right) \\ (w,v)(s) = (w(s_1), v(s_1)) = (w_1, w'(s_1)) = (w_1, v_1) \end{cases}$$

has a unique solution on some open interval $J \subseteq (c,d)$ such that $s_1 \in J$. □

**Lemma 1.5.3.** *For any curve $\alpha : [a,b] \longrightarrow \mathbb{R}^3$ parametrized by arclength and having curvature $k$ and torsion $\tau$, there exists an orthogonal linear function $\theta$ of $\mathbb{R}^3$, with $\det\theta > 0$ such that the binormal vector $b$ of $\theta \circ \alpha$ satisfies $\langle b, (0,0,1)\rangle > 0$ in a neighborhood of $s \in (a,b)$, where $s$ is the arclength function knowing that $k$ and $\tau$ are invariant under a rigid motion.*

*Proof.* Consider the map $\theta$ given by

$$\begin{aligned} \theta : \mathbb{R}^3 &\longrightarrow M_{3\times 3} \\ (T,N,B) &\longrightarrow \begin{pmatrix} \langle T,e_1\rangle & \langle T,e_2\rangle & \langle T,e_3\rangle \\ \langle N,e_1\rangle & \langle N,e_2\rangle & \langle N,e_3\rangle \\ \langle B,e_1\rangle & \langle B,e_2\rangle & \langle B,e_3\rangle \end{pmatrix}, \end{aligned}$$

where $T$,$N$, and $B$ are the Frenet-Serret frame of $\alpha$ and $\{e_1, e_2, e_3\}$ the orthonormal basis of $\mathbb{R}^3$. Now, $\theta$ is linear and orthogonal since its columns are orthonormal vectors, so $\det(\theta) = 1 > 0$. If we take $\theta \circ \alpha$, we know from Proposition 1.3.18 that $\theta \circ \alpha$ has the same curvature and torsion as $\alpha$ and that the Frenet-Serret frame of $\theta \circ \alpha$ is $(\theta T, \theta N, \theta B)$. Let's calculate $\theta B$. We have

$$\begin{aligned}
\theta B &= \begin{pmatrix} \langle T, e_1\rangle & \langle T, e_2\rangle & \langle T, e_3\rangle \\ \langle N, e_1\rangle & \langle N, e_2\rangle & \langle N, e_3\rangle \\ \langle B, e_1\rangle & \langle B, e_2\rangle & \langle B, e_3\rangle \end{pmatrix} \begin{pmatrix} \langle B, e_1\rangle \\ \langle B, e_2\rangle \\ \langle B, e_3\rangle \end{pmatrix} \\
&= \begin{pmatrix} \langle T, e_1\rangle . \langle B, e_1\rangle + \langle T, e_2\rangle . \langle B, e_2\rangle + \langle T, e_3\rangle . \langle B, e_3\rangle \\ \langle N, e_1\rangle . \langle B, e_1\rangle + \langle N, e_2\rangle . \langle B, e_2\rangle + \langle N, e_3\rangle \langle B, e_3\rangle \\ \langle B, e_1\rangle^2 + \langle B, e_2\rangle^2 + \langle B, e_3\rangle^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \|B\|^2 \end{pmatrix}.
\end{aligned}$$

Thus, $\langle \theta B, (0,0,1)\rangle = \|B\|^2 > 0$. □

Now, we are ready to prove Theorem 1.5.1.

*Proof of Theorem 1.5.1.* We need to find a curve $\alpha$ parametrized by arclength, such that its curvature $k_\alpha$ is equal to $k$, and its torsion $\tau_\alpha$ is equal to $\tau$. Let us write the tangent vector $T(s)$ in spherical coordinates $(\rho, \phi, \theta)$ where $\rho = 1$, $\phi$ the angle between $T(s)$ and the $z$-axis, and $\theta$ the rotation angle. We have

$$\begin{aligned}
T(s) &= (\rho \sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi), \\
T'(s) &= (\phi'\cos\phi\cos\theta - \theta'\sin\phi\sin\theta, \phi'\cos\phi\sin\theta + \theta'\sin\phi\cos\theta, -\phi'\sin\phi), \\
N(s) &= \frac{1}{k}T'(s) = \Big(\frac{\phi'\cos\phi\cos\theta - \theta'\sin\phi\sin\theta}{k}, \frac{\phi'\cos\phi\sin\theta + \theta'\sin\phi\cos\theta}{k}, \frac{-\phi'\sin\phi}{k}\Big), \\
B(s) &= T(s) \times N(s) \\
&= \Big(\frac{-\phi'\sin\theta\sin^2\phi - \phi'\sin\theta\cos^2\phi - \theta'\sin\phi\cos\phi\cos\theta}{k}, \\
&\quad -\frac{-\phi'\cos\theta\sin^2\phi - \phi'\cos\theta\cos^2\phi + \theta'\sin\phi\sin\theta\cos\phi}{k}, \\
&\quad +\frac{\phi'\sin\phi\cos\theta\cos\phi\sin\theta + \theta'\sin^2\phi\cos^2\theta - \phi'\sin\phi\sin\theta\cos\phi\cos\theta + \theta'\sin^2\phi\sin^2\theta}{k}\Big) \\
&= \Big(-\frac{\phi'\sin\phi}{k} - \frac{\phi'\cos\theta\sin 2\phi}{2k}, -\frac{\phi'\cos\theta}{k} + \frac{\theta'\sin\theta\sin 2\phi}{2k}, \frac{\theta'\sin^2\phi}{k}\Big).
\end{aligned}$$

We know that the Frenet-Serret trihedron $\Big(T(s), N(s), B(s)\Big)$ forms an orthonormal basis for $\mathbb{R}^3$ and satisfies:

$$\begin{cases} \dfrac{\partial T}{\partial s} = kN, \\ \dfrac{\partial N}{\partial s} = -kT + \tau B, \\ \dfrac{\partial B}{\partial s} = -\tau N. \end{cases}$$

Therefore, for $w = \langle T, u\rangle$, where $u$ is a fixed unit vector, we have:

$$\frac{\partial w}{\partial s} = \Big\langle \frac{\partial T}{\partial s}, u\Big\rangle + \Big\langle T, \frac{\partial u}{\partial s}\Big\rangle = \Big\langle \frac{\partial T}{\partial s}, u\Big\rangle = \langle kN, u\rangle = k\langle N, u\rangle.$$

Hence, $\dfrac{1}{k}\Big(\dfrac{\partial w}{\partial s}\Big) = \langle N, u\rangle$, and so

$$\begin{aligned} \frac{\partial}{\partial s}\Big\{\frac{1}{k}\frac{\partial w}{\partial s}\Big\} &= \langle \frac{\partial N}{\partial s}, u\rangle + \langle N, \frac{\partial u}{\partial s}\rangle = \langle \frac{\partial N}{\partial s}, u\rangle = \langle -kT + \tau B, u\rangle \\ &= \langle -kT, u\rangle + \langle \tau B, u\rangle = -k\langle T, u\rangle + \tau\langle B, u\rangle. \qquad (1.5.3) \end{aligned}$$

Also, since $u = \langle u, T\rangle T + \langle u, N\rangle N + \langle u, B\rangle B$, we get that

$$\langle u, u\rangle = \langle u, T\rangle^2 + \langle u, N\rangle^2 + \langle u, B\rangle^2,$$

and hence we have $\langle u, T\rangle^2 + \langle u, N\rangle^2 + \langle u, B\rangle^2 = 1$. Then, Equation (1.5.3) can be written as

$$\begin{aligned} \frac{\partial}{\partial s}\Big(\frac{1}{k}.\frac{\partial w}{\partial s}\Big) &= -k\langle T, u\rangle \pm \tau\sqrt{1 - \langle u, T\rangle^2 - \langle u, N\rangle^2} \\ &= -kw \pm \tau\sqrt{1 - w^2 - \Big(\frac{1}{k}.\frac{\partial w}{\partial s}\Big)^2}. \end{aligned}$$

By Lemma 1.5.3, we can always choose $\alpha$ such that $\langle B, (0,0,1)\rangle > 0$. Let $u = (0,0,1)$ and thus, $\langle B, (0,0,1)\rangle = \langle B, u\rangle = +\sqrt{1 - \langle N, u\rangle^2 - \langle T, u\rangle^2}$. So, we can consider the initial value problem:

$$\frac{\partial}{\partial s}\Big(\frac{1}{k}.\frac{\partial w}{\partial s}\Big) = -kw + \tau\sqrt{1 - w^2 - \Big(\frac{1}{k}.\frac{\partial w}{\partial s}\Big)^2} \tag{1.5.4}$$

with $w(s_0) = \langle T(s_0), u\rangle = \langle T(s_0), (0,0,1)\rangle = w_0$. We have

$$\frac{1}{k}\frac{\partial w}{\partial s}(s_0) = \langle N, u\rangle \Longrightarrow w'(s_0) = k\langle N(s_0), (0,0,1)\rangle = v_0.$$

Now, we calculate

$$\langle T, (0,0,1)\rangle = \cos\phi \quad \text{and} \quad \langle b, (0,0,1)\rangle = \frac{\theta' \sin^2\phi}{k}.$$

Thus, we have

$$\phi = \cos^{-1}(\langle T, (0,0,1)\rangle) = \cos^{-1}\xi, \tag{1.5.5}$$

and

$$\begin{aligned}
\theta &= \int \frac{k}{\sin^2\phi}\langle b, (0,0,1)\rangle ds \\
&= \int \frac{k}{\sin^2\phi}\sqrt{1 - \langle T, (0,0,1)\rangle^2 - \langle N, (0,0,1)\rangle^2} ds \\
&= \int \frac{k}{\sin^2\phi}\sqrt{1 - \xi^2 - \Big(\frac{1}{k}\frac{\partial \xi}{\partial s}\Big)^2} ds \\
&= \int \frac{k\sqrt{1 - \xi^2 - (\frac{1}{k}\frac{\partial \xi}{\partial s})^2}}{1 - \xi^2} ds. \qquad (1.5.6)
\end{aligned}$$

By replacing (1.5.5) and (1.5.6) in the expression of the tangent vector, we get

$$\begin{aligned} & T(s) \\ = & \left(\sqrt{1-\cos^2\phi}\cos\theta, \sqrt{1-\cos^2\phi}\sin\theta, \cos\phi\right) \\ = & \left(\sqrt{1-\xi^2}\cos\left(\int \frac{k\sqrt{1-\xi^2-(\frac{1}{k}\frac{\partial\xi}{\partial s})^2}}{1-\xi^2}ds\right), \sqrt{1-\xi^2}\sin\left(\int \frac{k\sqrt{1-\xi^2-(\frac{1}{k}\frac{\partial\xi}{\partial s})^2}}{1-\xi^2}ds\right), \xi\right)\end{aligned}$$

Hence, we find a curve $\alpha$ given by $\alpha(s) = (x(s), y(s), z(s))$, where

$$\begin{aligned} x(s) &= \int \left[\sqrt{1-\xi^2}\cos\left(\int \frac{k\sqrt{1-\xi^2-(\frac{1}{k}\frac{\partial\xi}{\partial s})^2}}{1-\xi^2}ds\right)\right]ds, \\ y(s) &= \int \left[\sqrt{1-\xi^2}\sin\left(\int \frac{k\sqrt{1-\xi^2-(\frac{1}{k}\frac{\partial\xi}{\partial s})^2}}{1-\xi^2}ds\right)\right]ds, \\ z(s) &= \int \xi ds.\end{aligned}$$

Now, the curve $\alpha$ is parametrized by arclength because

$$\begin{aligned} \|\alpha'(s)\| = \|T(s)\| &= \sqrt{(1-\cos^2\phi)\cos^2\theta + (1-\cos^2\phi)\sin^2\theta + \cos^2\phi} \\ &= \sqrt{1-\cos^2\phi+\cos^2\phi} = \sqrt{1} = 1.\end{aligned}$$

Let's calculate the curvature and the torsion of the curve $\alpha$. We have:

$$
\begin{aligned}
& T'(s) \\
= & \Bigg( -\frac{\xi\xi'}{\sqrt{1-\xi^2}} \cos\left( \int \frac{k\sqrt{1-\xi^2-(\frac{1}{k}\frac{\partial \xi}{\partial s})^2}}{1-\xi^2} ds \right) - \sqrt{1-\xi^2}\, \frac{k\sqrt{1-\xi^2-(\frac{1}{k}\frac{\partial \xi}{\partial s})^2}}{1-\xi^2} \\
& \times \sin\left( \int \frac{k\sqrt{1-\xi^2-(\frac{1}{k}\frac{\partial \xi}{\partial s})^2}}{1-\xi^2} ds \right), -\frac{\xi\xi'}{\sqrt{1-\xi^2}} \sin\left( \int \frac{k\sqrt{1-\xi^2-(\frac{1}{k}\frac{\partial \xi}{\partial s})^2}}{1-\xi^2} ds \right) \\
& + \sqrt{1-\xi^2}\, \frac{k\sqrt{1-\xi^2-(\frac{1}{k}\frac{\partial \xi}{\partial s})^2}}{1-\xi^2} \cos\left( \int \frac{k\sqrt{1-\xi^2-(\frac{1}{k}\frac{\partial \xi}{\partial s})^2}}{1-\xi^2} ds \right), \xi' \Bigg)
\end{aligned}
$$

Thus, we have :

$$
\begin{aligned}
k_\alpha &= \|T'(s)\| = \sqrt{\frac{\xi^2(\xi')^2}{1-\xi^2} + \frac{(1-\xi^2)k^2(1-\xi^2-(\frac{1}{k}\frac{\partial \xi}{\partial s})^2)}{(1-\xi^2)^2} + (\xi')^2} \\
&= \sqrt{\frac{\xi^2(\xi')^2}{1-\xi^2} + \frac{k^2(1-\xi^2-(\frac{1}{k}\frac{\partial \xi}{\partial s})^2)}{1-\xi^2} + (\xi')^2} \\
&= \sqrt{\frac{\xi^2(\xi')^2 + k^2 - k^2\xi^2(\xi')^2 + (\xi')^2 - \xi^2(\xi')^2}{1-\xi^2}} \\
&= \sqrt{\frac{k^2(1-\xi^2)}{1-\xi^2}} = \sqrt{k^2} = |k| = k
\end{aligned}
$$

Now, using Lemma 1.3.15, we know that $T_\alpha = \dfrac{\langle \alpha' \times \alpha'', \alpha''' \rangle}{\|\alpha' \times \alpha''\|^2}$, so let's calculate $\alpha'$, $\alpha^{''}$, $\alpha^{'''}$ and $\alpha' \times \alpha''$. We have

$$
\begin{aligned}
\alpha' &= (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi), \\
\alpha'' &= (\phi'\cos\phi\cos\theta - \theta'\sin\theta\sin\phi, \phi'\cos\phi\sin\theta + \theta'\cos\theta\sin\phi, -\phi'\sin\phi),
\end{aligned}
$$

$$
\begin{aligned}
& \alpha' \times \alpha'' \\
= \; & (-\phi' \sin^2\phi \sin\theta - \phi' \cos^2\phi \sin\theta - \theta' \cos\theta \sin\phi \cos\phi, \phi' \sin^2\phi \cos\theta + \phi' \cos^2\phi \cos\theta \\
& -\theta' \sin\theta \sin\phi \cos\phi, \phi' \sin\phi \sin\theta \cos\phi \cos\theta + \theta' \cos^2\theta \sin^2\phi - \phi' \cos\phi \cos\theta \sin\phi \sin\theta \\
& +\theta' \sin^2\theta \sin^2\phi) \\
= \; & (\underbrace{-\phi' \sin\theta - \theta' \cos\theta \sin\phi \cos\phi}_{B_1}, \underbrace{\phi' \cos\theta - \theta' \sin\theta \sin\phi \cos\phi}_{B_2}, \underbrace{\theta' \sin^2\phi}_{B_3}).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
& \|\alpha' \times \alpha''\|^2 \\
= \; & (\phi')^2 \sin^2\theta + (\theta')^2 \cos^2\theta \sin^2\phi \cos^2\phi + 2\phi'\theta' \sin\phi \sin\theta \cos\phi \cos\theta \\
& +(\phi')^2 \cos^2\theta + (\theta')^2 \sin^2\theta \sin^2\phi \cos^2\phi - 2\phi'\theta' \cos\theta \cos\phi \sin\phi \sin\theta + (\theta')^2 \sin^4\phi \\
= \; & (\phi')^2 + (\theta')^2 \sin^2\phi \cos^2\phi + (\theta')^2 \sin^4\phi.
\end{aligned}
$$

Using that

$$
\begin{cases}
\phi = \cos^{-1}\xi \\
\phi' = \frac{-\xi}{\sqrt{1-\xi^2}} \\
\phi'' = \frac{-\xi''\sqrt{1-\xi^2} + \xi' \frac{-2\xi\xi'}{2\sqrt{1-\xi^2}}}{1-\xi^2} = \frac{\xi''(1-\xi^2) - \xi(\xi')^2}{(1-\xi^2)\sqrt{1-\xi^2}} = \frac{-\xi'' + \xi^2\xi'' - \xi(\xi')^2}{(1-\xi^2)\sqrt{1-\xi^2}} \\
\theta = \int \frac{k\sqrt{1-\xi^2-(\frac{1}{k}\xi')^2}}{1-\xi^2} ds \\
\theta' = \frac{k\sqrt{1-\xi^2-(\frac{1}{k}\xi')^2}}{1-\xi^2} \\
\sin\phi = \sqrt{1-\xi^2} \\
\cos\phi = \xi
\end{cases}
, \qquad (1.5.7)
$$

we obtain

$$
\begin{aligned}
& \|\alpha' \times \alpha''\|^2 \\
= & \frac{(\xi')^2}{1-\xi^2} + \frac{k^2(1-\xi^2-\frac{1}{k^2}(\xi')^2)}{(1-\xi^2)^2}(1-\xi^2)\xi^2 + \frac{k^2(1-\xi^2-\frac{1}{k^2}(\xi')^2)}{(1-\xi^2)}(1-\xi^2) \\
= & \frac{(\xi')^2 + k^2\xi^2 - k^2\xi^4 - \xi^2(\xi')^2 + (k^2-k^2\xi^2)(1-\xi^2-\frac{1}{k^2}(\xi')^2)}{1-\xi^2} \\
= & \frac{(\xi')^2 + k^2\xi^2 - k^2\xi^4 - \xi^2(\xi')^2 + k^2 - k^2\xi^2 - (\xi')^2 - k^2\xi^2 + k^2\xi^4 + \xi^2(\xi')^2}{1-\xi^2} = k^2.
\end{aligned}
$$

We calculate now

$$\langle \alpha' \times \alpha'', \alpha''' \rangle = C_1B_1 + C_2B_2 + C_3B_3,$$

where $C_1$, $C_2$ and $C_3$ denote the three components of $\alpha'''$ given by

$$
\begin{aligned}
C_1 &= (\phi''\cos\phi - (\phi')^2\sin\phi)\cos\theta - \phi'\theta'\sin\theta\cos\phi \\
&\quad -\Big[(\theta''\sin\theta + (\theta')^2\cos\theta)\sin\phi + \phi'\theta'\sin\theta\cos\phi\Big] \\
&= \phi''\cos\phi\cos\theta - (\phi')^2\sin\phi\cos\theta - \phi'\theta'\sin\theta\cos\phi - \theta''\sin\theta\sin\phi - (\theta')^2\cos\theta\sin\phi \\
&\quad -\phi'\theta'\sin\theta\cos\phi,
\end{aligned}
$$

$$
\begin{aligned}
C_2 &= (\phi''\cos\phi - (\phi')^2\sin\phi)\sin\theta + \theta'\phi'\cos\phi\cos\theta + (\theta''\cos\theta - (\theta')^2\sin\theta)\sin\phi \\
&\quad +\theta'\phi'\cos\theta\cos\phi \\
&= \phi''\cos\phi\sin\theta - (\phi')^2\sin\phi\sin\theta + \theta'\phi'\cos\phi\cos\theta + \theta''\cos\theta\sin\phi - (\theta')^2\sin\theta\sin\phi \\
&\quad +\phi'\theta'\cos\theta\cos\phi,
\end{aligned}
$$

$$C_3 = -\phi'' \sin\phi - (\phi')^2 \cos\phi$$

First, we will calculate $B_iC_i$ for $i = 1, 2, 3$. We have

$$\begin{aligned}
& C_1B_1 \\
= \; & (\phi'' \cos\phi \cos\theta - (\phi')^2 \sin\phi \cos\theta - \phi'\theta' \sin\theta \cos\phi - \theta'' \sin\theta \sin\phi - (\theta')^2 \cos\theta \sin\phi \\
& -\phi'\theta' \sin\theta \cos\phi) \times (-\phi' \sin\theta - \theta' \cos\theta \sin\phi \cos\phi) \\
= \; & \phi'\phi'' \sin\theta \cos\phi \cos\theta - \theta'\phi'' \cos^2\theta \sin\phi \cos^2\phi + (\phi')^3 \sin\phi \cos\theta \sin\theta \\
& +\theta'(\phi')^2 \sin^2\phi \cos^2\theta \cos\phi + (\phi')^2\theta' \sin^2\theta \cos\phi + (\theta')^2\phi' \cos\theta \sin\phi \cos^2\phi \sin\theta \\
& +\phi'\theta'' \sin^2\theta \sin\phi + \theta''\theta' \cos\theta \sin^2\phi \cos\phi \sin\theta + (\theta')^2\phi' \sin\theta \cos\theta \sin\phi \\
& +(\theta')^3 \cos^2\theta \sin^2\phi \cos\phi + (\phi')^2\theta' \sin^2\theta \cos\phi + (\theta')^2\phi' \cos\theta \sin\phi \cos^2\phi \sin\theta,
\end{aligned}$$

$$\begin{aligned}
& C_2B_2 \\
= \; & (\phi'' \cos\phi \sin\theta - (\phi')^2 \sin\phi \sin\theta + \theta'\phi' \cos\phi \cos\theta + \theta'' \cos\theta \sin\phi - (\theta')^2 \sin\theta \sin\phi \\
& +\phi'\theta' \cos\theta \cos\phi) \times (\phi' \cos\theta - \theta' \sin\theta \sin\phi \cos\phi) \\
= \; & \phi'\phi'' \cos\theta \cos\phi \sin\theta - \theta'\phi'' \sin^2\theta \sin\phi \cos^2\phi - (\phi')^3 \sin\phi \sin\theta \cos\theta \\
& +\theta'(\phi')^2 \sin^2\phi \sin^2\theta \cos\phi + \theta'(\phi')^2 \cos^2\theta \cos\phi - (\theta')^2\phi' \cos^2\phi \cos\theta \sin\theta \sin\phi \\
& +\phi'\theta'' \cos^2\theta \sin\phi - \theta''\theta' \cos\theta \sin^2\phi \sin\theta \cos\phi - \phi'(\theta')^2 \sin\theta \sin\phi \cos\theta \\
& +(\theta')^3 \sin^2\theta \sin^2\phi \cos\phi + (\phi')^2\theta' \cos^2\theta \cos\phi - (\theta')^2\phi' \sin\theta \sin\phi \cos^2\phi \cos\theta,
\end{aligned}$$

$$\begin{aligned}
& C_3B_3 \\
= \; & (-\phi'' \sin\phi - (\phi')^2 \cos\phi) \times \theta' \sin^2\phi = -\theta'\phi'' \sin^3\phi - \theta'(\phi')^2 \cos\phi \sin^2\phi.
\end{aligned}$$

So, we get

$$
\begin{aligned}
& C_1B_1 + C_2B_2 + C_3B_3 \\
= \ & -\theta'\phi''\sin\phi\cos^2\phi + \theta'(\phi')^2\sin^2\phi\cos\phi + \theta'(\phi')^2\cos\phi \\
& +\phi'\theta''\sin\phi + (\theta')^3\sin^2\phi\cos\phi + \theta'(\phi')^2\cos\phi - \theta'\phi''\sin^3\phi \\
& -\theta'(\phi')^2\cos\phi\sin^2\phi \\
= \ & \theta'\phi''\sin\phi\cos^2\phi + 2\theta'(\phi')^2\cos\phi + \phi'\theta''\sin\phi + (\theta')^3\sin^2\phi\cos\phi \\
& -\theta'\phi''\sin^3\phi \\
= \ & \theta'\phi''\sin\phi(\cos^2\phi + \sin^2\phi) + 2\theta'(\phi')^2\cos\phi + \phi'\theta''\sin\phi + (\theta')^3\sin^2\phi\cos\phi \\
= \ & \underbrace{-\theta'\phi''\sin\phi}_{q_1} + \underbrace{2\theta'(\phi')^2\cos\phi}_{q_2} + \underbrace{\phi'\theta''\sin\phi}_{q_3} + \underbrace{(\theta')^3\sin^2\phi\cos\phi}_{q_4}.
\end{aligned}
$$

Using again (1.5.7), we get

$$
\begin{aligned}
q_1 &= -\theta'\phi''\sin\phi = \frac{k\sqrt{1-\xi^2-(\frac{\xi'}{k})^2}}{1-\xi^2} . \frac{\xi''-\xi^2\xi''+\xi(\xi')^2}{(1-\xi^2)\sqrt{1-\xi^2}}\sqrt{1-\xi^2} \\
&= \frac{k\sqrt{1-\xi^2-(\frac{\xi'}{k})^2}(\xi''-\xi^2\xi''+\xi(\xi')^2)}{(1-\xi^2)^2}, \\
q_2 &= 2\theta'(\phi')^2\cos\phi = \frac{2k\sqrt{1-\xi^2-\left(\frac{\xi'}{k}\right)^2}}{1-\xi^2}\frac{(\xi')^2}{1-\xi^2}\xi = \frac{2\xi(\xi')^2k\sqrt{1-\xi^2-\left(\frac{\xi'}{k}\right)^2}}{(1-\xi^2)}, \\
q_3 &= \phi'\theta''\sin\phi = \frac{-\xi'}{\sqrt{1-\xi^2}}\sqrt{1-\xi^2}\left(\frac{k\sqrt{1-\xi^2-(\frac{\xi'}{k})^2}}{1-\xi^2}\right)' \\
&= -\xi'\frac{(1-\xi^2)\left(k'\sqrt{1-\xi^2-\left(\frac{\xi'}{k}\right)^2} + k\frac{-2\xi\xi'-2\frac{\xi'}{k}(\frac{\xi''k-k'\xi'}{k^2})}{2\sqrt{1-\xi^2-(\frac{\xi'}{k})^2}}\right) + 2k\xi\xi'\sqrt{1-\xi^2-\left(\frac{\xi'}{k}\right)^2}}{(1-\xi^2)^2}
\end{aligned}
$$

$$
\begin{aligned}
&= -\xi' \frac{(1-\xi^2)\left[k'(1-\xi^2-(\frac{\xi'}{k})^2) - k\xi\xi' - \xi'\left(\frac{\xi''k-k'\xi'}{k^2}\right)\right] + 2k\xi\xi'\left(1-\xi^2-\left(\frac{\xi'}{k}\right)^2\right)}{\sqrt{1-\xi^2-(\frac{\xi'}{k})^2}(1-\xi^2)^2} \\
&= -\xi' \frac{(1-\xi^2)(k'k - k'k\xi^2 - (\xi')^2\frac{k'}{k} - k^2\xi\xi' - \frac{\xi'\xi''k-k'(\xi')^2}{k}) + 2k^2\xi\xi' - 2k^2\xi^3\xi' - 2\xi(\xi')^3}{\sqrt{(1-\xi^2)k^2-(\xi')^2}(1-\xi^2)^2} \\
&= -\xi' \frac{(1-\xi^2)(kk' - kk'\xi^2 - k^2\xi\xi' - \xi'\xi'') + 2\xi\xi'(k^2 - k^2\xi^2 - (\xi')^2)}{\sqrt{(1-\xi^2)k^2-(\xi')^2}(1-\xi^2)^2} \\
&= \xi' \frac{(1-\xi^2)(\xi\xi'k^2 - kk'(1-\xi^2) + \xi'\xi'') - 2\xi\xi'(k^2(1-\xi^2) - (\xi')^2)}{(1-\xi^2)^2\sqrt{(1-\xi^2)k^2-(\xi')^2}} \\
&= \xi' \frac{\xi\xi'k^2 - kk' + kk'\xi^2 + \xi'\xi'' - \xi^3\xi'k^2 + \xi^2kk' - kk'\xi^4 - \xi'\xi''\xi^2}{(1-\xi^2)^2\sqrt{(1-\xi^2)k^2-(\xi')^2}} \\
&\quad \frac{-2\xi\xi'k^2 + 2\xi^3\xi'k^2 + 2\xi(\xi')^3}{(1-\xi^2)^2\sqrt{(1-\xi^2)k^2-(\xi')^2}} \\
&= \xi' \frac{-\xi\xi'k^2 - kk' + 2\xi^2kk' + \xi'\xi'' + \xi^3\xi'k^2 - kk'\xi^4 - \xi'\xi''\xi^2 + 2\xi(\xi')^3}{(1-\xi^2)^2\sqrt{(1-\xi^2)^2k^2-(\xi')^2}},
\end{aligned}
$$

$$
q_4 = (\theta')^3\sin^2\phi\cos\phi = \frac{k^3(1-\xi^2-\left(\frac{\xi'}{k}\right)^2)^{\frac{3}{2}}}{(1-\xi^2)^3}.(1-\xi^2)\xi = \frac{((1-\xi^2)k^2-(\xi')^2)^{\frac{3}{2}}\xi}{(1-\xi^2)^2}.
$$

Finally,

$$
\begin{aligned}
&C_1B_1 + C_2B_2 + C_3B_3 \\
&= q_1 + q_2 + q_3 + q_4 \\
&= \frac{\sqrt{k^2(1-\xi^2)-(\xi')^2}(\xi'' - \xi^2\xi'' + \xi(\xi')^2)}{(1-\xi^2)^2} + \frac{2\xi(\xi')^2\sqrt{k^2(1-\xi^2)-(\xi')^2}}{(1-\xi^2)^2} \\
&\quad + \xi' \frac{\xi^3k^2\xi' - k^2\xi\xi' + 2\xi(\xi')^3 - kk' - kk'\xi^4 + \xi'\xi'' + 2\xi^2kk' - \xi^2\xi'\xi''}{(1-\xi^2)^2\sqrt{k^2(1-\xi^2)-(\xi')^2}} \\
&\quad + \frac{\xi(k^2(1-\xi^2)-(\xi')^2)^{\frac{3}{2}}}{(1-\xi^2)^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(k^2-k^2\xi^2-(\xi')^2)(\xi''-\xi^2\xi''+\xi(\xi')^2)+2\xi(\xi')^2(k^2-k^2\xi^2-(\xi')^2)+\xi(k^2-k^2\xi^2-(\xi')^2)^2}{(1-\xi^2)^2\sqrt{k^2(1-\xi^2)-(\xi')^2}} \\
&\quad + \frac{\xi^3k^2(\xi')^2-k^2\xi(\xi')^2+2\xi(\xi')^4-kk'\xi'-kk'\xi'\xi^4+\xi''(\xi')^2+2\xi^2\xi'kk'-\xi^2\xi''(\xi')^2}{(1-\xi^2)^2\sqrt{k^2(1-\xi^2)-(\xi')^2}} \\
&= \frac{k^2\xi''-\xi^2\xi''k^2+\xi(\xi')^2k^2-k^2\xi''\xi^2+k^2\xi^4\xi''-k^2\xi^3(\xi')^2-\xi''(\xi')^2+\xi''(\xi')^2\xi^2-\xi(\xi')^4}{(1-\xi^2)^2\sqrt{k^2(1-\xi^2)-(\xi')^2}} \\
&\quad + \frac{2k^2\xi(\xi')^2-2k^2\xi^3(\xi')^2-2\xi(\xi')^4+\xi(k^4+k^4\xi^4+(\xi')^4-2k^4\xi^2-2k^2(\xi')^2+2k^2\xi^2(\xi')^2)}{(1-\xi^2)^2\sqrt{k^2(1-\xi^2)-(\xi')^2}} \\
&\quad + \frac{\xi^3k^2(\xi')^2-k^2\xi(\xi')^2+2\xi(\xi')^4-kk'\xi'-kk'\xi^4\xi'+\xi''(\xi')^2+2\xi^2\xi'kk'-\xi^2\xi''(\xi')^2}{(1-\xi^2)^2\sqrt{k^2(1-\xi^2)-(\xi')^2}} \\
&= \frac{k^2\xi''-\xi^2\xi''k^2+\xi(\xi')^2k^2-k^2\xi''\xi^2+k^2\xi^4\xi''-k^2\xi^3(\xi')^2-\xi''(\xi')^2+\xi^2\xi''(\xi')^2-\xi(\xi')^4}{(1-\xi^2)^2\sqrt{k^2(1-\xi^2)-(\xi')^2}} \\
&\quad + \frac{2k^2\xi(\xi')^2-2k^2\xi^3(\xi')^2-2\xi(\xi')^4+\xi k^4+k^4\xi^5+\xi(\xi')^4-2k^4\xi^3-2k^2\xi(\xi')^2+2k^2\xi^3(\xi')^2}{(1-\xi^2)^2\sqrt{k^2(1-\xi^2)-(\xi')^2}} \\
&\quad + \frac{\xi^3k^2(\xi')^2-k^2\xi(\xi')^2+2\xi(\xi')^4-kk'\xi'-kk'\xi^4\xi'+\xi''(\xi')^2+2\xi^2\xi'kk'-\xi^2\xi''(\xi')^2}{(1-\xi^2)^2\sqrt{k^2(1-\xi^2)-(\xi')^2}} \\
&= \frac{k^2\xi''-2k^2\xi^2\xi''+k^2\xi^4\xi''+k^4\xi+k^4\xi^5-2k^4\xi^3-kk'\xi'-kk'\xi'\xi^4+2\xi^2\xi'kk'}{(1-\xi^2)^2\sqrt{k^2(1-\xi^2)-(\xi')^2}} \\
&= \frac{k(k\xi''-2k\xi^2\xi''+k\xi^4\xi''+k^3\xi+k^3\xi^5-2k^3\xi^3-k'\xi'-k'\xi^4\xi'+2k'\xi^2\xi')}{(1-\xi^2)^2\sqrt{k^2(1-\xi^2)-(\xi')^2}} \\
&= \frac{k\left[\xi k^3(1+\xi^4-2\xi^2)-k'\xi'(1+\xi^4-2\xi^2)+k\xi''(1-2\xi^2+\xi^4)\right]}{(1-\xi^2)^2\sqrt{k^2(1-\xi^2)-(\xi')^2}} \\
&= \frac{k(1-2\xi^2+\xi^4)(\xi k^3-k'\xi'+k\xi'')}{(1-\xi^2)^2\sqrt{k^2(1-\xi^2)-(\xi')^2}} = \frac{k(\xi k^3-k'\xi'+k\xi'')}{\sqrt{k^2(1-\xi^2)-(\xi')^2}}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\tau_\alpha &= \frac{\langle \alpha'\times\alpha'', \alpha'''\rangle}{\|\alpha'\times\alpha''\|^2} \\
&= \frac{(\xi k^3-k'\xi'+k\xi'')}{k\sqrt{k^2(1-\xi^2)-(\xi')^2}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(\xi k^3 - k'\xi' + k\xi'')}{k^2\sqrt{(1-\xi^2) - \left(\frac{\xi'}{k}\right)^2}} \\
&= \frac{\xi k - \frac{k'}{k^2}\xi' + \frac{k}{k^2}\xi''}{\sqrt{(1-\xi^2) - \left(\frac{\xi'}{k}\right)^2}} \\
&= \frac{\xi k + \frac{\xi'' k - \xi' k'}{k^2}}{\sqrt{(1-\xi^2) - \left(\frac{\xi'}{k}\right)^2}} \\
&= \frac{\xi k + \left(\frac{\xi'}{k}\right)'}{\sqrt{(1-\xi^2) - \left(\frac{\xi'}{k}\right)^2}} \\
&= \frac{\xi k + \frac{\partial}{\partial s}\left\{\frac{1}{k}\frac{\partial \xi}{\partial s}\right\}}{\sqrt{(1-\xi^2) - \left(\frac{1}{k}\frac{\partial \xi}{\partial s}\right)^2}} = \tau
\end{aligned}
$$

□

## 1.6 Observations and Applications

In this section, we will discuss some applications of the new proof of the Fundamental Theorem of Curves. In particular, we will characterize general and slant helices. First, let's restate the Fundamental Theorem of Curves which was proved in the previous section.

Let $k : [a,b] \longrightarrow \mathbb{R}$ be a function always positive of class $C^1$ and let $\tau : [a,b] \longrightarrow \mathbb{R}$ be a function of class $\mathbb{C}^0$. If $\xi = \xi(s)$ is a solution of:

$$\frac{\partial}{\partial s}\left\{\frac{1}{k}\frac{\partial w}{\partial s}\right\} = -kw + \tau\sqrt{1 - w^2 - \left(\frac{1}{k}\frac{\partial w}{\partial s}\right)^2}$$

$$with \quad \begin{cases} w(s_0) = w_0 \\ w'(s_0) = v_0 \end{cases}$$

where $s_0 \in (a+\epsilon, b-\epsilon)$, we have $(w_0, v_0) \in \left\{(w,v) \in \mathbb{R}^2 / w^2 + \frac{v^2}{k_0^2} < 1\right\}$ where $k_0 = min\left\{k(s)/s \in [a+\epsilon, b-\epsilon]\right\}$ for some $\epsilon > 0$. Then, $\alpha(s) = (x(s), y(s), z(s))$ where:

$$\begin{aligned} x(s) &= \int \sqrt{1-\xi^2}\cos\left(\int \frac{\sqrt{(1-\xi^2)k^2 - (\xi')^2}}{1-\xi^2} ds\right) ds \\ y(s) &= \int \sqrt{1-\xi^2}\sin\left(\int \frac{\sqrt{(1-\xi^2)k^2 - (\xi')^2}}{1-\xi^2} ds\right) ds \\ z(s) &= \int \xi ds \end{aligned}$$

is a curve parametrized by arclength $s$ where $k = k(s)$ is the curvature and $\tau = \tau(s)$ is the torsion of $\alpha$.

Reciprocally, let $\alpha : I \longrightarrow \mathbb{R}^3$ be a curve parametrized by arclength $s$, where $k = k(s)$ is the curvature and $\tau = \tau(s)$ is the torsion of $\alpha$, and let $T_0, N_0, B_0$ be the Frenet frame of $\alpha$ at $s = s_0 \in I$. Also, consider the canonical basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$. Then, there exists an orthogonal linear map $\sigma$ of $\mathbb{R}^3$ with positive determinant such that the components $\langle T_B(s), e_1\rangle, \langle T_B(s), e_2\rangle$, and $\langle T_B(s), e_3\rangle$ of the tangent vector $T_B$ of the curve $\beta = \sigma \circ \alpha$ satisfy the intial value problem:

$$
\begin{aligned}
\frac{\partial}{\partial s}\left\{\frac{1}{k}\frac{\partial w}{\partial s}\right\} &= -kw + \tau\sqrt{1 - w^2 - \left(\frac{1}{k}\frac{\partial w}{\partial s}\right)^2} \\
w(s_0) &= \langle \sigma \circ T_0, e_i\rangle \quad for \quad i = 1, 2, 3 \\
w'(s_0) &= \langle k(s_0)\sigma \circ N_0 e_i\rangle \quad for \quad i = 1, 2, 3
\end{aligned}
$$

in some neighborhoods $I_i$ of $s_0 \in I$ for $i = 1, 2, 3$ respectively.

### 1.6.1 General Helices

In this subsection, we define and characterize general helices.

**Definition 1.6.1.** *A curve $\alpha$, with $k(s) \neq 0$, is called a general helix if the principal tangent lines of $\alpha$ make a constant angle with a fixed direction.*

**Theorem 1.6.2.** *Let $\alpha$ be a unit speed curve in $\mathbb{R}^3$ (i.e. $\|\alpha'\| = 1$) with curvature $k = k(s) \neq 0$ and torsion $\tau = \tau(s)$. The following statements are equivalent*

1. *$\alpha$ is a general helix.*
2. *$\frac{\tau}{k}(s)$ is a constant.*
3. *The curve $\alpha$ is given by $\alpha(s) = (x(s), y(s), z(s))$, where*

$$
\begin{cases}
x(s) = \frac{1}{\sqrt{1+m^2}} \int \cos\left(\sqrt{1+m^2}\int k ds\right) ds, \\
y(s) = \frac{1}{\sqrt{1+m^2}} \int \sin\left(\sqrt{1+m^2}\int k ds\right) ds, \\
z(s) = \frac{ms}{\sqrt{1+m^2}}.
\end{cases}
$$

*Proof.* **1⟹2:** Assume that $\alpha$ is a general helix with a Frenet frame $(T, N, B)$. Then, the principal tangent lines of $\alpha$ form a constant angle with a fixed direction. So, there exists a fixed unit vector $U$ such that $\langle T, U\rangle = \delta$ where $\delta$ is a constant. From the proof of Theorem 1.5.1), we know that if $w = \langle T, D\rangle$ where $D$ is any unit vector, we have

$$\frac{\partial}{\partial s}\left\{\frac{1}{k}\frac{\partial w}{\partial s}\right\} = -kw \pm \tau\sqrt{1 - w^2 - \left(\frac{1}{k}\frac{\partial w}{\partial s}\right)^2}. \tag{1.6.1}$$

By taking $D = U$, we get

$$\frac{\partial}{\partial s}\left\{\frac{1}{k}\frac{\partial w}{\partial s}\right\} = -kw \pm \tau\sqrt{1 - w^2 - \left(\frac{1}{k}\frac{\partial w}{\partial s}\right)^2}. \tag{1.6.2}$$

Since $\delta$ is a constant, we have $\dfrac{\partial w}{\partial s} = \dfrac{\partial \delta}{\partial s} = 0$, and so Equation (1.6.2) becomes

$$0 \quad = \quad -k\delta \pm \tau\sqrt{1 - \delta^2 - 0}.$$

Thus, $\dfrac{\tau}{k} = \pm\dfrac{\delta}{\sqrt{1-\delta^2}}$, which means that $\frac{\tau}{k}$ is a constant.

**2 ⟹ 3:** Assume $\dfrac{\tau}{k}(s) = m$, where $m$ is a constant. Now, replace $\tau = km$ in the Equation 1.6.1 to get:

$$\frac{\partial}{\partial s}\left\{\frac{1}{k}\frac{\partial w}{\partial s}\right\} = -kw \pm +km\sqrt{1 - w^2 - \left(\frac{1}{k}\frac{\partial w}{\partial s}\right)^2}. \tag{1.6.3}$$

Now, let's find a solution of the form $\xi(s) = \delta$ where $\delta$ is a constant and $1 - \delta^2 > 0$ to match the definition of a general helix. $\xi(s) = \delta$ is a solution of the Equation: 1.6.3 but $\xi(s) = \delta$ is a constant

$$\begin{aligned}
&\Longrightarrow \quad 0 = -k\delta \pm km\sqrt{1-\delta^2} \Longrightarrow \delta^2 = m^2(1-\delta^2) \\
&\Longrightarrow \quad \delta^2 = m^2 - \delta^2 m^2 \Longrightarrow \delta^2(1+m^2) \\
&\Longrightarrow \quad \delta^2 = \frac{m^2}{1+m^2} \Longrightarrow \delta = \frac{m}{\sqrt{1+m^2}} = \xi(s)
\end{aligned}$$

is a solution of the differential equation. Now using the proof of the fundamental theorem of curves, we know that any curve $\alpha(s) = (x(s), y(s), z(s))$ where

$$\begin{aligned}
x(s) &= \int \left[ \sqrt{1-\xi^2} \cos \left( \int \frac{k\sqrt{1-\xi^2 - \left(\frac{1}{k}\frac{\partial \xi}{\partial s}\right)^2}}{1-\xi^2} ds \right) \right] ds, \\
y(s) &= \int \left[ \sqrt{1-\xi^2} \sin \left( \int \frac{k\sqrt{1-\xi^2 - \left(\frac{1}{k}\frac{\partial \xi}{\partial s}\right)^2}}{1-\xi^2} ds \right) \right] ds, \\
z(s) &= \int \xi ds.
\end{aligned}$$

We know that:

$$\begin{aligned}
\int \frac{k\sqrt{1-\xi^2 - \left(\frac{1}{k}\frac{\partial \xi}{\partial s}\right)^2}}{1-\xi^2} ds &= \int \frac{k\sqrt{1-\delta^2}}{1-\delta^2} ds = \frac{1}{\sqrt{1-\delta^2}} \int kds = \frac{1}{\sqrt{1-\frac{m^2}{1+m^2}}} \int kds \\
&= \frac{\sqrt{1+m^2}}{1+m^2-m^2} \int kds = \sqrt{1+m^2} \int kds
\end{aligned}$$

$$\sqrt{1-\delta^2} = \sqrt{1-\frac{m^2}{1+m^2}} = \sqrt{\frac{1+m^2-m^2}{1+m^2}} = \frac{1}{\sqrt{1+m^2}}$$

$$\Longrightarrow \begin{cases} x(s) = \dfrac{1}{\sqrt{1+m^2}} \displaystyle\int \cos\left(\sqrt{1+m^2}\int k ds\right) ds, \\ y(s) = \dfrac{1}{\sqrt{1+m^2}} \displaystyle\int \sin\left(\sqrt{1+m^2}\int k ds\right) ds, \\ z(s) = \dfrac{ms}{\sqrt{1+m^2}}. \end{cases}$$

So, 3. is satisfied.

**3 $\Longrightarrow$ 1:** Assume that the statement 3 holds. Then, its tangent vector $T_\alpha$ has the following components

$$\begin{aligned} x'(s) &= \frac{1}{\sqrt{1+m^2}} \cos\left(\sqrt{1+m^2}\int k ds\right), \\ y'(s) &= \frac{1}{\sqrt{1+m^2}} \sin\left(\sqrt{1+m^2}\int k ds\right), \\ z'(s) &= \frac{m}{\sqrt{1+m^2}}. \end{aligned}$$

Then, $\langle T_\alpha, (0,0,1)\rangle = \dfrac{m}{\sqrt{1+m^2}}$, which is a constant. So, the principal tangent lines of $\alpha$ make a constant angle with the fixed direction $U = (0,0,1)$. Hence, $\alpha$ is a general helix.

□

### 1.6.2 Slant Helices

In this subsection, we define and characterize slant helices.

**Definition 1.6.3.** *A curve $\alpha$ with $k(s) \neq 0$ is called a slant helix if the principal normal lines of $\alpha$ make a constant angle with a fixed direction.*

**Theorem 1.6.4.** *Let $\alpha$ be a unit speed curve in $\mathbb{R}^3$ with curvature $k = k(s) \neq 0$ and torsion $\tau = \tau(s)$. Then the following are equivalent*

1. $\alpha$ *is a slant helix.*

2. *The function* $\sigma(s) = \left(\frac{k^2}{(k^2+\tau^2)^{\frac{3}{2}}}\left(\frac{\tau}{k}\right)'\right)(s)$ *is constant.*

3. *The curve* $\alpha$ *is given by* $\alpha(s) = (x(s), y(s), z(s))$ *where*

$$\begin{cases} x(s) = \dfrac{1}{\sqrt{1+m^2}} \displaystyle\int \left( \int \sin\left[\frac{\sqrt{1+m^2}\sin^{-1}(m\int_0^s kds)}{m}\right] k(s)ds \right) ds, \\ \\ y(s) = \dfrac{1}{\sqrt{1+m^2}} \displaystyle\int \left( \int \cos\left[\frac{\sqrt{1+m^2}\sin^{-1}(m\int_0^s kds)}{m}\right] k(s)ds \right) ds, \\ \\ z(s) = \dfrac{|m|}{\sqrt{1+m^2}} \displaystyle\int \Big( \int_0^s kds \Big) ds. \end{cases}$$

*Proof.* **1 $\Longrightarrow$ 2:** Assume $\alpha$ is a slant helix with a Frenet frame $\Big(T, N, B\Big)$. Then the principal normal lines of $\alpha$ make a constant angle with a fixed direction. So, there exists a fixed unit vector $U$ such that $\langle N, U\rangle = \delta$, where $\delta$ is a constant. From the proof of Theorem 1.5.1), we know that if $w = \langle T, D\rangle$ where $D$ is any unit vector, then

$$\frac{\partial}{\partial s}\left\{\frac{1}{k}\frac{\partial w}{\partial s}\right\} = -kw \pm \tau\sqrt{1 - w^2 - \left(\frac{1}{k}\frac{\partial w}{\partial s}\right)^2}$$

Now taking $D = U$ we have:

$$\frac{\partial w}{\partial s} = \langle\frac{\partial T}{\partial s}, U\rangle + \langle T, \frac{\partial U}{\partial s}\rangle \quad = \quad \langle k(s).N(s), U\rangle = k\langle N, U\rangle = k\delta.$$

Because $\delta$ is constant, we get $\dfrac{\partial}{\partial s}\left\{\dfrac{1}{k}\dfrac{\partial w}{\partial s}\right\} = \dfrac{\partial}{\partial s}(\delta) = 0$ and hence we have,

$$-kw \pm \tau\sqrt{1 - w^2\delta^2} = 0.$$

Thus,

$$\begin{aligned}
& kw = \pm\tau\sqrt{1 - w^2 - \delta^2} \\
\implies \quad & w^2 = \frac{\tau^2(1 - w^2 - \delta^2)}{k^2} \\
\implies \quad & k^2 w^2 = \tau^2 - \tau^2 w^2 - \tau^2\delta^2 \\
\implies \quad & w^2 = \frac{\tau^2 - \tau^2\delta^2}{k^2 + \tau^2} = \frac{\tau^2(1 - \delta^2)}{k^2(1 + \frac{\tau^2}{k^2})} = \left(\frac{\tau}{k}\right)^2 \frac{1 - \delta^2}{1 + (\frac{\tau}{k})^2}.
\end{aligned}$$

Now, $\dfrac{\partial w}{\partial s} = k\delta \implies w = \delta \displaystyle\int_0^s k ds$. Also, we have $w = (\dfrac{\tau}{k})\dfrac{\sqrt{1 - \delta^2}}{\sqrt{1 + (\frac{\tau}{k})^2}}$. Thus, deriving with respect to $s$, we get:

$$\begin{aligned}
& \frac{\delta k}{\sqrt{1 - \delta^2}} = \frac{(\frac{\tau}{k})'\sqrt{1 + (\frac{\tau}{k})^2} - (\frac{\tau}{k})\frac{(\frac{\tau}{k})'(\frac{\tau}{k})'}{\sqrt{1 + (\frac{\tau}{k})^2}}}{1 + (\frac{\tau}{k})^2} \\
\implies \quad & \frac{\delta k}{\sqrt{1 - \delta^2}} = \frac{(\frac{\tau}{k})'(1 + (\frac{\tau}{k})^2 - (\frac{\tau}{k})^2)}{(1 + (\frac{\tau}{k})^2)^{\frac{3}{2}}} \\
\implies \quad & \frac{\delta}{\sqrt{1 - \delta^2}} = \frac{(\frac{\tau}{k})'}{k(1 + (\frac{\tau}{k})^2)^{\frac{3}{2}}} \times \frac{k^3}{k^3} \\
\implies \quad & \frac{k^2(\frac{\tau}{k})'}{(k^2 + \tau^2)^{\frac{3}{2}}} = \frac{\delta}{\sqrt{1 - \delta^2}}.
\end{aligned}$$

We get that $\dfrac{k^2(\frac{\tau}{k})'}{(k^2 + \tau^2)^{\frac{3}{2}}}$ is a constant and hence the statement 2 is satisfied.

**2 $\Longrightarrow$ 3:** Suppose $\dfrac{k^2}{(k^2+\tau^2)^{\frac{3}{2}}}\left(\dfrac{\tau}{k}\right)' = m$, where $m$ is a constant. We have

$$\begin{aligned}
\frac{(\frac{\tau}{k})'}{k(1+(\frac{\tau}{k})^2)^{\frac{3}{2}}} &= m \\
\Longrightarrow \int_0^s \frac{(\frac{\tau}{k})'}{(1+(\frac{\tau}{k})^2)^{\frac{3}{2}}} ds &= \int_0^s kmds \\
\Longrightarrow \int_0^s \frac{(\frac{\tau}{k})'}{(1+(\frac{\tau}{k})^2)^{\frac{3}{2}}} ds &= m\int_0^s kds
\end{aligned}$$

Now, we calculate the integral $I = \displaystyle\int_0^2 \frac{(\frac{\tau}{k})'}{(1+(\frac{\tau}{k})^2)^{\frac{3}{2}}} ds$. Consider the change of variable $u = \dfrac{\tau}{k}$. We get

$$I = \int_0^s \frac{du}{(1+u^2)^{\frac{3}{2}}} ds.$$

Now we consider the trigonometric substitution $u = \tan\theta$. We have

$$\begin{aligned}
I &= \int_0^2 \frac{\sec^2\theta}{\sec^3\theta} d\theta \\
&= \int_0^s \cos\theta d\theta = [\sin\theta]_0^s \\
&= \left[\frac{u}{\sqrt{1+u^2}}\right]_0^s \\
&= \left[\frac{\frac{\tau}{k}}{(1+(\frac{\tau}{k})^2)^{\frac{1}{2}}}\right]_0^s
\end{aligned}$$

Thus, we have

$$\left[\frac{\frac{\tau}{k}}{(1+(\frac{\tau}{k})^2)^{\frac{1}{2}}}\right]_0^s = m\int_0^s kds,$$

and finally

$$\frac{\frac{\tau}{k}}{(1+(\frac{\tau}{k})^2)^{\frac{1}{2}}} = m\int_0^s kds + A,$$

where A is the integration constant. Hence, we have

$$\begin{aligned}
& \frac{\tau^2}{k^2} = \Big(m\int_0^s kds + A\Big)^2\Big(1+\Big(\frac{\tau}{k}\Big)^2\Big) \\
\Longrightarrow\quad & \frac{\tau^2}{k^2} = \Big(m\int_0^s kds + A\Big)^2 + \Big(\frac{\tau}{k}\Big)^2\Big(m\int_0^s kds + A\Big)^2 \\
\Longrightarrow\quad & \frac{\tau^2}{k^2}\Big(1-\Big(m\int_0^s kds + A\Big)^2\Big) = \Big(m\int_0^s kds + A\Big)^2 \\
\Longrightarrow\quad & \frac{\tau^2}{k^2} = \frac{(m\int_0^s kds + A)^2}{1-(m\int_0^s kds + A)^2} \\
\Longrightarrow\quad & \tau = k\left(\frac{(m\int_0^s kds + A)^2}{1-(m\int_0^s kds + A)^2}\right)^{\frac{1}{2}}.
\end{aligned}$$

Now, using the differential equation 1.6.1 and replacing $\tau$ by the quantity above, we get

$$\frac{\partial}{\partial s}\Big\{\frac{1}{k}\frac{\partial w}{\partial s}\Big\} = -kw + k\left(\frac{(m\int_0^s kds + A)^2}{1-(m\int_0^s kds + A)^2}\right)^{\frac{1}{2}}\sqrt{1-w^2-\Big(\frac{1}{k}\frac{\partial w}{\partial s}\Big)^2} \tag{1.6.4}$$

We need to find a solution of the form $\epsilon = \delta\int_0^s kds$ with $\frac{1}{k}\frac{\partial \epsilon}{\partial s} = \delta$ where $\delta$ is a constant and $\epsilon$ satisfies $1-\epsilon^2-\Big(\frac{1}{k}\frac{\partial \epsilon}{\partial s}\Big)^2 > 0$. Replacing in the differential equation above, we get:

$$0 = -k\delta\int_0^s kds + k\frac{|m\int_0^s kds + A|}{\sqrt{1-(m\int_0^s kds + A)^2}}\sqrt{1-\Big(\delta\int_0^s kds\Big)^2-\delta^2}.$$

If $m > 0$, we have $\delta = \frac{m}{\sqrt{1+m^2}}$ and $A = 0$ (for simplicity of calculation) so we get:

$$- \quad k\left(\frac{m}{\sqrt{1+m^2}}\right)\int_0^s kds + k\frac{|m\int_0^s kds|}{\sqrt{1-(\int_0^s kds)^2}}\sqrt{1-\frac{m^2}{1+m^2}\Big(\int_0^s kds\Big)^2-\frac{m^2}{1+m^2}}$$

$$= \quad - \quad k\left(\frac{m}{\sqrt{1+m^2}}\right)\int_0^s kds + k\frac{|m\int_0^s kds|}{\sqrt{1-(\int_0^s kds)^2}}\sqrt{\frac{1-(m\int_0^s kds)^2}{1+m^2}}$$

$$= \quad - \quad k\left(\frac{m}{\sqrt{1+m^2}}\right)\int_0^s kds + k\frac{m\int_0^s kds}{\sqrt{1-(\int_0^s kds)^2}}\frac{\sqrt{1-(\int_0^s kds)^2}}{\sqrt{1+m^2}} = 0$$

If $m < 0$, we have $\delta = \dfrac{-m}{\sqrt{1+m^2}}$. $A = 0$ and similarly, we get

$$-k\delta\int_0^s kds + k\frac{|m\int_0^s kds|}{\sqrt{1-(m\int_0^s kds)^2}}\sqrt{1-\Big(\delta\int_0^s kds\Big)^2-\delta^2} = 0.$$

It means that $\epsilon = \dfrac{m}{\sqrt{1+m^2}}\displaystyle\int_0^s kds$ (for $m > 0$) is a solution of the differential equation:

$$\frac{\partial}{\partial s}\Big\{\frac{1}{k}\frac{\partial w}{\partial s}\Big\} = -kw + k\frac{(m\int_0^s kds)}{\sqrt{1-(m\int_0^s kds)^2}}\sqrt{1-w^2-\Big(\frac{1}{k}\frac{\partial w}{\partial s}\Big)^2}$$

and $\epsilon = \dfrac{-m}{\sqrt{1+m^2}}\displaystyle\int_0^s kds$ (for $m < 0$) is a solution of the differential equation:

$$\frac{\partial}{\partial s}\Big\{\frac{1}{k}\frac{\partial w}{\partial s}\Big\} = -kw + k\frac{(m\int_0^s kds)}{\sqrt{1-(m\int_0^s kds)^2}}\sqrt{1-w^2-\Big(\frac{1}{k}\frac{\partial w}{\partial s}\Big)^2}.$$

We need to calculate the coordinates of $\alpha$ using the following:

$$\begin{aligned} x(s) &= \int \left[ \sqrt{1-\xi^2} \cos \left( \int \frac{k\sqrt{1-\xi^2-\left(\frac{1}{k}\frac{\partial \xi}{\partial s}\right)^2}}{1-\xi^2} ds \right) \right] ds, \\ y(s) &= \int \left[ \sqrt{1-\xi^2} \sin \left( \int \frac{k\sqrt{1-\xi^2-\left(\frac{1}{k}\frac{\partial \xi}{\partial s}\right)^2}}{1-\xi^2} ds \right) \right] ds, \\ z(s) &= \int \xi ds. \end{aligned}$$

Let's consider the case $m < 0$:

$$\begin{aligned} \text{We have } J = \int \frac{k\sqrt{1-\xi^2-\left(\frac{1}{k}\frac{\partial \xi}{\partial s}\right)^2}}{1-\xi^2} ds &= \int \frac{k\sqrt{1-\frac{m^2}{1+m^2}(\int_0^s kds)^2-\frac{m^2}{1+m^2}}}{1-\frac{m^2}{1+m^2}(\int_0^s kds)^2} ds \\ &= \frac{1}{\sqrt{1+m^2}} \int \frac{k\sqrt{1-m^2(\int_0^s kds)^2}}{1-\frac{m^2}{1+m^2}(\int_0^s kds)^2} ds \end{aligned}$$

Let $u = m \displaystyle\int_0^s kds$, $du = mkds$ and $ds = \frac{du}{mk}$. Then,

$$J = \frac{1}{m\sqrt{1+m^2}} \int \frac{\sqrt{1-u^2}}{1-\frac{u^2}{1+m^2}} du = -\frac{\sqrt{1+m^2}}{m} \int \frac{\sqrt{1-u^2}}{u^2-(1+m^2)} du.$$

Let $u = \sin\theta$, so $du = \cos\theta d\theta$ and

$$\begin{aligned} J &= \frac{-\sqrt{1+m^2}}{m} \int \frac{\cos^2\theta}{\sin^2\theta-(1+m^2)} d\theta \\ &= \frac{-\sqrt{1+m^2}}{m} \int \sec^2\theta \frac{-1}{(\tan^2\theta+1)(m^2+1+m^2\tan^2\theta)} d\theta. \end{aligned}$$

For $w = \tan\theta$, $dw = \sec^2\theta d\theta$ and

$$J = \frac{\sqrt{1+m^2}}{m}\int \frac{1}{(w^2+1)(m^2+1+m^2w^2)}dw.$$

Using the decomposition into partial fractions, we have

$$\begin{aligned}
& \frac{1}{(w^2+1)(m^2+1+m^2w^2)} \\
= \; & \frac{Aw+B}{w^2+1} + \frac{Cw+D}{m^2w^2+m^2+1} = \frac{(Aw+B)(m^2w^2+m^2+1)+(Cw+D)(w^2+1)}{(w^2+1)(m^2+1+m^2w^2)} \\
= \; & \frac{Am^2w^3+Am^2w+Aw+Bm^2w^2+Bm^2+B+Cw^3+Cw+Dw^2+D}{(w^2+1)(m^2+1+m^2w^2)} \\
= \; & \frac{w^3(Am^2+C)+w^2(D+Bm^2)+w(Am^2+A+C)+(Bm^2+B+D)}{(w^2+1)(m^2+1+m^2w^2)}
\end{aligned}$$

Hence, $\begin{cases} Am^2+C=0 \\ D+Bm^2=0 \\ A(m^2+1)+C=0 \\ B(m^2+1)+D=1 \end{cases}$.

This means that $A=0,\ B=1,\ C=0$, and $D=-m^2$. So, we get:

$$\begin{aligned}
J &= \frac{\sqrt{1+m^2}}{m}\int\left(\frac{1}{w^2+1} - \frac{m^2}{m^2+1+m^2w^2}\right)dw \\
&= \frac{\sqrt{1+m^2}}{m}\left(\tan^{-1}w - \int \frac{m^2(\frac{1}{m^2+1})}{1+\frac{m^2}{1+m^2}w^2}dw\right) \\
&= \frac{\sqrt{1+m^2}}{m}\left(\tan^{-1}w - \frac{m^2}{m^2+1}\times\frac{1}{\frac{m}{\sqrt{1+m^2}}}\tan^{-1}\left(\frac{m}{\sqrt{1+m^2}}w\right)\right)
\end{aligned}$$

$$= \frac{\sqrt{1+m^2}}{m}\left(\tan^{-1} w - \frac{m}{\sqrt{1+m^2}}\tan^{-1}\left(\frac{m}{\sqrt{1+m^2}}w\right)\right)$$
$$= \frac{\sqrt{1+m^2}}{m}\left(\theta - \frac{m}{\sqrt{1+m^2}}\tan^{-1}\left(\frac{m}{\sqrt{1+m^2}}\tan\theta\right)\right)$$
$$= \frac{\sqrt{1+m^2}}{m}\left(\sin^{-1} u - \frac{m}{\sqrt{1+m^2}}\tan^{-1}\left(\frac{m}{\sqrt{1+m^2}}.\frac{u}{\sqrt{1-u^2}}\right)\right)$$
$$= \frac{\sqrt{1+m^2}}{m}\sin^{-1}\left(m\int_0^s kds\right) - \tan^{-1}\left(\frac{m^2}{\sqrt{1+m^2}}.\frac{\int_0^s kds}{\sqrt{1-m^2(\int_0^s kds)^2}}\right).$$

Now, we calculate $x(s)$. We have

$$x(s)$$
$$= \int\sqrt{1-\frac{m^2}{1+m^2}\Big(\int_0^s kds\Big)^2}$$
$$\times\cos\left(\frac{\sqrt{1+m^2}}{m}\sin^{-1}\left(m\int_0^s kds\right) - \tan^{-1}\left(\frac{m^2}{\sqrt{1+m^2}}.\frac{\int_0^s kds}{\sqrt{1-(m\int_0^s ds)^2}}\right)\right)$$
$$= \sqrt{1-\frac{m^2}{m^2+1}\Big(\int_0^s kds\Big)^2}$$
$$\times\left(\cos\left(\frac{\sqrt{1+m^2}}{m}\sin^{-1}\left(m\int_0^s kds\right)\right)\cos\left(\tan^{-1}\left(\frac{m^2}{\sqrt{1+m^2}}.\frac{\int_0^s kds}{\sqrt{1-(m\int_0^s kds)^2}}\right)\right)\right.$$
$$\left.+\sin\left(\frac{\sqrt{1+m^2}}{m}\sin^{-1}\left(m\int_0^s kds\right)\right)\sin\left(\tan^{-1}\left(\frac{m^2}{\sqrt{1+m^2}}.\frac{\int_0^s kds}{\sqrt{1-(m\int_0^s kds)^2}}\right)\right)\right)$$
$$= \sqrt{1-\frac{m^2}{m^2+1}\Big(\int_0^s kds\Big)^2}\cos\left(\frac{\sqrt{1+m^2}}{m}\sin^{-1}\left(m\int_0^s kds\right)\right).\frac{\sqrt{1+m^2}\sqrt{1-(m\int_0^s kds)^2}}{\sqrt{1+m^2-(m\int_0^s kds)^2}}$$
$$+\sqrt{1-\frac{m^2}{m^2+1}\Big(\int_0^s kds\Big)^2}\sin\left(\frac{\sqrt{1+m^2}}{m}\sin^{-1}\left(m\int_0^s kds\right)\right)$$
$$\times\frac{m^2\int_0^s kds}{\sqrt{1+m^2}}.\frac{1}{\sqrt{1-\frac{m^2}{m^2+1}(\int_0^s kds)^2}}$$

$$
\begin{aligned}
= & \int \sqrt{1-\left(m\int_0^s kds\right)^2}\cos\left(\frac{\sqrt{1+m^2}}{m}\sin^{-1}\left(m\int_0^s kds\right)\right) \\
& +\int \frac{m^2}{\sqrt{1+m^2}}\int_0^s kds\sin\left(\frac{\sqrt{1+m^2}}{m}\sin^{-1}\left(m\int_0^s kds\right)\right)
\end{aligned}
$$

The derivative of :

$$
\begin{aligned}
& \sqrt{1-m^2\Big(\int_0^s kds\Big)^2}\cos\left[\frac{\sqrt{1+m^2}}{m}\sin^{-1}\left(m\int_0^s kds\right)\right] \\
& +\frac{m^2}{\sqrt{1+m^2}}\Big(\int_0^s kds\Big)\sin\left[\frac{\sqrt{1+m^2}}{m}\sin^{-1}\left(m\int_0^s kds\right)\right]
\end{aligned}
$$

is given by

$$
\begin{aligned}
- & \frac{2m^2k(\int_0^s kds)}{2\sqrt{1-(m\int_0^s kds)^2}}\cos\left[\frac{\sqrt{1+m^2}}{m}\sin^{-1}\left(m\int_0^s kds\right)\right] \\
- & \sqrt{1-\left(m\int_0^s kds\right)^2}\frac{\sqrt{1+m^2}}{m}\frac{mk}{\sqrt{1-(m\int_0^s kds)^2}}\sin\left[\frac{\sqrt{1+m^2}}{m}\sin^{-1}\left(m\int_0^s kds\right)\right] \\
+ & \frac{m^2}{\sqrt{1+m^2}}k\sin\left[\frac{\sqrt{1+m^2}}{m}\sin^{-1}\left(m\int_0^s kds\right)\right] \\
+ & \frac{m^2}{\sqrt{1+m^2}}\Big(\int_0^s kds\Big)\frac{\sqrt{1+m^2}}{m}\frac{mk}{\sqrt{1-(m\int_0^s kds)^2}}\cos\left[\frac{\sqrt{1+m^2}}{m}\sin^{-1}\left(m\int_0^s kds\right)\right] \\
= & \frac{-1}{\sqrt{1+m^2}}k(s)\sin\left[\frac{\sqrt{1+m^2}\sin^{-1}(m\int_0^s kds)}{m}\right]
\end{aligned}
$$

Hence,

$$
x(s) \quad = \quad \frac{-1}{\sqrt{1+m^2}}\int\int\left(\sin\left[\frac{\sqrt{1+m^2}\sin^{-1}(m\int_0^s kds)}{m}\right]k(s)ds\right)ds
$$

Similarly, we have

$$\begin{cases} y(s) = \dfrac{-1}{\sqrt{1+m^2}} \displaystyle\int\int \left( \cos\left[ \dfrac{\sqrt{1+m^2}\sin^{-1}(m\int_0^s kds)}{m} \right] k(s)ds \right) ds, \\ z(s) = \displaystyle\int \xi ds = \int \dfrac{m}{\sqrt{1+m^2}} \int_0^s kds = \dfrac{m}{\sqrt{1+m^2}} \int\int kds. \end{cases}$$

**3 $\Longrightarrow$ 1:** For the cases $m < 0$ or $m > 0$, the components of $T(s)$ are given by:

$$\begin{cases} x'(s) = \dfrac{1}{\sqrt{1+m^2}} \displaystyle\int \sin\left[ \dfrac{\sqrt{1+m^2}\cos^{-1}(m\int_0^s kds)}{m} \right] k(s)ds, \\ y'(s) = \dfrac{1}{\sqrt{1+m^2}} \displaystyle\int \cos\left[ \dfrac{\sqrt{1+m^2}\cos^{-1}(m\int_0^s kds)}{m} \right] k(s)ds, \\ z'(s) = \dfrac{|m|}{\sqrt{1+m^2}} \displaystyle\int_0^s kds. \end{cases}$$

Therefore, the components of $T'(s)$ are given by:

$$\begin{cases} x''(s) = \dfrac{1}{\sqrt{1+m^2}} \sin\left[ \dfrac{\sqrt{1+m^2}\cos^{-1}(m\int_0^s kds)}{m} \right] k(s), \\ y''(s) = \dfrac{1}{\sqrt{1+m^2}} \cos\left[ \dfrac{\sqrt{1+m^2}\cos^{-1}(m\int_0^s kds)}{m} \right] k(s), \\ z''(s) = \dfrac{|m|}{\sqrt{1+m^2}}\, k. \end{cases}$$

Now, we have: $N(s) = \dfrac{1}{k} T'(s)$, so

$$N(s) = \begin{pmatrix} \dfrac{1}{\sqrt{1+m^2}} \sin\left[ \dfrac{\sqrt{1+m^2}\cos^{-1}(m\int_0^s kds)}{m} \right] \\ \dfrac{1}{\sqrt{1+m^2}} \cos\left[ \dfrac{\sqrt{1+m^2}\cos^{-1}(m\int_0^s kds)}{m} \right] \\ \dfrac{|m|}{\sqrt{1+m^2}} \end{pmatrix}.$$

Thus, $\langle N(s), (0,0,1)\rangle = \dfrac{|m|}{\sqrt{1+m^2}}$, which is constant, so the principal normal lines of $\alpha$ form a constant angle with a fixed direction $u = (0,0,1)$, and hence $\alpha$ is a slant helix.

$\square$

# Chapter 2

# Principal, Gaussian, and Mean Curvatures of Parametrized Surfaces

In this chapter, we first give a quick review on quadratic forms and linear operators. Then, we define a parametrized surface, its tangent planes and normal vectors. Finally, we introduce the first and second fundamental forms of a surface in order to define the principal, Gaussian, and mean curvatures. See [1, 2, 4].

## 2.1 Review on Linear Algebra

In this section, we recall basic facts on linear operators, bilinear and quadratic forms.

### 2.1.1 Bilinear and Quadratic Forms

A bilinear form on a vector space $V$ is a map $b : V \times V \longrightarrow \mathbb{R}$ such that:

$$\begin{cases} b(c_1 v_1 + c_2 v_2, v) = c_1 b(v_1, v) + c_2 b(v_2, v), \\ b(v, c_1 v_1 + c_2 v_2) = c_1 b(v, v_1) + c_2 b(v, v_2), \end{cases}$$

for all $v_1$, $v_2$, $v \in V$ and $c_1$, $c_2 \in \mathbb{R}$. The bilinear form $b$ is called symmetric if $b(u, v) = b(v, u)$ for all $u, v \in V$.

**Definition 2.1.1.** *A real-valued function $Q$ on a vector space $V$ is called a quadratic form if it can be written in the form $Q(v) = b(v, v)$ for some symmetric bilinear form $b$ on $V$. It is then called the quadratic form associated to $b$.*

**Remark** Let $b$ be a bilinear symmetric form on $V$ and $Q$ the quadratic form associated to $b$. We have

$$\begin{aligned} b(v + w, w + v) &= b(v, w) + b(v, v) + b(w, w) + b(w, v) \\ &= b(v, v) + 2b(v, w) + b(w, w). \end{aligned}$$

Hence, $Q(v + w) = Q(v) + 2b(v, w) + Q(w)$ and thus

$$b(v, w) = b(v, w) = \frac{1}{2}\Big[Q(v + w) - Q(v) - Q(w)\Big].$$

It means that we can recover $b$ from the below formula.

**Example 2.1.2.** *Let $A$ be $n \times n$ matrix, $\mathbb{R}^n$ be the space of $n \times 1$ real matrices* *and* $b(X,Y) = X^TAY$ *for all* $X, Y \in \mathbb{R}^n$. *Then, b is a bilinear form since*

$$\begin{aligned} b(c_1X_1 + c_2X_2, Y) &= (c_1X_1 + c_2X_2)^TAY = (c_1X_1^T + c_2X_2^T)AY \\ &= c_1X_1^TAY + c_2X_2^TAY = c_1b(X_1, Y) + c_2b(X_2, Y) \\ b(Y, c_1X_1 + c_2X_2) &= Y^TA(c_1X_1 + c_2X_2) = Y^TAc_1X_1 + Y^TAc_2X_2 \\ &= c_1Y^TAX_1 + c_2Y^TAX_2 = c_1b(Y, X_1) + c_2b(Y, X_2) \end{aligned}$$

If $X = (x_1, x_2, ....., x_n)^T$, $Y = (y_1, y_2, .....y_n)^T$ and $A = (a_{ij})$, then:

$$\begin{aligned} b(X,Y) &= (x_1, x_2, ...., x_n) \begin{pmatrix} a_{11} & a_{12} & .. & .. & a_{1n} \\ a_{21} & .. & .. & .. & .. \\ .. & .. & .. & .. & .. \\ a_{1n} & .. & .. & .. & a_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ .. \\ y_n \end{pmatrix} \\ &= \Big( \sum_{i=1}^{n} x_1 a_{i1}, ..., \sum_{i=1}^{n} x_n a_{in} \Big) \begin{pmatrix} y_1 \\ y_2 \\ .. \\ .. \\ y_n \end{pmatrix} \\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} x_i a_{ij} y_j = \sum_{i,j=1}^{n} a_{ij} x_i y_j. \end{aligned}$$

Let $\{v_1, v_2, ..., ..., v_n\}$ be a basis for $V$ and $b$ be a bilinear form on $V$ with $b_{ij} = b(v_i, v_j)$ for $1 \leq i, j \leq n$. The matrix $B = (b_{ij})$ is called the coefficient matrix of $b$ with respect to $\{v_1, v_2, ..., ..., v_n\}$. If $w_1 = (x_1, x_2, ..., ..., x_n)$ and $w_2 = (y_1, y_2, ..., ..., y_n)$, then $b(w_1, w_2) = \sum_{i,j=1}^{n} b_{ij} x_i y_j$.

If $b$ is symmetric, then $b_{ij} = b_{ji}$ and $b(w_1, w_1) = Q(w_1) = \sum_{i,j=1}^{n} b_{ij} x_i x_j$.

### 2.1.2 Linear Operators

Let $V$ be a vector space with basis $\{v_1, v_2, ..., .., v_n\}$ and $T$ be a linear map $T : V \longrightarrow V$. Then for all $1 \leq j \leq n$, $T(v_j)$ can be written as a linear combination of $v_1, v_2, \cdots, v_n$. In other words, $T(v_j) = \sum_{i=1}^{n} a_{ij} v_i$. We denote by $A = (a_{ij})$ the matrix associated to the linear map $T$. If $v = \sum_{i=1}^{n} x_i v_i$ and $T(v) = \sum_{i=1}^{n} y_i v_i$, then we have

$$Y = AX,$$

where $Y = (y_1, y_2, ..., ..., y_n)^T$ and $X = (x_1, x_2, ..., ..., x_n)^T$. In fact, we have

$$\begin{aligned} T(v) &= T\Big(\sum_{i=1}^{n} x_i v_i\Big) = \sum_{i=1}^{n} x_i T(v_i) \\ &= \sum_{i,j=1}^{n} x_i a_{ji} v_j = \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} x_i a_{ji}\Big) v_j. \end{aligned}$$

Since $T(v) = \sum_{j=1}^{n} y_j v_j$, we get, $y_j = \sum_{i=1}^{n} x_i a_{ji}$ and $Y = AX$.

**Proposition 2.1.3.** *Let $T : V \longrightarrow V$ be a linear map and $A, B$ be the matrices of $T$ associated to bases $\{v_1, .., .., .., v_n\}$ and $\{u_1, .., .., u_n\}$ respectively. If $u_i = \sum_{j=1}^{n} c_{ji} v_j$ for $1 \leq i \leq n$, then, $B = C^{-1}AC$ where $C = (c_{ij})$ is a matrix of size $n \times n$.*

*Proof.* On one hand, we have:

$$T(u_i) = \sum_{k=1}^{n} b_{k_i} u_k = \sum_{k=1}^{n} b_{k_i} \sum_{m=1}^{n} c_{mk} v_m = \sum_{m=1}^{n} \left( \sum_{k=1}^{n} b_{k_i} c_{mk} \right) v_m$$

On the other hand, we have

$$T(u_i) = T\left( \sum_{j=1}^{n} c_{ji} v_j \right) = \sum_{j=1}^{n} c_{ji} T(v_j) = \sum_{j=1}^{n} c_{ji} \sum_{m=1}^{n} a_{mj} v_m = \sum_{m=1}^{n} \left( \sum_{j=1}^{n} c_{ji} a_{mj} \right) v_m.$$

But $T(u_i)$ can be uniquely written as a linear combination of $v_1, v_2, ...., v_n$. Then:

$$\sum_{k=1}^{n} b_{ki} c_{mk} = \sum_{j=1}^{n} c_{ji} a_{mj}$$

So, we have that the $(mi)^{th}$ entry of $CB$ equals to the $(mi)^{th}$ entry of $AC$. Hence, $CB = AC$ and $B = C^{-1}AC$. □

Suppose $A = (a_{ij})$ and $B = (b_{ij})$ are the matrices associated to the linear maps $S : V \longrightarrow V$ and $T : V \longrightarrow V$ with respect to the basis $\{v_1, v_2, ..., v_n\}$. In other words:

$$S(v_j) = \sum_{i=1}^{n} a_{ij} v_i \quad \text{and} \quad T(v_j) = \sum_{i=1}^{n} b_{ij} v_i \quad \text{for} \quad 1 \leq j \leq n.$$

We denote by $S \circ T : V \longrightarrow V$ the composition of $S$ and $T$.

**Proposition 2.1.4.** *IF $A = (a_{ij})$ aand $B = (b_{ij})$ are the matrices associated to the linear maps $S : V \longrightarrow V$ and $T : V \longrightarrow V$ with respect to the basis $\{v_1, v_2, ..., .., v_n\}$. Then, $AB$ is the matrix associated to $S \circ T$ with respect to* *the basis* $\{v_1, v_2, ..., v_n\}$.

*Proof.* Suppose $C = (c_{ij})$ is the matrix associated to $S \circ T$. We have

$$S \circ T(v_j) = S(T(v_j)) = \sum_{i=1}^{n} c_{ij} v_i.$$

Also, we have

$$\begin{aligned} S(T(v_j)) &= S\left(\sum_{i=1}^{n} b_{ij} v_i\right) = \sum_{i=1}^{n} b_{ij} S(v_i) \\ &= \sum_{i=1}^{n} b_{ij} \sum_{k=1}^{n} a_{ki} v_k = \sum_{k=1}^{n} (\sum_{i=1}^{n} b_{ij} a_{ki}) v_k. \end{aligned}$$

Hence, $c_{ij} = \sum_{i=1}^{n} a_{ki} b_{ij}$ and $C = AB$. □

**Definition 2.1.5.** *Let* $\langle ., . \rangle$ *be an inner product on* $V$. *A linear operator*

$$T : V \longrightarrow V$$

*is self adjoint if* $\langle Tv, w \rangle = \langle v, Tw \rangle$ *for all* $v, w \in V$.

**Proposition 2.1.6.** *Let* $T : V \longrightarrow V$ *be a linear map and* $A = (a_{ij})$ *the matrix associated to* $T$ *with respect to an orthonormal basis* $\{v_1, v_2, ..., v_n\}$. *Then,* $a_{ij} = \langle T(v_j), v_i \rangle$ *for all* $1 \leq i, j \leq n$. *If* $T$ *is self adjoint, then* $A$ *is symmetric.*

*Proof.* We have $T(v_j) = \sum_{i=1}^{n} a_{ij} v_i$. Then,

$$\begin{aligned}
\langle T(v_j), v_i \rangle &= \langle \sum_{i=1}^{n} a_{ij} v_i, v_i \rangle = \langle a_{1j} v_1 + ... + a_{ij} v_i + .... + a_{nj} v_n, v_i \rangle \\
&= a_{1j} \langle v_1, v_i \rangle + ..... + a_{ij} \langle v_i, v_i \rangle + .... + a_{nj} \langle v_n, v_i \rangle \\
&= 0 + 0 + ... + a_{ij}.1 + ... + 0 = a_{ij}.
\end{aligned}$$

Now, if $T$ is self adjoint, then

$$a_{ij} = \langle T(v_j), v_i \rangle = \langle v_j, T(v_i) \rangle = a_{ji}.$$

Therefore, $A$ is symmetric. $\square$

**Proposition 2.1.7.** *Let $(V, \langle ., . \rangle)$ be an inner product space with $\{v_1, v_2, ..., v_n\}$ as a basis for $V$. Let $T : V \longrightarrow V$ be a self adjoint linear operator. We denote by $A = (a_{ij})$ the matrix associated to $T$ with respect to the basis $\{v_1, v_2, ..., v_n\}$. Let $b_{ij} = \langle T(v_i), v_j \rangle$, $g_{ij} = \langle v_i, v_j \rangle$, $B = (b_{ij})$, $G = (g_{ij})$, and $G^{-1} = (g^{ij})$ be the inverse of $G$. Then:*

1. $B = A^T G$
2. $A = G^{-1} B^T = G^{-1} B$
3. $\det A = \dfrac{\det B}{\det G}$ *and* $\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i,j=1}^{n} b_{ij} g^{ij}$

*Proof.*

1. We have

$$\begin{aligned} b_{ij} &= \langle T(v_i), v_j\rangle = \langle \sum_{k=1}^{n} a_{ki} v_k, v_j\rangle \\ &= a_{1i}\Big\langle v_1, v_j\rangle + a_{2i}\langle v_2, v_j\Big\rangle + .... + a_{ji}\langle v_j, v_j\rangle + ... + a_{ni}\langle v_n, v_j\rangle \\ &= a_{1i}g_{1j} + a_{2i}g_{2j} + .... + a_{ni}g_{nj} \\ &= \sum_{k=1}^{n} a_{ki}g_{kj}. \end{aligned}$$

So, $B = A^T G$.

2. From Proposition 2.1.6, $B$ is symmetric, and $G$ is symmetric since $g_{ij} = \langle v_i, v_j\rangle = \langle v_j, v_i\rangle = g_{ji}$. Thus, $B^T = B$, $G^T = G$, $(G^{-1})^T = G^{-1}$, and $(B^{-1})^T = B^{-1}$. Now, we have $B = A^T G$ and so

$$A^T = BG^{-1}.$$

Taking the transpose of both sides, we get $(A^T)^T = (BG^{-1})^T$ and so

$$A = (G^{-1})^T B^T = G^{-1}B.$$

3. We have

$$\det A = \det(G^{-1}B) = \det(G^{-1}). \det B = \frac{1}{\det G}. \det B.$$

Hence, $\det A = \dfrac{\det B}{\det G}$.

□

**Proposition 2.1.8.** *Let $\langle ., .\rangle$ be an inner product on $V$ and $S : V \longrightarrow V$ a* *self adjoint linear operator. Define $b^S : V \times V \longrightarrow \mathbb{R}$ by $b^S(v, w) = \langle S(v), w \rangle$. Then:*

1. *$b^S$ is a symmetric bilinear form.*

2. *The coefficient matrix of $b^S$ is $(s_{ij})$ given by $s_{ij} = \langle S(v_i), v_j \rangle$.*

*Proof.* 1. We have:

$$\begin{aligned} b^S(c_1v_1 + c_2v_2, w) &= \langle S(c_1v_1c_2v_2), w \rangle = \langle c_1S(v_1) + c_2S(v_2), w \rangle \\ &= c_1\langle S(v_1), w \rangle + c_2\langle S(v_2), w \rangle = c_1b^S(v_1, w) + c_2b^S(v_2, w) \\ b^S(w, c_1v_1 + c_2v_2) &= \langle S(w), c_1v_1 + c_2v_2 \rangle = \langle S(w), c_1v_1 \rangle + \langle S(w), c_2v_2 \rangle \\ &= c_1\langle S(w), v_1 \rangle + c_2\langle S(w), v_2 \rangle = c_1b^S(w, v_1) + c_2b^S(w, v_2). \end{aligned}$$

So, $b^S$ is a bilinear form. Moreover,

$$b^S(v, w) = \langle S(v), w \rangle = \langle v, S(w) \rangle = \langle S(w), v \rangle = b^S(w, v)$$

So, $b^S$ is symmetric.

2. Trivial

□

**Proposition 2.1.9.** *Let $\langle ., . \rangle$ be an inner product on $V$ and $b$ be a symmetric bilinear form on $V$. Then, there exists a self adjoint operator $S : V \longrightarrow V$ such that $b = b^S$.*

*Proof.* Let $\{v_1, v_2, ..., v_n\}$ br an orthonormal basis for $V$, $b_{ij} = b(v_i, v_j)$ and $S(v_i) = \sum_{j=1}^{n} b_{ij}v_j$. Now, $b(v_i, v_j) = b_{ij}$, and since $b$ symmetric and the basis is orthonormal, we have

$$b^S(v_i, v_j) = \langle S(v_i), v_j \rangle = \langle \sum_{j=1}^{n} b_{ji} v_j, v_j \rangle = \langle \sum_{j=1}^{n} b_{ij} v_j, v_j \rangle = b_{ij}.$$

Thus, $b = b^S$. □

### 2.1.3 Eigenvalues and Eigenvectors

Let $V$ be a vector space and $S : V \longrightarrow V$ be a linear map. A non-zero vector $u \in V$ is an eigenvector of $S$ with eigenvalue $\lambda$ if $S(u) = \lambda u$. Note that if $A$ is a real $n \times n$ matrix, then a number $\lambda_0 \in \mathbb{R}$ is called an eigenvalue of $A$ if

$$\exists u \in \mathbb{R}^n / Au = \lambda_0 u.$$

In this case, $u$ is the eigenvector of $A$. If $S : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear map defined by $S(u) = Au$, then eigenvalues and eigenvectors of $A$ are the same as that of the linear operator $S$.

**Proposition 2.1.10.** *Let $A$ be the matrix associated to the linear operator $S : V \longrightarrow V$ and denote by $\{v_1, v_2, ..., v_n\}$ a basis of $V$.*

1. *$\lambda_0 \in \mathbb{R}$ is an eigenvalue of $S$* $\iff \det(A - \lambda_0 \mathrm{Id}) = 0$.

2. *If $Au = \lambda_0 u$, then $v = u_1 v_1 + u_2 v_2 + ... + u_n v_n$ is an eigenvector of $S$ where $u = \begin{pmatrix} u_1 & u_2 & \cdots & \cdots & u_n \end{pmatrix}^T$.*

*Proof.* 1. $\lambda_0$ is an eigenvalue of $S$

$\iff \exists v \neq 0 \in V$ such that $S(v) = \lambda_0 v$

$\iff Av = \lambda_0 v \iff Av - \lambda_0 v = 0$

$\iff (A - \lambda_0 \mathrm{Id})v = 0$

$\iff \det(A - \lambda_0 \mathrm{Id}) = 0$ (since $v \neq 0$).

2. we have $Au = \lambda_0 u$. Now,

$$\begin{aligned} S(v_i) &= a_{1i}v_1 + a_{2i}v_2 + ... + a_{ni}v_n = \sum_{j=1}^{n} a_{ji}v_j \\ S(v) &= S(u_1v_1 + u_2v_2 + ... + u_nv_n) = S(u_1v_1) + S(u_2v_2) + ... + S(u_nv_n) \\ &= u_1S(v_1) + u_2S(v_2) + ... + u_nS(v_n) = u_1 \sum_{j=1}^{n} a_{j1}v_j + ..... + u_n \sum_{j=1}^{n} a_{jn}v_j \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ji}u_iv_j. \end{aligned}$$

But $\sum_{i=1}^{n} a_{ji}u_i$ is the $j1^{th}$ entry of $Au$ which is $\lambda_0 u$. So,

$$\begin{pmatrix} a_{11} & a_{12} & .. & .. & a_{1n} \\ .. & .. & .. & .. & .. \\ a_{j1} & a_{j2} & .. & .. & a_{jn} \\ .. & .. & .. & .. & .. \\ a_{n1} & a_{n2} & .. & .. & a_{nn} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ .. \\ .. \\ u_n \end{pmatrix} = Au = \lambda_0 u = \lambda_0 \begin{pmatrix} u_1 \\ .. \\ u_j \\ .. \\ u_n \end{pmatrix}$$

Thus $\sum_{i=1}^{n} a_{ji}u_i = \lambda_0 u_j$ and

$$S(v) = S(\sum_{j=1}^{n} u_jv_j) = \lambda_0 \sum_{j=1}^{n} u_jv_j.$$

Hence $v = \sum_{i=1}^{n} u_iv_i$ is an eigenvector of $S$ with eigenvalue $\lambda_0$.

$\square$

**Proposition 2.1.11.** *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be a* $2 \times 2$ *real symmetric matrix. Then,*

1. *$A$ has 2 real eigenvalues $\lambda_1$ and $\lambda_2$.*

2. *$\lambda_1 \lambda_2 = \det(A)$ and $\lambda_1 + \lambda_2 = \mathrm{tr}(A)$.*

3. *There exists an orthonormal basis $\{v_1, v_2\}$ of $\mathbb{R}^2$ such that $v_i$ is an eigenvector with eigenvalue $\lambda_i$ where $i = 1, 2$.*

*We call $\{v_1, v_2\}$ an orthonormal eigenbase of $A$.*

*Proof.* 1. We have $\det(A - \lambda \mathrm{Id}) = 0$. Thus,

$$\begin{vmatrix} q - \lambda & b \\ b & c - \lambda \end{vmatrix} = 0.$$

Hence $(a - \lambda)(c - \lambda) - b^2 = 0$, which implies that

$$ac - b^2 - (a + c)\lambda + \lambda^2 = 0. \tag{2.1.1}$$

We now calculate the discriminant of the last equation and get

$$\Delta = (a+c)^2 - 4(ac - b^2) = a^2 + 2ac + c^2 - 4ac + 4b^2 = (a-c)^2 + 4b^2 \geq 0.$$

So, $\lambda_1 = \dfrac{(a + c) - \sqrt{(a - c)^2 + 4b^2}}{2}$ and $\lambda_2 = \dfrac{(a + c) + \sqrt{(a - c)^2 + 4b^2}}{2}$.

2. From Proposition 2.1.1, we get:

$$\begin{cases} \lambda_1 \lambda_2 = \frac{ac-b^2}{1} = \det(A) \\ \lambda_1 + \lambda_2 = \frac{a+c}{1} = \operatorname{tr}(A) \end{cases}$$

3. If $\Delta = 0$, then $(a-c)^2 + 4b^2 = 0$ and so $a = c$ and $b = 0$. Hence,

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = aI.$$

The vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the orthonormal basis for $A$ where $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the eigenvectors.

If $\Delta > 0$, then $Av_1 = \lambda_1 v_1$ and we have

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Longrightarrow \begin{cases} ax_1 + bx_2 = \lambda_1 x_1 \\ bx_1 + cx_2 = \lambda_1 x_2 \end{cases}$$

So, $x_2 = \dfrac{(\lambda_1 - a)}{b} x_1$ and $x_2 = \dfrac{b}{\lambda_1 - c} x_1$. But, $\dfrac{\lambda_1 - a}{b} = \dfrac{b}{\lambda_1 - c}$ since $(\lambda_1 - a)(\lambda_1 - c) - b^2 = 0$. Therefore, for $x_1 = -b$,

$$x_2 = a - \lambda_1 = a - \frac{a + c - \sqrt{(a-c)^2 + 4b^2}}{2} = \frac{a - c + \sqrt{(a-c)^2 + 4b^2}}{2}$$

and

$$v_1 = \begin{pmatrix} -b \\ \frac{a-c+\sqrt{(a-c)^2+4b^2}}{2} \end{pmatrix}.$$

Similarly, $v_2 = \begin{pmatrix} -b \\ \frac{a-c-\sqrt{(a-c)^2+4b^2}}{2} \end{pmatrix}$. Thus, $v_1$ and $v_2$ are eigenvectors of $A$ with eigenvalues $\lambda_1$ and $\lambda_2$ respectively, and

$$\langle v_1, v_2 \rangle = b^2 + \frac{(a-c)^2 - ((a-c)^2 + 4b^2)}{4} = 0$$

Consider $u_1 = \dfrac{v_1}{\|v_1\|}$ and $u_2 = \dfrac{v_2}{\|v_2\|}$. We have that $\{u_1, u_2\}$ is the eigenbase for $A$.

□

**Theorem 2.1.12.** *[7]* **(Spectral Theorem).** *Let $\langle . \, , \, . \rangle$ be an inner product on $V$ with $dim(V) = n$ and $S : V \longrightarrow V$ be a linear self adjoint operator. Then,*

1. *$S$ has $n$ real eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$.*
2. *There exists an orthonormal basis $\{v_1, v_2, ..., v_n\}$ of $V$ such that $S(v_i) = \lambda_i v_i$ for $i = 1, 2, ..., n$.*

**Proposition 2.1.13.** *Let $\langle \; , \; \rangle$ be an inner product on $V$ with $dim(V) = n$ and $S : V \longrightarrow V$ a linear self adjoint operator. Denote by $\{v_1, v_2, ..., v_n\}$ an orthonormal eigenbase of $S$ and $S(v_i) = \lambda_i v_i$ for $i = 1, 2, ..., n$. Let $b^S : V \times V \longrightarrow \mathbb{R}$ be the symmetric bilinear form associated to $S$. In other words,*

$b^S(u,v) = \langle S(u), v\rangle$. *Then,*

$$\begin{cases} \min\limits_{\|v\|=1} b^S(v,v) = \lambda_1 = b^S(v_1), \\ \max\limits_{\|v\|=1} b^S(v,v) = \lambda_n = b^S(v_n). \end{cases}$$

*Proof.* Let $\{v_1, v_2, ..., v_n\}$ be an orthonormal eigenbase of $S$ with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ respectively. Then, for $v \in V$, $v = \sum\limits_{i=1}^{n} x_i v_i$ and

$$\begin{aligned} b^S(v,v) &= b^S\Big(\sum_{i=1}^{n} x_i v_i, \sum_{i=1}^{n} x_i v_i\Big) = \Big\langle S\Big(\sum_{i=1}^{n} x_i v_i\Big), \sum_{i=1}^{n} x_i v_i\Big\rangle \\ &= \Big\langle \sum_{i=1}^{n} x_i S(v_i), \sum_{i=1}^{n} x_i v_i\Big\rangle = \langle \sum_{i=1}^{n} x_i \lambda_i v_i, \sum_{i=1}^{n} x_i v_i\rangle \\ &= \begin{pmatrix} x_1\lambda_1 \\ .. \\ .. \\ x_n\lambda_n \end{pmatrix} \begin{pmatrix} x_1 \\ .. \\ .. \\ x_n \end{pmatrix} = \lambda_1 x_1^2 + ... + \lambda_n x_n^2 = \sum_{i=1}^{n} \lambda_i x_i^2 \end{aligned}$$

Now, $\min\limits_{\|v\|=1} b^S(v,v) = \min\limits_{\|v\|=1} (\sum\limits_{i=1}^{n} \lambda_i x_i^2)$. But, since $\|v\| = 1$, we have $\sum_{i=1}^{n} x_i^2 = 1$ and so

$$\min_{\|v\|=1} b^S(v,v) = \min_{\|v\|=1} (\lambda_1 x_1^2 + ... + \lambda_n x_n^2) = \lambda_1$$

and $\max\limits_{\|v\|=1} = \lambda_n$. Also,

$$\begin{cases} b^S(v_1) = b^S(v_1, v_1) = \langle S(v_1), v_1\rangle = \langle \lambda_1 v_1, v_1\rangle = \lambda_1\langle v_1, v_1\rangle = \lambda_1\|v_1\|^2 = \lambda_1, \\ b^S(v_n) = b^S(v_n, v_n) = \langle S(v_n), v_n\rangle = \langle \lambda_n v_n, v_n\rangle = \lambda_n. \end{cases}$$

□

## 2.2 Parametrized Surfaces in $\mathbb{R}^3$

In this section, we define parametrized surfaces in $\mathbb{R}^3$ and give various examples. We then define the tangent plane and the unit normal vector at a given point of the parametrized surface.

**Definition 2.2.1.** *Let $O$ be an open subset of $\mathbb{R}^2$. A smooth map $f$*

$$\begin{array}{rccl} f: & O & \longrightarrow & \mathbb{R}^3 \\ & (u,v) & \longrightarrow & f(u,v) = \Big(f_1(u,v), f_2(u,v), f_3(u,v)\Big) \end{array}$$

*is called a parametrized surface in $\mathbb{R}^3$ if $\dfrac{\partial f}{\partial u}(u,v)$ and $\dfrac{\partial f}{\partial v}(u,v)$ are linearly independent for all $(u,v) \in O$.*

**Example 2.2.2. (The graph of a smooth function)**. *Consider a function $h : O \longrightarrow \mathbb{R}$ and the map $f$ given by*

$$\begin{array}{rccl} f: & O & \longrightarrow & \mathbb{R}^3 \\ & (u,v) & \longrightarrow & f(u,v) = \Big(u, v, h(u,v)\Big). \end{array}$$

*We have*

$$\frac{\partial f}{\partial u}(u,v) = \Big(1, 0, \frac{\partial f}{\partial u}(u,v)\Big) \quad \textit{and} \quad \frac{\partial f}{\partial v}(u,v) = \Big(1, 0, \frac{\partial f}{\partial v}(u,v)\Big).$$

*Thus,*

$$\frac{\partial f}{\partial u}(u,v) \times \frac{\partial f}{\partial v}(u,v) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial h}{\partial u}(u,v) \\ 0 & 1 & \frac{\partial h}{\partial v}(u,v) \end{vmatrix} = \Big( - \frac{\partial h}{\partial u}(u,v), \frac{\partial h}{\partial v}(u,v), 1 \Big) \neq (0,0,0).$$

*This means that* $\frac{\partial h}{\partial u}(u,v)$ *and* $\frac{\partial h}{\partial v}(u,v)$ *are linearly independent, and* $f$ *is a parametrized surface in* $\mathbb{R}^3$.

**Example 2.2.3. (Surface of Revolution)**. *Let* $h : \mathbb{R} \longrightarrow \mathbb{R}$ *be a smooth function and* $(C)$ *a curve in the plane given by* $z = h(y)$ *and* $O$ *be an open subset of* $\mathbb{R}^2$. *We denote by* $M$ *the surface obtained by rotating* $(C)$ *along the* $y$*-axis. It can be parametrized by*

$$\begin{array}{rccl} f: & O & \longrightarrow & \mathbb{R}^3 \\ & (y,\theta) & \longrightarrow & f(y,\theta) = \Big( h(y)\cos\theta, y, h(y)\sin\theta \Big). \end{array}$$

*We calculate*

$$\begin{cases} \dfrac{\partial f}{\partial y}(y,\theta) = \Big( h'(y)\cos\theta, 1, h'(y)\sin\theta \Big) \\ \dfrac{\partial f}{\partial \theta}(y,\theta) = \Big( - h(y)\sin\theta, 0, h(y)\cos\theta \Big) \end{cases}$$

*and*

$$\begin{aligned}\frac{\partial f}{\partial y}\times\frac{\partial f}{\partial \theta} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ h'(y)\cos\theta & 1 & h'(y)\sin\theta \\ -h(y)\sin\theta & 0 & h(y)\cos\theta \end{vmatrix} \\ &= \Big(h(y)\cos\theta, -h(y)h'(y)\cos^2\theta - h(y)h'(y)\sin^2\theta, h(y)\sin\theta\Big) \\ &= \Big(h(y)\cos\theta, -h(y)h'(y), h(y)\sin\theta\Big).\end{aligned}$$

*Clearly, it is* $0$ *for* $h(y) = 0$*, so* $f$ *is not a parametrized surface unless* $h(y) \neq 0$ *for all* $y \in \mathbb{R}$*.*

From now on, we denote $\dfrac{\partial f}{\partial u}$ by $f_u$, $\dfrac{\partial f}{\partial v}$ by $f_v$, $\dfrac{\partial}{\partial u}\Big(\dfrac{\partial f}{\partial v}\Big)$ by $f_{uv}$, etc...

**Definition 2.2.4.** *Let* $f : O \longrightarrow \mathbb{R}^3$ *be a parametrized surface and* $(u_0, v_0) \in O$*. Suppose that, for some* $\epsilon > 0$*, the map*

$$\begin{aligned} c: \quad (-\epsilon, +\epsilon) &\longrightarrow O \\ t &\longrightarrow c(t) = (u(t), v(t)), \end{aligned}$$

*is smooth and* $c(0) = (u(0), v(0)) = (u_0, v_0)$*. We call the vector* $(f \circ c)'(0)$ *a tangent vector of* $f$ *at* $(u_0, v_0)$*.*

By the Chain rule, we have

$$(f \circ c)'(0) = f_u(u_0, v_0)u'(0) + f_v(u_0, v_0)v'(0).$$

If $c(t) = (u_0 + t, v_0)$, then $(f \circ c)'(0) = f_u(u_0, v_0)$ and if $c(t) = (u_0, v_0 + t)$, then $(f \circ c)'(0) = f_v(u_0, v_0)$. Thus, the space of all tangent vectors of $f$ at $(u_0, v_0)$

is a two-dimensional linear subspace of $\mathbb{R}^3$ with $\left\{f_u(u_0, v_0), f_v(u_0, v_0)\right\}$ as a basis. The space $T_{f_p}$ of all tangent vectors of $f$ at $P = (u_0, v_0)$ is called the tangent plane of $f$ at P. The vector

$$N(u_0, v_0) = \frac{f_u(u_0, v_0) \times f_v(u_0, v_0)}{\|f_u(u_0, v_0) \times f_v(u_0, v_0)\|}$$

is a unit normal vector to $f$ at $(u_0, v_0)$ (perpendicular to the $T_{f_p}$). Hence, we now have a basis $\{f_u, f_v, N\}$ for the parametrized surface $f : O \longrightarrow \mathbb{R}^3$. The vectors $f_{uu}$, $f_{vu}$, $N_u$, $f_{vu}$, $f_{vv}$, and $N_v$ can all be written as linear combinations of $f_u$, $f_v$, and $N$. Moreover,

$$\begin{aligned}
(f_u, f_v, N)_u &= (f_{uu}, f_{vu}, N_u) \\
&= (p_{11}f_u + p_{21}f_v + p_{31}N, p_{12}f_u + p_{22}f_v + p_{32}N, p_{13}f_u + p_{23}f_v + p_{33}N) \\
&= (f_u, f_v, N)\begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \\
&= (f_u, f_v, N)P
\end{aligned}$$

and

$$\begin{aligned}
(f_u, f_v, N)_v &= (f_{uv}, f_{vv}, N_v) \\
&= (f_u, f_v, N)\begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix} \\
&= (f_u, f_v, N)Q,
\end{aligned}$$

where $P$ and $Q$ are $3 \times 3$ matrices $P, Q : O \longrightarrow M_{3\times 3}$. Since $f_{uv} = f_{vu}$, the first column of $Q$ and the second column of $P$ are the same.

## 2.3 The First Fundamental Form

We are now ready to define the first fundamental form of a parametrized surface.

**Definition 2.3.1.** *Suppose that $f : O \longrightarrow \mathbb{R}^3$ is a parametrized surface in $\mathbb{R}^3$. A quadratic form $Q$ on $f$ is a function $Q : p \longrightarrow Q_p$ that assigns to each $p$ in $O$ a quadratic form $Q_p$ on the tangent plane $T_{f_p}$ of $f$ at $p$. In other words, at each $p$ in $O$, $Q_p : T_{f_p} \times T_{f_p} \longrightarrow \mathbb{R}$ is a quadratic form.*

**Remark** The quadratic form $Q$ on $f$ is described by the symmetric $2 \times 2$ matrix of real valued functions: $Q_{uv} : O \longrightarrow \mathbb{R}$ defined by $Q_{uv}(p) = Q(f_u(p), f_v(p))$. $Q_{ij}$ are the coefficients of the quadratic form $Q$. Now, $Q_{11}$, $Q_{12} = Q_{21}$, and $Q_{22}$ determine the quadratic form $Q$ on $f$ uniquely. In fact, If $w \in T_{f_p}$, then $w = \alpha f_u(p) + \beta f_v(p)$, and

$$\begin{aligned} Q_p(w) &= Q_p(w, w) = Q_p(\alpha f_u(p) + \beta f_v(p), \alpha f_u(p) + \beta f_v(p)) \\ &= Q_p(\alpha f_u(p), \alpha f_u(p)) + Q_p(\alpha f_u(p), \beta f_v(p)) + Q_p(\beta f_v(p), \alpha f_u(p)) \\ &\quad + Q_p(\beta f_v(p), \beta f_v(p)) \\ &= \alpha^2 Q_{11}(p) + 2\alpha\beta Q_{12}(p) + \beta^2 Q_{22}. \end{aligned}$$

Because of that, it is convenient to have a simple way of referring to the quadratic form $Q$ on a surface having the three coefficients $A$, $B$, and $C$. The classical notation is as follows:

$$Q = A(u,v)du^2 + 2B(u,v)dudv + C(u,v)dv^2.$$

To better understand this, consider a curve defined as $\alpha(t) = f(u(t), v(t))$. We have

$$\alpha'(t) \quad = \quad f_u u'(t) + f_v v'(t) = f_u(u,v)u'(t) + f_v(u,v)v'(t)$$

and

$$\begin{aligned}
Q(\alpha'(t)) &= Q(\alpha'(t), \alpha'(t)) \\
&= Q(f_u(u,v)u'(t) + f_v(u,v)v'(t), f_u(u,v)u'(t) + f_v(u,v)v'(t)) \\
&= Q(f_u, f_u)(u'(t))^2 + 2Q(f_u, f_v)u'(t)v'(t) + Q(f_v, f_v)(v'(t))^2 \\
&= A(u,v)(u')^2 + B(u,v)u'v' + C(u,v)(v')^2.
\end{aligned}$$

This explains the classical notation of the first fundamental form.

**Definition 2.3.2. (The First Fundamental Form of a Surface $f$)** *Let $I_p : T_{f_p} \times T_{f_p} \longrightarrow \mathbb{R}$ denote the inner product $I_p(u,v) = \langle u, v \rangle$ where $u, v \in T_{f_p}$. $I$ is the first fundamental form on $f$.*

The coefficient matrix for $I$ with respect to the basis $\{f_u, f_v\}$ is $(g_{ij})$ where:

$$\begin{cases} g_{11} = I_p(f_u, f_u) = \langle f_u, f_u\rangle \\ g_{12} = I_p(f_u, f_v) = \langle f_u, f_v\rangle \\ g_{21} = I_p(f_v, f_u) = \langle f_v, f_u\rangle = \langle f_u, f_v\rangle \\ g_{22} = I_p(f_v, f_v) = \langle f_v, f_v\rangle. \end{cases}$$

So, from the previous notation:

$$I = g_{11}(u,v)du^2 + 2g_{12}(u,v)dudv + g_{22}(u,v)dv^2.$$

Now, if $u_1$ and $u_2$ are tangent vectors at $p_0 = (u_0, v_0)$, then $u_1$ and $u_2$ can be written as:

$$u_1 = a_1 f_u(p_0) + a_2 f_v(p_0),$$

$$u_2 = b_1 f_u(p_0) + b_2 f_v(p_0),$$

where $a_1, a_2, b_1, b_2 \in \mathbb{R}$). We then have

$$\begin{aligned} \langle u_1, u_2\rangle &= \langle a_1 f_u(p_0) + a_2 f_v(p_0), b_1 f_u(p_0) + b_2 f_v(p_0)\rangle \\ &= a_1 b_1 \langle f_u(p_0), f_u(p_0)\rangle + a_1 b_2 \langle f_u(p_0), f_v(p_0)\rangle \\ &\quad + a_2 b_1 \langle f_v(p_0), f_u(p_0)\rangle + a_2 b_2 \langle f_v(p_0), f_v(p_0)\rangle \\ &= g_{11}(p_0) a_1 b_1 + g_{12}(p_0)(a_1 b_2 + a_2 b_1) + g_{22}(p_0) a_2 b_2. \end{aligned}$$

So, we can compute the inner product between two vectors in $T_{f_p}$ from $(g_{ij})$ and deduce the length and the angle between them.

**Definition 2.3.3. (The Arclength of a Curve on a Surface)** *Let $f : O \longrightarrow \mathbb{R}^3$ be a parametrized surface and let $\alpha : t \longrightarrow (u(t), v(t))$ be a parametric curve in $\mathbb{R}^2$ where $t \in [a, b]$. The length of the curve $\alpha$ is given by:*

$$L = \int_a^b \|\alpha'(t)\|dt = \int_a^b \sqrt{g_{11}(u')^2 + 2g_{12}u'v' + g_{22}(v')^2}dt.$$

**Definition 2.3.4. (The Angle Between Two Curves)** *Let $f : O \longrightarrow \mathbb{R}^3$ be a parametrized surface. For $t \in [a, b]$, we define $c_1(t) = (x_1(t), x_2(t))$ and $c_2(t) = (y_1(t), y_2(t))$ two smooth curves in $\mathbb{R}^2$ such that $c_1(0) = c_2(0) = p_0 = (u_0, v_0)$. The angle $\theta$ between $\alpha_1 = f \circ c_1$ and $\alpha_2 = f \circ c_2$ is defined to be the angle between $\alpha_1'(0)$ and $\alpha_2'(0)$. We have*

$$\begin{aligned} \cos\theta &= \frac{\alpha'(0)\alpha_2'(0)}{\|\alpha_1'(0)\|.\|\alpha_2'(0)\|} = \frac{\langle f_{x_1}x_1'(0) + f_{x_2}x_2'(0)\rangle.\langle f_{y_1}y_1'(0) + f_{y_2}y_2'(0)\rangle}{\|\alpha_1'(0)\|.\|\alpha_2'(0)\|} \\ &= \frac{\sum_{i,j=1}^{2} g_{ij}(p_0)x_i'(0)y_j'(0)}{\sqrt{\sum_{i,j=1}^{2} g_{ij}(p_0)x_i'(0)x_j'(0)}\sqrt{\sum_{i,j=1}^{2} g_{ij}(p_0)y_i'(0)y_j'(0)}}. \end{aligned}$$

## 2.4 The Shape Operator and the Second Fundamental Form

Assume that $f : O \longrightarrow \mathbb{R}^3$ is a parametrized surface and $N : O \longrightarrow \mathbb{R}^3$ is the unit normal vector to $f$. Since $\langle N, N\rangle = 1$, we have $\frac{\partial}{\partial u}\langle N, N\rangle = 0$. Thus,

$$\langle N_u, N\rangle + \langle N, N_u\rangle = 0.$$

It means that $2\langle N, N_u\rangle = 0$ and hence $N$ is perpendicular to $N_u$. Similarly, $\langle N, N_v\rangle = 0$ and $N$ is perpendicular to $N_v$. Thus, $N_u$ and $N_v$ are tangent to $f$.

**Definition 2.4.1.** *The shape operator $S_p$ at the point $p \in O$ of the parametrized surface $f : O \longrightarrow \mathbb{R}^3$ is the linear map given by*

$$\begin{aligned} S_p : T_{f_p} &\longrightarrow T_{f_p} \\ f_u(p) &\longrightarrow S_p(f_u(p)) = -N_u(p) \\ f_v(p) &\longrightarrow S(f_v(p)) = -N_v(p) \end{aligned}$$

*For any vector $w \in T_{f_p}$, we have $w = c_1 f_u(p) + c_2 f_v(p)$ for some $c_1, c_2 \in \mathbb{R}$. Hence,*

$$S_p(w) = c_1 S_p(f_u(p)) + c_2 S_p(f_v(p)) = -c_1 N_u(p) - c_2 N_v(p).$$

**Proposition 2.4.2.** *The shape operator $S_p$ is self adjoint.*

*Proof.* Let $p \in O$ and $w$ and $z$ be two vectors in $T_{f_p}$. We have

$$\begin{cases} w = c_1 f_u(p) + c_2 f_v(p) \\ z = a_1 f_u(p) + a_2 f_v(p) \end{cases},$$

where $c_1, c_2, a_1, a_2$ are real constants. We have

$$\begin{aligned}
\langle S_p(w), z\rangle &= \langle -c_1 N_u - c_2 N_v, a_1 f_u + a_2 f_v\rangle \\
&= -c_1 a_1 \langle N_u, f_u\rangle - c_1 a_2 \langle N_u, f_v\rangle - c_2 a_1 \langle N_v, f_u\rangle - c_2 a_2 \langle N_v, f_v\rangle \\
\langle w, S_p(z)\rangle &= \langle c_1 f_u + c_2 f_v, -a_1 N_u - a_2 N_v\rangle \\
&= -c_1 a_1 \langle f_u, N_u\rangle - c_1 a_2 \langle f_u, N_v\rangle - c_2 a_1 \langle f_v, N_u\rangle - c_2 a_2 \langle f_v, N_v\rangle.
\end{aligned}$$

But $\langle N, f_v\rangle = \langle N, f_u\rangle = 0$ so we get

$$\langle N_u, f_v\rangle + \langle N, f_{vu}\rangle = 0,$$

$$\langle N_v, f_u\rangle + \langle N, f_{uv}\rangle = 0.$$

Because $f$ is smooth, we have $f_{uv} = f_{vu}$. Thus,

$$\langle N_u, f_v\rangle = \langle N_v, f_u\rangle.$$

Finally, we get $\langle S_p(w), z\rangle = \langle w, S_p(z)\rangle$. Therefore, $S_p$ is self adjoint. $\square$

At $p \in O$, let $II_p$ denote the symmetric bilinear form on $T_{f_p}$ associated to the self adjoint operator $S_p$. We have

$$\begin{aligned}
II_p : T_{f_p} \times T_{f_p} &\longrightarrow \mathbb{R} \\
(u, v) &\longrightarrow \langle S(u), v\rangle
\end{aligned}$$

The entries of the coefficient matrix $(l_{ij})$ of $II_p$ with respect to the basis

$\{f_u, f_v\}$ of $T_{f_p}$ are:

$$\begin{cases} l_{11} = II_p(f_u, f_v) = \langle S_p(f_u), f_u \rangle = \langle -N_u, f_u \rangle = \langle N, f_{uu} \rangle, \\ l_{12} = l_{21} = II_p(f_u, f_v) = \langle S_p(f_u), f_v \rangle = \langle -N_u, f_v \rangle = \langle N, f_{uv} \rangle, \\ l_{22} = II_p(f_v, f_v) = \langle S_p(f_v), f_v \rangle = \langle -N_v, f_v \rangle = \langle N, f_{vv} \rangle. \end{cases}$$

For any $w$, we have,

$$\begin{aligned} II(w) &= II(w, w) = II(a_1 f_u + a_2 f_v, b_1 f_u + b_2 f_v) = \langle S_p(a_1 f_u + a_2 f_v), b_1 f_u + b_2 f_v \rangle \\ &= b_1 a_1 \langle S_p(f_u), f_u \rangle + b_2 a_1 \langle S_p(f_u), f_v \rangle + b_1 a_2 \langle S_p(f_v), f_u \rangle + a_2 2 b_2 \langle S_p(f_v), f_v \rangle \end{aligned}$$

Hence, $II = l_{11} du^2 + 2 l_{12} du dv + l_{22} dv^2$ and it is called the second fundamental form of $f$.

**Example 2.4.3. (The Graph of a function)** *Let $g : O \longrightarrow \mathbb{R}$ be a smooth function and $f : O \longrightarrow \mathbb{R}^3$ be the graph of $g$ given by $f(u, v) = (u, v, g(u, v))$. We have*

$$\begin{cases} f_u = (1, 0, g_u), \\ f_v = (0, 1, g_v), \\ f_{uv} = (0, 0, g_{uv}), \\ N = \dfrac{1}{\sqrt{1 + g_u^2 + g_v^2}} (-g_u, -g_v, 1). \end{cases}$$

*Let's calculate the first fundamental form. We have*

$$\begin{aligned} g_{11} &= \langle f_u, f_v \rangle = 1 + g_u^2, \\ g_{12} &= \langle f_u, f_v \rangle = g_u g_v, \\ g_{22} &= \langle f_v, f_v \rangle = 1 + g_v^2. \end{aligned}$$

*Hence,*

$$I = g_{11}du^2 + 2g_{12}dudv + g_{22}dv^2 = (1 + g_u^2)du^2 + 2g_u g_v dudv + (1 + g_v^2)dv^2.$$

*Now, we can also calculate*

$$\begin{aligned} l_{11} &= \langle N, f_{uu} \rangle = \frac{g_{uu}}{\sqrt{1 + g_u^2 + g_v^2}}, \\ l_{12} &= \langle N, f_{uv} \rangle = \frac{g_{uv}}{\sqrt{1 + g_u^2 + g_v^2}}, \\ l_{22} &= \langle N, f_{vv} \rangle = \frac{g_{vv}}{\sqrt{1 + g_u^2 + g_v^2}} \end{aligned}$$

*Hence,*

$$\begin{aligned} II &= l_{11}du^2 + 2l_{12}dudv + l_{22}dv^2 \\ &= \frac{g_{uu}}{\sqrt{1 + g_u^2 + g_v^2}}du^2 + \frac{2g_{uv}}{\sqrt{1 + g_u^2 + g_v^2}}dudv + \frac{g_{vv}}{\sqrt{1 + g_u^2 + g_v^2}}dv^2. \end{aligned}$$

*The area of the surface $f(D)$ is*

$$\begin{aligned} A &= \int_D \int \sqrt{g_{11}g_{22} - g_{12}^2}dudv = \int_D \int \sqrt{(1 + g_u^2)(1 + g_v^2) - g_u^2 g_v^2}dudv \\ &= \int_D \int \sqrt{1 + g_u^2 + g_v^2}dudv \end{aligned}$$

**Definition 2.4.4.** *Fix $p_0 = (u_0, v_0) \in O$ and fix $\xi \in T_{f_{p_0}}$. Let $\sigma$ denote the intersection between the surface $f(O)$ and the plane $E$ spanned by $\xi$ and $N(p_0)$. Then, $\sigma$ is a curve belonging to $E$. We will call it the plane section of $f$ at $p_0$ defined by $\xi$.*

**Theorem 2.4.5. (Meusnier's Theorem).** *The curvature of a plane section of a parametrized surface $f : O \longrightarrow \mathbb{R}^3$ at $p_0$ defined by a unit tangent vector $\xi$ in $T_{f_{p_0}}$ is equal to $II_{p_0}(\xi, \xi)$.*

*Proof.* Assume that there exists $c$:

$$\begin{aligned} c : (-\epsilon, \epsilon) &\longrightarrow O \\ s &\longrightarrow (u(s), v(s)), \end{aligned}$$

such that $c(0) = (u_0, v_0) = p_0$. We have

$$\begin{cases} \sigma(s) = f(c(s)) = f(u(s), v(s)), \\ T(s) = \sigma'(s) = f_u u'(s) + f_v v'(s), \\ T'(s) = \sigma''(s) = f_{uu}(u'(s))^2 + f_{vv}(v'(s))^2 + 2f_{uv}u'(s)v'(s). \end{cases}$$

Now, the curvature of $\sigma$ at $p_0$ is given by

$$\begin{aligned} k(0) &= T'(0) \cdot N(p_0) \\ &= f_{uu}(p_0)u'(0)^2 \cdot N(p_0) + f_{vv}(p_0)v'(0)^2 \cdot N(p_0) + 2f_{uv}(p_0)u'(0)v'(0) \cdot N(p_0) \\ &= l_{11}(p_0)u'(0)^2 + 2l_{12}u'(0)v'(0) + l_{22}v'(0)^2 \\ &= II_{p_0}(\xi, \xi) \end{aligned}$$

$\square$

**Example 2.4.6.** *Let* $f(x,y) = (x,y,0)$ *(also known as the xy-plane). Then, we have*

$$\begin{cases} f_x = (1,0,0) \quad , \quad f_{xx} = (0,0,0), \\ f_y = (0,1,0) \quad , \quad f_{yy} = (0,0,0), \\ N = (0,0,1) \quad , \quad f_{xy} = (0,0,0). \end{cases}$$

*Then, the unit normal vector is* $N = (0,0,1)$ *and any plane section is a straight line.*

$$\begin{cases} l_{11} = \langle N, f_{xx} \rangle = 0 \\ l_{12} = \langle N, f_{xy} \rangle = 0 \\ l_{22} = \langle N, f_{yy} \rangle = 0 \end{cases}$$

*Hence,* $II(\xi,\xi) = l_{11}dx^2 + 2l_{12}dxdy + l_{22}dy^2 = 0$ *for any* $p_0 \in O$.

**Example 2.4.7.** *For the cylinder of equation* $x^2 + y^2 = 1$, *we have* $f(x,y) = (\cos x, \sin x, y)$. *Then,*

$$\begin{cases} f_x = (-\sin x, \cos x, 0), \\ f_y = (0,0,1), \\ N = (\cos x, \sin x, 0). \end{cases}$$

*For the top of the cylinder, the plane section defined by* $\xi = f_y = (0,0,1)$ *is a straight line, so* $k = 0$ *and* $II_{p_0}(\xi,\xi) = 0$. *For the lateral part of the cylinder,* *the plane section of* $f$ *defined by* $\eta = f_x = (-\sin x, \cos x, 0)$ *is a circle of radius 1, so* $k = 1$ *and* $II_{p_0}(\eta, \eta) = 1$.

**Example 2.4.8.** *Let* $O = \{(x, y)/x^2+y^2 < 1\}$ *and* $f(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$. *Every plane section is a circle of radius 1, so* $k = 1$ *and* $II_{p_0}(\xi, \xi) = 1$ *for all* $\xi \in T_{f_{p_0}}$.

**Example 2.4.9.** *Let* $O = \{(x_1, x_2)/x_1 \in (0, 2\pi), x_2 \in \mathbb{R}\}$ *and define* $f : O \longrightarrow \mathbb{R}^3$ *and* $h : O \longrightarrow \mathbb{R}^3$ *by*

$$f(x_1, x_2) = (x_1, x_2, 0),$$

$$h(x_1, x_2) = (\cos x_1, \sin x_1, x_2),$$

*For* $f$, *we have:*

$$\begin{cases} f_{x_1} = (1, 0, 0) \quad , \quad f_{x_1x_1} = (0, 0, 0), \\ f_{x_2} = (0, 1, 0) \quad , \quad f_{x_2x_2} = (0, 0, 0), \\ f_{x_1x_2} = (0, 0, 0) \quad , \quad N = (0, 0, 1). \end{cases}$$

*So,* $g_{11} = 1$, $g_{22} = 1$, *and* $g_{12} = 0$. *Thus,*

$$I = dx_1^2 + dx_2^2.$$

*Also,* $l_{11} = 0$, $l_{22} = 0$, *and* $l_{12} = 0$. *Thus,* $II = 0$. *For* $h$, *we have*

$$\begin{cases} h_{x_1} = (-\sin x_1, \cos x_1, 0) \quad , \quad h_{x_2} = (0,0,1), \\ h_{x_1x_1} = (-\cos x_1, -\sin x_1, 0) \quad , \quad h_{x_2x_2} = (0,0,0), \\ N = (\cos x_1, \sin x_1, 0) \quad , \quad h_{x_1x_2} = (0,0,0). \end{cases}$$

*So, $g_{11} = 1$, $g_{22} = 1$, and $g_{12} = 0$. Hence, $\tilde{I} = dx_1^2 + dx_2^2$. Also, $l_{11} = -1$, $l_{22} = 0$, and $l_{12} = 0$. So, $\tilde{II} = -dx_1^2$. Note that $f$ and $h$ have the same first fundamental form, but different second fundamental forms.*

## 2.5 Principal, Gaussian, and Mean Curvatures

In this section, we will introduce Principal, Gaussian, and Mean curvatures which are linked to the shape operator $S$. Since the shape operator $S$ of a parametrized surface $f : O \longrightarrow \mathbb{R}^3$ is a self adjoint operator on $T_{f_p}$, we have as a consequence of the Spectral Theorem, Theorem 2.1.12, we have

**Proposition 2.5.1.** *The shape operator of a parametrized surface $f : O \longrightarrow \mathbb{R}^3$ at $f(p)$ has two real eigenvalues and an orthonormal eigenbase.*

The eigenvalues $k_1$ and $k_2$ of the shape operator $S$ of the parametrized surface $f : O \longrightarrow \mathbb{R}^3$ at $p$ are called the principal curvatures. The Gaussian curvature of $f$ is defined by $K = k_1 k_2$. The Mean curvature of $f$ is defined by $H = k_1 + k_2$. Finally, the principal directions of $f$ at $p$ are the unit eigenvectors $v_1$ and $v_2$ of $S$.

We have already shown in Proposition 2.1.7 that the matrix associated to $S$ is given by $A = G^{-1}L$ where $G = (g_{ij})$, $L = (l_{ij})$ and $g_{ij}$ (resp. $l_{ij}$) are the coefficients of the first (resp. second) fundamental form $I$ (resp. $II$). Note that if $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $B^{-1} = \frac{a}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. By Proposition 2.1.11, we have that $K = k_1k_2 = \det(A)$ and $H = k_1 + k_2 = \operatorname{tr}(A)$, so,

$$k = \det(A) = \det(G^{-1})\det(L) = \frac{1}{\det(G)}.\det(L) = \frac{l_{11}l_{22} - l_{12}^2}{g_{11}g_{22} - g_{12}^2},$$

and we also have

$$\begin{aligned} A = G^{-1}L &= \frac{1}{g_{11}g_{22} - g_{12}^2}\begin{pmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{pmatrix}\begin{pmatrix} l_{11} & l_{12} \\ l_{12} & l_{22} \end{pmatrix} \\ &= \frac{1}{g_{11}g_{22} - g_{12}^2}\begin{pmatrix} g_{22}l_{11} - g_{12}l_{12} & g_{22}l_{12} - g_{12}l_{22} \\ -l_{11}g_{12} + g_{11}l_{12} & -g_{12}l_{12} + g_{11}l_{22} \end{pmatrix} \\ \text{Thus, } H &= \frac{1}{g_{11}g_{22} - g_{12}^2}(g_{22}l_{11} - 2g_{12}l_{12} + g_{11}l_{22}). \end{aligned}$$

If follows that

1. The principal curvatures $k_1$ and $k_2$ are eigenvalues of $A = G^{-1}L$.
2. If $\begin{pmatrix} r_1 \\ s_1 \end{pmatrix}$ and $\begin{pmatrix} r_2 \\ s_2 \end{pmatrix}$ are unit eigenvectors of $A$ with eigenvalue $k_1$ and $k_2$, then $v_1 = r_1 f_u + s_1 f_v$ and $v_2 = r_2 f_u + s_2 f_v$ are its principal directions.

**Example 2.5.2.** *For the cylinder* $f(x_1, x_2) = (\cos x_1, \sin x_1, x_2)$, *we have*

$$\begin{cases} f_{x_1} = (-\sin x_1, \cos x_1, 0) \;\;,\;\; f_{x_2} = (0, 0, 1), \\ f_{x_1x_1} = (-\cos x_1, -\sin x_1, 0) \;\;,\;\; f_{x_2x_2} = (0, 0, 0), \\ N = (\cos x_1, \sin x_1, 0) \;\;,\;\; f_{x_1x_2} = (0, 0, 0). \end{cases}$$

*So,*

$$\begin{cases} g_{11} = \langle f_{x_1}, f_{x_1} \rangle = 1 \\ g_{22} = \langle f_{x_2}, f_{x_2} \rangle = 1 \\ g_{12} = \langle f_{x_1}, f_{x_2} \rangle = 0 \end{cases} \quad and \quad \begin{cases} l_{11} = \langle N, f_{x_1 x_1} \rangle = -1 \\ l_{12} = \langle N, f_{x_1 x_2} \rangle = 0 \\ l_{22} = \langle N, f_{x_2 x_2} \rangle = 0 \end{cases}$$

*Hence,*

$$A = \frac{1}{1-0} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

*Thus,* $k_1 = -1$ *and* $k_2 = 0$. *Moreover,*

$$\begin{pmatrix} -1+1 & 0 \\ 0 & 0+1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

*We have* $x_1 = t$ *and* $x_2 = 0$, *then* $\begin{pmatrix} t \\ 0 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

*Therefore,* $v_1 = f_{x_1}$ *is an eigenvector corresponding to* $k_1$. *Similarly,* $v_2 = f_{x_2}$ *is an eigenvector corresponding to* $k_2$. *Thus,* $f_{x_1}$ *and* $f_{x_2}$ *are the principal directions (unit vectors with* $\langle f_{x_1}, f_{x_2} \rangle = 0$*).*

**Example 2.5.3.** *Let* $f(x_1, x_2) = (x_1, x_2, \sqrt{R^2 - x_1^2 - x_2^2})$ *be the sphere of radius* $R$. *We have*

$$\begin{cases} f_{x_1} = \Big(1, 0, \dfrac{-x_1}{\sqrt{R^2 - x_1^2 - x_2^2}}\Big), \\ f_{x_2} = \Big(0, 1, \dfrac{-x_2}{\sqrt{R^2 - x_1^2 - x_2^2}}\Big), \\ N = \Big(\dfrac{x_1}{R}, \dfrac{x_2}{R}, \dfrac{\sqrt{R^2 - x_1^2 - x_2^2}}{R}\Big) = \dfrac{1}{R} f. \end{cases}$$

*Then,* $S(f_{x_i}) = -N_{x_i} = \dfrac{-1}{R} f_{x_i}$ *for* $i = 1, 2$ *and* $S = \dfrac{-1}{R}\mathrm{Id}$ *where* Id *denotes the corresponding identity matrix. We have*

$$II(\xi, \eta) \quad = \quad \langle S(\xi), \eta\rangle = \langle \frac{-1}{R}\xi, \eta\rangle = \frac{-1}{R}\langle \xi, \eta\rangle = \frac{-1}{R}\mathrm{Id} = \begin{pmatrix} \frac{-1}{R} & 0 \\ 0 & \frac{-1}{R} \end{pmatrix}.$$

*So,* $k_1 = \dfrac{-1}{R}$ *and* $k_2 = \dfrac{-1}{R}$ *are eigenvalues of* $S$*. Then,* $H = \dfrac{-2}{R}$ *and* $K = \dfrac{1}{R^2}$.

**Example 2.5.4.** *For* $f(x, y) = (x, y, x^2 + y^2)$*, we have*

$$\begin{cases} f_x = (1, 0, 2x) \ , \ f_{xx} = (0, 0, 2), \\ f_y = (0, 1, 2y) \ , \ f_{yy} = (0, 0, 2) \ , \ f_{xy} = (0, 0, 0), \\ N = (\dfrac{-2x}{\sqrt{1 + 4x^2 + 4y^2}}, \dfrac{-2y}{\sqrt{1 + 4x^2 + 4y^2}}, \dfrac{1}{\sqrt{1 + 4x^2 + 4y^2}}), \\ g_{11} = \langle f_x, f_x\rangle = 1 + 4x^2, \\ g_{12} = \langle f_x, f_y\rangle = 4xy, \\ g_{22} = \langle f_y, f_y\rangle = 1 + 4y^2, \end{cases}$$

$$\begin{cases} l_{11} = \langle N, f_{xx} \rangle = \dfrac{2}{\sqrt{1+4x^2+4y^2}}, \\ l_{12} = \langle N, f_{xy} \rangle = 0, \\ l_{22} = \langle N, f_{yy} \rangle = \dfrac{2}{\sqrt{1+4x^2+4y^2}}. \end{cases}$$

*Thus, we get*

$$\begin{aligned} K &= k_1 k_2 = \det(A) \\ &= \frac{\det(L)}{\det(G)} \\ &= \frac{l_{11}l_{22} - l_{12}^2}{g_{11}g_{22} - g_{12}^2} \\ &= \frac{\frac{4}{1+4x^2+4y^2} - 0}{(1+4x^2)(1+4y^2) - 16x^2y^2} \\ &= \frac{\frac{4}{1+4x^2+4y^2}}{1+4x^2+4y^2+16x^2y^2-16x^2y^2} \\ &= \frac{4}{(1+4x^2+4y^2)^2}, \end{aligned}$$

*and*

$$\begin{aligned} H &= \mathrm{tr}(A) \\ &= \frac{1}{1+4x^2+4y^2}\Big((1+4y^2)\frac{2}{\sqrt{1+4x^2+4y^2}} - 0 + (1+4x^2)\frac{2}{\sqrt{1+4x^2+4y^2}}\Big) \\ &= \frac{1}{1+4x^2+4y^2}\Big(\frac{2+8y^2+2+8x^2}{\sqrt{1+4x^2+4y^2}}\Big) \\ &= \frac{4(1+2y^2+2x^2)}{(1+4x^2+4y^2)^{\frac{3}{2}}}. \end{aligned}$$

# Chapter 3

# Fundamental Theorem of Surfaces in $\mathbb{R}^3$

In this chapter, We state and prove the Frobenius Theorem, and we also define line of curvature coordinates. Then, we derive the Gauss-Codazzi Equations and state the Fundamental Theorem of Surfaces and the Gauss Theorem in line of curvature coordinates. Finally, we write the Gauss-Codazzi Equations in local and orthogonal coordinates. [1, 2, 4]

## 3.1 The Frobenius Theorem

We start by stating some results needed to prove the Frobenius Theorem.

**Theorem 3.1.1.** *Let $O$ be an open subset of $\mathbb{R}^2$. Consider $f, g : O \longrightarrow \mathbb{R}$ be smooth maps with $(x_0, y_0) \in O$ and $c_0 \in \mathbb{R}$. The Initial Value Problem for the* *partial differential equations system, with* $u : O \longrightarrow \mathbb{R}$*), given by*

$$\begin{cases} \dfrac{\partial u}{\partial x} = f(x,y), \\ \dfrac{\partial u}{\partial y} = g(x,y), \\ u(x_0, y_0) = c_0, \end{cases}$$

*has a smooth solution defined in some disk centered at* $(x_0, y_0)$ *fo any given* $(x_0, y_0) \in O$ *if and only if* $f$ *and* $g$ *satisfy the compatibility relation* $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$ *in* $O$*.*

Moreover, one can use integration to find the solution as follows: Suppose $u(x,y)$ is a solution, then using the Fundamental Theorem of Calculus, we have

$$\frac{\partial u}{\partial x} = f(x,y) \Longrightarrow u(x,y) = \int_{x_0}^{x} \frac{\partial g}{\partial s} ds + v'(y),$$

for some $v(y)$ such that $v(y_0) = c_0$, but

$$\frac{\partial u}{\partial y} = \int_{x_0}^{x} \frac{\partial f}{\partial y} ds + v'(y) = \int_{x_0}^{x} \frac{\partial g}{\partial x} ds + v'(y) = v'(y) + g(x,y) - g(x_0, y)$$

and $\frac{\partial u}{\partial y} = g(x,y)$ so $v'(y) = g(x_0, y)$ and

$$v(y) = \int_{y_0}^{y} g(x_0, t) dt + c.$$

Since $v(y_0) = c_0$, we have $v(y) = c_0 + \displaystyle\int_{y_0}^{y} g(x_0, t)dt$ and hence,

$$u(x, y) = \int_{x_0}^{x} f(s, y)ds + \int_{y_0}^{y} g(x_0, t)dt + c_0.$$

Given smooth maps $A, B : O \times \mathbb{R} \longrightarrow \mathbb{R}$, we now consider the first order PDE system for $u : O \longrightarrow \mathbb{R}$:

$$\begin{cases} \dfrac{\partial u}{\partial x} = A(x, y, u(x, y)), \\ \dfrac{\partial u}{\partial y} = B(x, y, u(x, y)). \end{cases} \tag{3.1.1}$$

The system (3.1.1) has a smooth solution $u$ when $u_{xy} = u_{yx}$. We calculate

$$\begin{aligned} u_{xy} &= (A(x, y, u(x, y)))_y = A_x x_y + A_y y_y + A_u u_y = A_y + A_u B, \\ u_{yx} &= (B(x, y, u(x, y)))_x = B_x x_x + B_y y_x + B_u u_x = B_x + B_u A. \end{aligned}$$

Then, the system (3.1.1) has a smooth solution $u$ when $A$ and $B$ satisfy

$$A_y + A_u B = B_x + B_u A.$$

**Theorem 3.1.2. (Frobenius Theorem).** *(for $u : O \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^n$) Let $U_1 \subset \mathbb{R}^2$ and $U_2 \subset \mathbb{R}^2$ be two open subsets. Consider $A = (A_1, A_2, ...., A_n)$ and $B = (B_1, B_2, ..., B_n)$ two smooth maps given by $A, B : U_1 \times U_2 \longrightarrow \mathbb{R}^n$ with*

*$(x_0, y_0) \in U_1$ and $p_0 \in U_2$. The the following first order system*

$$\begin{cases} \dfrac{\partial u}{\partial x} = A(x, y, u(x, y)), \\ \dfrac{\partial u}{\partial y} = B(x, y, u(x, y)), \\ u(x_0, y_0) = p_0, \end{cases} \tag{3.1.2}$$

*has a smooth solution for $u$ in a neighborhood of $(x_0, y_0)$ for all possible $(x_0, y_0) \in U_1 \subset \mathbb{R}^2$ and $p_o \in U_2 \subset \mathbb{R}^n$ if and only if*

$$(A_i)_y + \sum_{j=1}^{n} \frac{\partial A_i}{\partial u_j} B_j = (B_i)_x + \sum_{j=1}^{n} \frac{\partial B_i}{\partial u_j} A_j, \tag{3.1.3}$$

*for $i = 1, 2, ..., n$.*

Now, the system (3.1.2) written in coordinates gives the following

$$\begin{cases} \dfrac{\partial u_i}{\partial x} = A_i(x, y, u_1(x, y), u_2(x, y), ..., u_n(x, y)), \\ \dfrac{\partial u_i}{\partial y} = B_i(x, y, u_1(x, y), u_2(x, y), ..., u_n(x, y)), \\ u_i(x_0, y_0) = p_{0_i}, \end{cases}$$

for $i = 1, 2, .., n$, where $p_0 = (p_{0_1}, p_{0_2}, ...., p_{0_n})$. We call Equation (3.1.3) the compatibility condition for the system (3.1.2). To prove the Frobenius Theorem, we need to solve a family of Ordinary Differential Equations depending smoothly on a parameter, and we need to know whether the solutions depend smoothly on the initial data and the parameter. This is answered in the following theorem.

**Theorem 3.1.3.** *[8] Let $O$ be an open subset of $\mathbb{R}^n$, $t_0 \in (a_0, b_0)$ and a smooth* *map* $f : [a_0, b_0] \times O \times [a_1, b_1] \longrightarrow \mathbb{R}^n$*. Suppose it is given that* $p \in O$ *and* $r \in [a_1, b_1]$*. Let* $y^{p,r}$ *denotes the solution of*

$$\begin{cases} \dfrac{\partial y}{\partial t} = f(t, y(t), r), \\ y(t_0) = p, \end{cases}$$

*and* $u(t, p, r) = y^{p,r}(t)$*. Then,* $u$ *is smooth in* $t$*,* $p$*, and* $r$*.*

*Proof of The Frobenius Theorem, Theorem 3.1.2.* If $u = (u_1, u_2, ..., u_n)$ is a smooth solution for the system (3.1.2) then,

$$\begin{aligned} \frac{\partial}{\partial y}\Big(\frac{\partial u_i}{\partial x}\Big) &= \frac{\partial}{\partial y} A_i\Big(x, y, u_1(x, y), ..., u_n(x, y)\Big) \\ &= \frac{\partial A_i}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial A_i}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial A_i}{\partial u_1}\frac{\partial u_1}{\partial y} + ... + \frac{\partial A_i}{\partial u_n}\frac{\partial u_n}{\partial y} \\ &= \frac{\partial A_i}{\partial y} + \sum_{j=1}^{n} \frac{\partial A_i}{\partial u_j}\frac{\partial u_j}{\partial y} = \frac{\partial A_i}{\partial y} + \sum_{j=1}^{n} \frac{\partial A_i}{\partial u_j} B_j, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial x}\Big(\frac{\partial u_i}{\partial y}\Big) &= \frac{\partial}{\partial x} B_i\Big(x, y, u_1(x, y), ..., u_n(x, y)\Big) \\ &= \frac{\partial B_i}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial B_i}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial B_i}{\partial u_1}\frac{\partial u_1}{\partial x} + ... + \frac{\partial B_i}{\partial u_n}\frac{\partial u_n}{\partial x} \\ &= \frac{\partial B_i}{\partial x} + \sum_{j=1}^{n} \frac{\partial B_i}{\partial u_j}\frac{\partial u_j}{\partial x} = \frac{\partial B_i}{\partial x} + \sum_{j=1}^{n} \frac{\partial B_i}{\partial u_j} A_j \end{aligned}$$

Then, $(A_i)_y + \sum_{j=1}^{n} \frac{\partial A_i}{\partial u_j} B_j = (B_i)_x + \sum_{j=1}^{n} \frac{\partial B_i}{\partial u_j} A_j$. Conversely, assume that $(A_i)_y + \sum_{j=1}^{n} \frac{\partial A_i}{\partial u_j} B_j = (B_i)_x + \sum_{j=1}^{n} \frac{\partial B_i}{\partial u_j} A_j$. To solve the system (3.1.2) we proceed as follows: The existence and uniqueness theorem of ODE, Theorem 1.4.1, states that there exists $\delta > 0$ and $\alpha : (x_0 - \delta, x_0 + \delta) \longrightarrow U_2$ satisfying the following

$$\begin{cases} \dfrac{\partial \alpha}{\partial x} = A(x, y_0, \alpha(x)), \\ \alpha(x_0) = p_0. \end{cases} \tag{3.1.4}$$

Now, fix $x \in (x_0 - \delta, x_0 + \delta)$ and let $\beta^x(y)$ be the unique solution of the following ODE in $y$:

$$\begin{cases} \dfrac{\partial \beta^x}{\partial y} = B(x, y, \beta^x(y)), \\ \beta^x(y_0) = \alpha(x). \end{cases} \tag{3.1.5}$$

Let $u(x, y) = \beta^x(y)$. We have that the system (3.1.5) is a family of ordinary differential equations in $y$ depending on the parameter $x$ and $B$ is smooth, then by Theorem 3.1.3, $u$ is smooth in $x$ and $y$. By construction, $u$ satisfies the second equation of the system (3.1.2). We will prove that $u$ satisfies the first equation of the system (3.1.2) for $n = 1$. The proof for general $n$ is similar. Let $z(x, y) = u_x - A(x, y, u(x, y))$. We have

$$\begin{aligned} z_y &= (u_x - A(x, y, u(x, y)))_y \\ &= u_{xy} - A_y - A_u u_y \\ &= u_{xy} - A_y - A_u B \\ &= u_{xy} - (A_y + A_u B) \\ &= u_{yx} - (A_y + A_u B) \\ &= B_x(x, y, u(x, y)) - (A_y + A_u B) \end{aligned}$$

$$\begin{aligned}
&= B_x + B_u u_x - (A_y + A_u B) \\
&= B_x + B_u u_x - (B_x + B_u A) \\
&= B_u(u_x - A) \\
&= B_u(x, y, u(x, y)) \cdot z.
\end{aligned}$$

For each $x$, $h^x(y) = z(x, y)$ is a solution for the differential equation

$$\frac{\partial h}{\partial y} = B_u\Big(x, y, u(x, y)\Big) \cdot h. \tag{3.1.6}$$

Now,

$$h^x(y_0) = z(x, y_0) = u_x(x, y_0) - A(x, y_0, u(x, y_0)) = \alpha'(x) - A(x, y_0, \alpha(x)) = 0.$$

Then, $h^x$ is a solution of Equation (3.1.6) with initial condition $h^x(y_0) = 0$. But 0 is also a solution of Equation (3.1.6) with $0(y_0) = 0$. So, by the existence and uniqueness Theorem of ODE, Theorem 1.4.1, we get

$$z(x, y) = h^x(y) = 0.$$

Thus,

$$u_x(x, y_0) = A(x, y_0, u(x, y_0)).$$

So, $u$ satisfies the first equation of the system (3.1.2). □

Let $\mathrm{Gl}(n)$ denotes the space of all real $n \times n$ matrices. For $P, Q \in \mathrm{Gl}(n)$, we denote by $[P, Q]$ the commutator of $P$ and $Q$ defined by

$$[P, Q] = PQ - QP.$$

**Proposition 3.1.4.** *Let $U$ be an open subset of $\mathbb{R}^2$ with $(x_0, y_0) \in U \subset \mathbb{R}^2$. $C \in \mathrm{Gl}(n)$ and $P, Q : U \longrightarrow \mathrm{Gl}(n)$ are smooth maps. Then, the following initial value problem for $u : U \longrightarrow \mathrm{Gl}(n)$*

$$\begin{cases} u_x = u(x,y)P(x,y), \\ u_y = u(x,y)Q(x,y), \\ u(x_0, y_0) = c, \end{cases}$$

*has a unique solution $u$ defined on a disk centered at $(x_0, y_0)$ for all $(x_0, y_0) \in U$ and $C \in \mathrm{Gl}(n)$ if and only if*

$$P_y - Q_x = [P, Q].$$

*Proof.* This proposition follows from Frobenius Theorem, Theorem 3.1.2, where $A = uP$ and $B = uQ$. The initial value problem has a smooth solution $u$ if and only if $u_{xy} = u_{yx}$. Now, we calculate

$$\begin{cases} u_{xy} = (u_x)_y = (uP)_y = u_y P + uP_y = (uQ)P + uP_y = u(QP + P_y), \\ u_{yx} = (uQ)_x = u_x Q + uQ_x = (uP)Q + uQ_x = u(PQ + Q_x). \end{cases}$$

Thus,

$$\begin{aligned} & u_{yx} = u_{xy} \\ \Longleftrightarrow\quad & u(QP + P_y) = u(PQ + Q_x) \\ \Longleftrightarrow\quad & QP + P_y = PQ + Q_x \\ \Longleftrightarrow\quad & P_y - Q_x = PQ - QP \\ \Longleftrightarrow\quad & P_y - Q_x = [P, Q]. \end{aligned}$$

□

**Remark** The equation $QP+P_y = PQ+Q_x$ is called the compatibility relation.

**Remark** Given $3 \times 3$ smooth matrices $P$ and $Q$ such that $\begin{cases} u_x = uP \\ u_y = uQ \end{cases}$ is equivalent to say that we have

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix}_x = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$
$$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix}_y = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix}$$

This is the type of equation we need for the Fundamental Theorem of Surfaces in $\mathbb{R}^3$. Now, if $P$ and $Q$ are skew symmetric (i.e: $P^T = -P$ and $Q^T = -Q$), then,

$$\begin{aligned}(PQ-QP)^T &= (PQ)^T-(QP)^T = Q^TP^T-P^TQ^T = (-Q)(-P)-(-P)(-Q)\\ &= QP-PQ = -(PQ-QP) = -[P,Q].\end{aligned}$$

It means that $[P,Q]$ is skew symmetric. For $P^T=-P$, we get

$$\begin{pmatrix} p_{11} & p_{21} & p_{31}\\ p_{12} & p_{22} & p_{32}\\ p_{13} & p_{23} & p_{33}\end{pmatrix} = \begin{pmatrix} -p_{11} & -p_{12} & -p_{13}\\ -p_{21} & -p_{22} & -p_{23}\\ -p_{31} & -p_{32} & -p_{33}\end{pmatrix}$$

Thus, the system $\begin{cases} u_x = uP \\ u_y = uQ \end{cases}$ becomes a system of three first order partial differential equations involving six functions $p_{12},\ p_{13},\ p_{23},\ q_{12},\ q_{13},$ and $q_{23}$ where $p_{11}=p_{22}=p_{33}=q_{11}=q_{22}=q_{33}=0$.

**Proposition 3.1.5.** *Let $O$ be an open subset of $\mathbb{R}^2$ and $P,Q: O \longrightarrow \mathrm{Gl}(n)$ be smooth maps such that $P$ and $Q$ are skew symmetric. Suppose $P$ and $Q$ satisfy the compatibility relation $QP+P_y = PQ+Q_x$ and $C$ is the initial data where $C$ is an orthogonal matrix. Let $O_0 \subset O$. If $u: O_0 \longrightarrow \mathrm{Gl}(n)$ is a solution of*

$$\begin{cases} u_x = u(x,y)P(x,y), \\ u_y = u(x,y)Q(x,y), \\ u(x_0,y_0) = C, \end{cases}$$

*then $u(x,y)$ is an orthogonal matrix for all $(x,y) \in O_0$*

*Proof.* Let $\xi(x,y) = u(x,y)^T u(x,y)$. Since the initial data is an orthogonal matrix, we have

$$\xi(x_0,y_0) = u(x_0,y_0)^T u(x_0,y_0) = \mathrm{Id}.$$

Now,

$$\begin{cases} \xi_x = (u_x)^T u + u^T u_x = (uP)^T u + u^T(uP) = P^T u^T u + u^T u P = P^T \xi + \xi P, \\ \xi_y = (u_y)^T u + u^T u_y = (uQ)^T u + u^T(uQ) = Q^T u^T u + u^T u Q = Q^T \xi + \xi Q. \end{cases}$$

So, $\xi$ satisfies

$$\begin{cases} \xi_x = P^T \xi + \xi P, \\ \xi_y = Q^T \xi + \xi Q, \\ \xi(x_0,y_0) = \mathrm{Id} \end{cases} .$$

We also know that Id is a solution of the above system since $\mathrm{Id}_x = 0$ and

$$P^T \mathrm{Id} + \mathrm{Id} P = P^T + P = -P + P = 0.$$

Thus $\mathrm{Id}_x = P^T \mathrm{Id} + \mathrm{Id} P$. Similarly, $Q^T \mathrm{Id} + \mathrm{Id} Q = -Q + Q = 0 = \mathrm{Id}_y$ and $\mathrm{Id}(x_0,y_0) = \mathrm{Id}$. So, by the uniqueness of solutions of the Frobenius Theorem, Theorem 3.1.2, we get that $\xi = \mathrm{Id}$, so $u^T u = \mathrm{Id}$ and $u$ is orthogonal for all $(x,y) \in O_0$. $\square$

**Example 3.1.6.** *Given* $c_0 > 0$*, consider the following PDE system*

$$\begin{cases} u_x = 2\sin u, \\ u_y = \frac{1}{2}\sin u, \\ u(0,0) = c_0. \end{cases}$$

*This is System (3.1.2) from the Frobenius Theorem where* $A = 2\sin u$*,* $B = \frac{1}{2}\sin u$*, and* $c_0 = p_0$*. Let's check the compatibility relation. We have*

$$\begin{aligned} A_y + A_u B &= \cos u \sin u + 2\cos u \times \frac{1}{2}\sin u = 2\cos u \sin u, \\ B_x + B_u A &= \cos u \sin u + \frac{1}{2}\cos u \times 2\sin u = 2\cos u \sin u. \end{aligned}$$

*Hence,* $A_y + A_u B = B_x + B_u A$*. Thus, by the Frobenius Theorem, Theorem 3.1.2, this system has a smooth solution. Now, let's solve the system using methods in Frobenius theorem's proof. First, let's solve the ODE*

$$\begin{cases} \dfrac{\partial \alpha}{\partial x} = 2\sin \alpha \\ \alpha(0) = c_0 \end{cases} .$$

*The equation* $\alpha_x = 2\sin x$ *is a separable ODE. Hence,*

$$\begin{aligned} & \int \frac{d\alpha}{\sin \alpha} = \int 2dx \\ \Longrightarrow \quad & -\ln|\csc \alpha + \cot \alpha| = 2x + c \\ \Longrightarrow \quad & \frac{1}{\csc \alpha + \cot \alpha} = e^{2x+c} \end{aligned}$$

$$\Longrightarrow \quad \frac{1}{\frac{1}{\sin\alpha}+\frac{\cos\alpha}{\sin\alpha}} = \frac{\sin\alpha}{1+\cos\alpha} = \frac{2\sin(\frac{\alpha}{2})\cos(\frac{\alpha}{2})}{2\cos(\frac{\alpha}{2})} = \tan(\frac{\alpha}{2})$$
$$\Longrightarrow \quad \tan(\frac{\alpha}{2}) = e^{2x+c} \Longrightarrow \frac{\alpha}{2} = \tan^{-1}(e^{2x+c})$$
$$\Longrightarrow \quad \alpha(x) = 2\tan^{-1}(e^{2x+c}).$$

*But* $\alpha(0) = c_0$, *so* $2\tan^{-1} e^c = c_0$ *and* $c = \ln\left(\tan\frac{c_0}{2}\right)$. *Finally, we have*

$$\alpha(x) = 2\tan^{-1}\left(e^{2x+\ln(\tan\frac{c_0}{2})}\right).$$

*Also, solving the separable ODE* $\begin{cases} \dfrac{\partial u}{\partial y} = \dfrac{1}{2}\sin u \\ u(x,0) = \alpha(x) = 2\tan^{-1} e^{2x+c} \end{cases}$, *we get*

$$\begin{aligned} \int \frac{du}{\sin u} &= \int \frac{1}{2}dy \\ \Longrightarrow u(x,y) &= 2\tan^{-1}(e^{\frac{y}{2}+k}) \\ \Longrightarrow u(x,0) &= \alpha(x) = 2\tan^{-1}(e^{2x+c}) \\ \Longrightarrow 2\tan^{-1}(e^k) &= 2\tan^{-1}(e^{2x+c}) \\ \Longrightarrow k = 2x+c. \end{aligned}$$

*Hence,* $u(x,y) = 2\tan^{-1}\left(e^{\frac{y}{2}+2x+c}\right)$, *where* $c = \ln(\tan\dfrac{c_0}{2})$. *Since* $u$ *is a solution of (3.1.2), then*

$$u_{xy} = (u_x)_y = (2\sin u)_y = 2u_y\cos u = 2\cos u(\frac{1}{2}\sin u) = \cos u \sin u.$$

*Thus, $u$ satisfies the sine Gordon wave equation (SGE):*

$$u_{xy} = \sin u \cos u.$$

The previous example is a special case of the following theorem:

**Theorem 3.1.7. (Theorem of Backlund)** *Given a smooth function $q : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $r \in \mathbb{R}$ such that $r \neq 0$. The following system of PDE:*

$$\begin{cases} u_s = -q_s + r\sin(u-q), \\ u_t = q_t + \frac{1}{r}\sin(u+q), \end{cases}$$

*has a solution for $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ if and only if $q$ satisfies the sine Gordon wave equation (SGE),*

$$q_{st} = \sin q \cos q.$$

*Moreover, the solution $u$ is also a solution of the SGE.*

*Proof.* Suppose that the system has a smooth solution $u \in C^2$, where $C^2$ is the set of all functions which are twice differentiable and having all partial derivatives continuous. Then, we have $u_{st} = u_{ts}$. Let's calculate $u_{st}$ and $u_{ts}$. We have

$$\begin{aligned} u_{st} &= (u_s)_t = (-q_s + r\sin(u-q))_t \\ &= -q_{st} + r(u_t - q_t)\cos(u-q) \\ &= -q_{st} + r(\frac{1}{r}\sin(u+q))\cos(u-q), \end{aligned} \qquad (3.1.7)$$

$$
\begin{aligned}
u_{ts} &= (u_t)_s = (q_t + \frac{1}{r}\sin(u+q))_s \\
&= q_{ts} + \frac{1}{r}(u_s + q_s)\cos(u+q) \\
&= q_{ts} + \frac{1}{r}(r\sin(u-q))\cos(u+q). \qquad (3.1.8)
\end{aligned}
$$

Now, $u_{st} = u_{ts}$ gives that

$$
-q_{st} + \sin(u+q)\cos(u-q) = q_{ts} + \sin(u-q)\cos(u+q).
$$

Thus,

$$
\begin{aligned}
2q_{st} &= \sin(u+q)\cos(u-q) - \sin(u-q)\cos(u+q) \\
&= \sin(u+q-u+q) = \sin(2q) = 2\sin q\cos q.
\end{aligned}
$$

Finally, $q_{st} = \sin q \cos q$ which means that $q$ satisfies the SGE. Now, adding Equation (3.1.7) to Equation (3.1.8) gives that

$$
\begin{aligned}
& u_{ts} + u_{st} = \sin(u+q)\cos(u-q) + \sin(u-q)\cos(u+q) \\
\Longrightarrow\ & 2u_{ts} = \sin(u+q+u-q) = \sin(2u) = 2\sin u\cos u \\
\Longrightarrow\ & u_{ts} = \sin u\cos u.
\end{aligned}
$$

So $u$ satisfies also the SGE. □

The above theorem states that if we know one solution $q$ of the SGE, we can solve the previous PDE system to get a family of solutions of the SGE (one for each $r$). Now, $q = 0$ is a trivial solution of the SGE. Theorem 3.1.7

states that the PDE system can be solved for $u$ with $q = 0$.

$$\begin{cases} u_s = r \sin u \\ u_t = \frac{1}{r} \sin u \end{cases}$$

This is solved exactly like example 3.1.6 where $r = 2 \Longrightarrow u(s,t) = 2\tan^{-1}(e^{rs+\frac{t}{r}})$. $u$ are also solutions of the SGE.

## 3.2 Line of Curvature Coordinates

In this section, we will introduce line of curvature coordinates. We will also show how this type of coordinates helps in facilitating the calculation of the first and second fundamental forms of a parametrized surface.

**Definition 3.2.1.** *A parametrized surface $f : O \longrightarrow \mathbb{R}^3$ is said to be parametrized by line of curvature coordinates if and only if $g_{12} = l_{12} = 0$. It means that it is parametrized by line of curvature coordinates if and only if the first and second fundamental forms are in diagonal form, i.e, $I = g_{11}dx_1^2 + g_{22}dx_2^2$ and $II = l_{11}dx_1^2 + l_{22}dx_2^2$.*

For surfaces parametrized by line of curvature coordinates, we have that

$$A = G^{-1}L = \frac{1}{g_{11}g_{22}} \begin{pmatrix} g_{22} & 0 \\ 0 & g_{11} \end{pmatrix} \begin{pmatrix} l_{11} & 0 \\ 0 & l_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{g_{11}} & 0 \\ 0 & \frac{1}{g_{22}} \end{pmatrix} \begin{pmatrix} l_{11} & 0 \\ 0 & l_{22} \end{pmatrix} = \begin{pmatrix} \frac{l_{11}}{g_{11}} & 0 \\ 0 & \frac{l_{22}}{g_{22}} \end{pmatrix}.$$

Then the principal curvatures are given by $k_1 = \dfrac{l_{11}}{g_{11}}$ and $k_2 = \dfrac{l_{22}}{g_{22}}$. The Gaussian curvature is given by $K = k_1k_2 = \dfrac{l_{11}l_{22}}{g_{11}g_{22}}$ and the mean curvature by

$H = k_1 + k_2 = \dfrac{l_{11}}{g_{11}} + \dfrac{l_{22}}{g_{22}}.$

**Example 3.2.2.** *Let $u : [a,b] \longrightarrow \mathbb{R}$ be a smooth function and consider the surface given by*

$$f(y,\theta) = (u(y)\sin\theta, 1, u'(y)\cos\theta).$$

*We have*

$$\begin{cases} f_y = (u'(y)\sin\theta, 1, u'(y)\cos\theta), \\ f_\theta = (u(y)\cos\theta, 0, -u(y)\sin\theta), \\ N = \frac{f_y\times f_\theta}{\|f_y\times f_\theta\|} = (\frac{-\sin\theta}{\sqrt{1+(u')^2}}, \frac{u'}{\sqrt{1+(u')^2}}, \frac{-\cos\theta}{\sqrt{1+(u')^2}}), \\ f_{yy} = (u''(y)\sin\theta, 0, u''(y)\cos\theta), \\ f_{\theta\theta} = (-u(y)\sin\theta, 0, -u(y)\cos\theta), \\ f_{y\theta} = (u'(y)\cos\theta, 0, -u'(y)\sin\theta), \\ g_{11} = \langle f_y, f_y\rangle = (u'(y))^2 + 1, \\ g_{22} = \langle f_\theta, f_\theta\rangle = (u(y))^2, \\ g_{12} = \langle f_y, f_\theta\rangle = 0, \\ l_{11} = \langle N, f_{yy}\rangle = \dfrac{-u''(y)}{\sqrt{1+(u')^2}}, \\ l_{22} = \langle N, f_{\theta\theta}\rangle = \dfrac{u(y)}{\sqrt{1+(u')^2}}, \\ l_{12} = \langle N, f_{y\theta}\rangle = 0 \end{cases}$$

*Then, $f(y,\theta)$ is parametrized by line of curvature coordinates since $l_{12} = g_{12} = 0$.*

**Proposition 3.2.3.** *For a surface, assume that the principal curvatures $k_1(p_0) \neq$*

*$k_2(p_0)$ for some $p_0 \in O$. Then, $\exists \delta > 0$ such that $B(p_0, \delta) \subset O$ where $B(p_0, \delta)$ is an open ball of center $p_0$ and radius $\delta$ and $k_1(p) \neq k_2(p)$ for all $p \in B(p_0, \delta)$.*

*Proof.* The Gaussian and mean curvatures $H$ and $K$ are smooth maps where $H = k_1 + k_2$ and $K = k_1 k_2$. Then, $k_1$ and $k_2$ are solutions of the equation

$$x^2 - Hx + K = 0.$$

We calulate $\Delta = H^2 - 4K$. Then, $k_1 = \dfrac{H + \sqrt{H^2 - 4K}}{2}$ and $k_2 = \dfrac{H - \sqrt{H^2 - 4K}}{2}$ are two distinct roots if and only if $H^2 - 4K = u > 0$. Now, since $k_1(p_0) \neq k_2(p_0)$, we get $u(p_0) > 0$. Also, $u$ is continuous at $p_0$ since $H$ and $K$ are smooth, then

$$\forall \epsilon > 0, \exists \delta > 0 / \|p - p_0\| < \delta \implies |u(p) - u(p_0)| < \epsilon.$$

Take $\epsilon = \dfrac{u(p_0)}{2}$, then $\dfrac{u(p_0)}{2} < u(p) < \dfrac{3u(p_0)}{2}$. But $u(p_0) > 0$, then $u(p) > 0$ for all $p \in B(p_0, \delta)$. Therefore, $k_1(p) \neq k_2(p)$ for all $p \in B(p_0, \delta)$. □

**Remark** A smooth map $V : O \longrightarrow \mathbb{R}^3$ is called a tangent vector field of the parametrized surface $f : O \longrightarrow \mathbb{R}^3$ if $v(p) \in T_{f_p}$ for all $p \in O$.

**Proposition 3.2.4.** *Let $f : O \longrightarrow \mathbb{R}^3$ be a parametrized surface. Assume that $k_1(p) \neq k_2(p)$ for all $p \in O$. There exists smooth orthonormal tangent vector fields $e_1, e_2$ of $f$. Moreover, $e_1(p)$ and $e_2(p)$ are eigenvectors for the shape operator $S_p$ for all $p \in O$.*

**Remark** We call $f(p_0)$ an umbilic point of the parametrized surface $f : O \longrightarrow \mathbb{R}^3$ if $k_1(p_0) = k_2(p_0)$.

## 3.3 The Gauss-Codazzi Equations in Line of Curvature Coordinates

Suppose $f : O \longrightarrow \mathbb{R}^3$ is a surface parametrized by line of curvature coordinates, then $g_{12} = \langle f_{x_1}, f_{x_2} \rangle = 0$ and $l_{12} = \langle f_{x_1x_2}, N \rangle = 0$. We define $A_1, A_2, r_1, r_2$ as follows:

$$\begin{cases} g_{11} = \langle f_{x_1}, f_{x_1} \rangle = A_1^2 \\ g_{22} = \langle f_{x_2}, f_{x_2} \rangle = A_2^2 \\ l_{11} = \langle f_{x_1x_1}, N \rangle = r_1 A_1 \\ l_{22} = \langle f_{x_2x_2}, N \rangle = r_2 A_2 \end{cases} \iff \begin{cases} A_1 = \sqrt{g_{11}} \\ A_2 = \sqrt{g_{22}} \\ r_1 = \frac{l_{11}}{A_1} \\ r_2 = \frac{l_{22}}{A_2} \end{cases}$$

Set $e_1 = \dfrac{f_{x_1}}{A_1}$, $e_2 = \dfrac{f_{x_2}}{A_2}$, and $e_3 = N$. The frame $\{e_1, e_2, e_3\}$ is an orthonormal moving frame on $f$. Then, any vector $v \in \mathbb{R}^3$ can be written as a linear combination of $e_1$, $e_2$, and $e_3$.

$$v = a_1 e_1 + a_2 e_2 + a_3 e_3 = \langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2 + \langle v, e_3 \rangle e_3.$$

Now, since $(e_1)_{x_1}$, $(e_1)_{x_2}$, $(e_2)_{x_1}$, $(e_2)_{x_2}$, $(e_3)_{x_1}$, and $(e_3)_{x_2}$ belong to $\mathbb{R}^3$, we can write them as a linear combination of $e_1$, $e_2$, and $e_3$.

$$\begin{cases} (e_1)_{x_1} = p_{11} e_1 + p_{21} e_2 + p_{31} e_3, \\ (e_2)_{x_1} = p_{12} e_1 + p_{22} e_2 + p_{32} e_3, \\ (e_3)_{x_1} = p_{13} e_1 + p_{23} e_2 + p_{33} e_3, \end{cases}$$

$$\begin{cases} (e_1)_{x_2} = q_{11}e_1 + q_{21}e_2 + q_{31}e_3, \\ (e_2)_{x_2} = q_{12}e_1 + q_{22}e_2 + q_{32}e_3, \\ (e_3)_{x_2} = q_{13}e_1 + q_{23}e_2 + q_{33}e_3, \end{cases}$$

where $p_{ij} = \langle (e_j)_{x_1}, e_i \rangle$ and $q_{ij} = \langle (e_j)_{x_2}, e_i \rangle$. The matrices $P = (p_{ij})$ and $Q = (q_{ij})$ are skew symmetric since

$$p_{ij} + p_{ji} = \langle (e_j)_{x_1}, e_i \rangle + \langle (e_i)_{x_1}, e_j \rangle = \langle e_j, e_i \rangle_{x_1} = 0.$$

Similarly, $q_{ij} = -q_{ji}$. Thus, we have

$$P = \begin{pmatrix} 0 & p_{12} & p_{13} \\ -p_{12} & 0 & p_{23} \\ -p_{13} & -p_{23} & 0 \end{pmatrix} \quad \textit{and} \quad Q = \begin{pmatrix} 0 & q_{12} & q_{13} \\ -q_{12} & 0 & q_{23} \\ -q_{13} & -q_{23} & 0 \end{pmatrix}.$$

Now,

$$\begin{aligned} p_{12} &= \langle (e_2)_{x_1}, e_1 \rangle = \left\langle \left( \frac{f_{x_2}}{A_2} \right)_{x_1}, \frac{f_{x_1}}{A_1} \right\rangle = \left\langle \frac{f_{x_2 x_1}}{A_2} - \frac{f_{x_2}(A_2)_{x_1}}{A_2^2}, \frac{f_{x_1}}{A_1} \right\rangle \\ &= \frac{\langle f_{x_2 x_1}, f_{x_1} \rangle}{A_1 A_2} - \frac{(A_2)_{x_1}}{A_2^2 A_1} \langle f_{x_2}, f_{x_1} \rangle = \frac{\frac{1}{2} \langle f_{x_1}, f_{x_2} \rangle_{x_2}}{A_1 A_2} = \frac{\frac{1}{2}(A_1)^2_{x_2}}{A_1 A_2} = \frac{A_1 (A_1)_{x_2}}{A_1 A_2} \end{aligned}$$

Hence,

$$p_{12} = \frac{(A_1)_{x_2}}{A_2},$$

$$
\begin{aligned}
p_{31} &= \langle (e_1)_{x_1}, e_3 \rangle = \left\langle \left(\frac{f_{x_1}}{A_1}\right)_{x_1}, N \right\rangle = \left\langle \frac{f_{x_1x_1}}{A_1} - \frac{f_{x_1}(A_1)_{x_1}}{A_1^2}, N \right\rangle \\
&= \frac{\langle f_{x_1x_1}, N\rangle}{A_1} - \frac{(A_1)_{x_1}}{A_2^2}\langle f_{x_1}, N\rangle = \frac{r_1A_1}{A_1} = r_1, \\
p_{32} &= \langle (e_2)_{x_1}, e_3 \rangle = \left\langle \left(\frac{f_{x_2}}{A_2}\right)_{x_1}, N \right\rangle = \left\langle \frac{f_{x_2x_1}}{A_2} - \frac{f_{x_2}(A_2)_{x_1}}{A_2^2}, N \right\rangle \\
&= \frac{\langle f_{x_2x_1}, N\rangle}{A_2} - \frac{(A_2)_{x_1}}{A_2^2}\langle f_{x_2}, N\rangle = 0.
\end{aligned}
$$

Similarly, $q_{12} = -\dfrac{(A_2)_{x_1}}{A_1}$, $q_{31} = 0$, and $q_{32} = r_2$, hence

$$
P = \begin{pmatrix} 0 & \frac{(A_1)_{x_2}}{A_2} & -r_1 \\ -\frac{(A_1)_{x_2}}{A_2} & 0 & 0 \\ r_1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & -\frac{(A_2)_{x_1}}{A_1} & 0 \\ \frac{(A_2)_{x_1}}{A_1} & 0 & -r_2 \\ 0 & r_2 & 0 \end{pmatrix}.
$$

Thus,
$$
\begin{cases}
(e_1)_{x_1} = -\frac{(A_1)_{x_2}}{A_2} e_2 + r_1 e_3 \\
(e_2)_{x_1} = \frac{(A_1)_{x_2}}{A_2} e_1 \\
(e_3)_{x_1} = -r_1 e_1 \\
(e_1)_{x_2} = \frac{(A_2)_{x_!}}{A_1} e_2 \\
(e_2)_{x_2} = -\frac{(A_2)_{x_1}}{A_1} e_1 + r_2 e_3 \\
(e_3)_{x_2} = -r_2 e_2.
\end{cases}
$$

In matrix form we have,

$$
\begin{cases}
(e_1, e_2, e_3)_{x_1} = (e_1, e_2, e_3)P \\
(e_1, e_2, e_3)_{x_2} = (e_1, e_2, e_3)Q
\end{cases}.
$$

From Proposition 3.1.4, we have that $PQ - QP = P_{x_2} - Q_{x_1}$. Set $p = \frac{(A_1)_{x_2}}{A_2}$ and $q = -\frac{(A_2)_{x_1}}{A_1}$, we get

$$\begin{aligned} PQ - QP &= \begin{pmatrix} 0 & p & -r_1 \\ -p & 0 & 0 \\ r_1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & q & 0 \\ -q & 0 & -r_2 \\ 0 & r_2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & q & 0 \\ -q & 0 & -r_2 \\ 0 & r_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & p & -r_1 \\ -p & 0 & 0 \\ r_1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -pq & -r_1 r_2 & -r_2 p \\ 0 & -pq & 0 \\ 0 & r_1 q & 0 \end{pmatrix} - \begin{pmatrix} -pq & 0 & 0 \\ -r_1 r_2 & -pq & r_1 q \\ -r_2 p & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -r_1 r_2 & -r_2 p \\ r_1 r_2 & 0 & -r_1 q \\ r_2 p & r_1 q & 0 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} P_{x_2} - Q_{x_1} &= \begin{pmatrix} 0 & p_{x_2} & (-r_1)_{x_2} \\ (-p)_{x_2} & 0 & 0 \\ (r_1)_{x_2} & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & q_{x_1} & 0 \\ (-q)_{x_1} & 0 & (-r_2)_{x_1} \\ 0 & (r_2)_{x_1} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & p_{x_2} - q_{x_1} & (-r_1)_{x_2} \\ -p_{x_2} + q_{x_1} & 0 & (r_2)_{x_1} \\ (r_1)_{x_2} & (-r_2)_{x_1} & 0 \end{pmatrix}. \end{aligned}$$

Since $PQ - QQ = P_{x_2} - Q_{x_1}$, we get

$$\begin{cases} p_{x_2} - q_{x_1} = -r_1 r_2, \\ (\frac{(A_1)_{x_2}}{A_2})_{x_2} + (\frac{(A_2)_{x_1}}{A_1})_{x_1} = r_1 r_2, \\ -p r_2 = -(r_1)_{x_2} \end{cases} \Longrightarrow \begin{cases} (r_1)_{x_2} = p r_2 = (\frac{(A_1)_{x_2}}{A_2}) r_1 \\ q r_1 = (-r_2)_{x_1} \\ (r_2)_{x_1} = -q r_1 = (\frac{(A_2)_{x_1}}{A_1}) r_1 \end{cases}$$

$$\Longrightarrow \begin{cases} (\frac{(A_1)_{x_2}}{A_2})_{x_2} + (\frac{(A_2)_{x_1}}{A_1})_{x_1} = -r_1 r_2, \text{(Gauss Equation)} \\ (r_1)_{x_2} = (\frac{(A_1)_{x_2}}{A_2}) r_2, \text{(Codazzi Equation)} \\ (r_2)_{x_1} = (\frac{(A_2)_{x_1}}{A_1}) r_1. \text{(Codazzi Equation)} \end{cases}$$

This system is called the Gauss-Codazzi Equations. We have proved the following theorem:

**Theorem 3.3.1.** *Let $f : O \longrightarrow \mathbb{R}^3$ be a surface parametrized by line of curvature coordinates. Set $A_1 = \sqrt{g_{11}}, A_2 = \sqrt{g_{22}}, r_1 = \dfrac{l_{11}}{A_1}, r_2 = \dfrac{l_{22}}{A_2}$, $e_1 = \dfrac{f_{x_1}}{A_1}$, $e_2 = \dfrac{f_{x_2}}{A_2}$, and $e_3 = N$. Then $A_1, A_2, r_1$, and $r_2$ satisfy the Gauss-Codazzi equations and we have*

$$\begin{cases} (f, e_1, e_2, e_3)_{x_1} = (e_1, e_2, e_3) \begin{pmatrix} A_1 & 0 & \frac{(A_1)_{x_2}}{A_2} & -r_1 \\ 0 & -\frac{(A_1)_{x_2}}{A_2} & 0 & 0 \\ 0 & r_1 & 0 & 0 \end{pmatrix} \\ \\ (f, e_1, e_2, e_3)_{x_2} = (e_1, e_2, e_3) \begin{pmatrix} 0 & 0 & -\frac{(A_2)_{x_1}}{A_1} & 0 \\ A_2 & \frac{(A_2)_{x_1}}{A_1} & 0 & -r_2 \\ 0 & 0 & r_2 & 0 \end{pmatrix} \end{cases} \quad (3.3.1)$$

## 3.4 Fundamental Theorem of Surfaces in Line of Curvature Coordinates

The converse of Theorem 3.3.1 is also true. It is known as the fundamental theorem of surfaces in $\mathbb{R}^3$ with respect to line of curvature coordinates.

**Theorem 3.4.1.** *Suppose $A_1$, $A_2$, $r_1$, and $r_2$ are smooth functions from $O \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ satisfying the Gauss-Codazzi equations with $A_1 > 0$ and $A_2 > 0$. Given $p_0 \in O$, $y_0 \in \mathbb{R}^3$, and $\{v_1, v_2, v_3\}$ an orthonormal basis of $\mathbb{R}^3$, there exists $O_0 \subseteq O$ such that $p_0 \in O_0$, and there exists a unique solution*

$$(f, e_1, e_2, e_3) : O_0 \longrightarrow (\mathbb{R}^3)^4$$

*of Equation (3.3.1) satisfying the initial condition*

$$(f, e_1, e_2, e_3)(p_0) = (y_0, v_1, v_2, v_3).$$

*Also, $f$ is a parametrized surface with*

$$\begin{aligned} I &= A_1^2 dx_1^2 + A_2^2 dx_2^2, \\ II &= r_1 A_1 dx_1^2 + r_2 A_2 dx_2^2. \end{aligned}$$

*Proof.* From Proposition 3.1.4, $A_1$, $A_2$, $r_1$, and $r_2$ satisfy the Gauss-Codazzi equation if and only if we have the compatibility relation $PQ - QP = P_{x_2} - Q_{x_1}$. From Frobenius Theorem, Theorem 3.1.2, this is equivalent to say that the following system

$$\begin{cases} (e_1, e_2, e_3)_{x_1} = (e_1, e_2, e_3)P \\ (e_1, e_2, e_3)_{x_2} = (e_1, e_2, e_3)Q \end{cases}$$

is solvable. Let $(e_1, e_2, e_3)$ be the solution with initial data $(e_1, e_2, e_3)(p_0) = (v_1, v_2, v_3)$. Since the initial data $(v_1, v_2, v_3)$ is orthogonal, $O \subset \mathbb{R}^2$ and $P, Q$ are smooth maps such that $P^T = -P$, $Q^T = -Q$ and $PQ - QP = P_{x_2} - Q_{x_1}$, then by Proposition 3.1.5), the solution $(e_1, e_2, e_3)(p)$ of the above system is an orthogonal matrix for all $p \in O$. Now, to construct $f$, we need to solve

$$\begin{cases} f_{x_1} = A_1 e_1 \\ f_{x_2} = A_2 e_2 \end{cases} \tag{3.4.1}$$

First, let us check that the System 3.1.5 is solvable. We have

$$\begin{aligned} f_{x_1x_2} &= (A_1e_1)_{x_2} = (A_1)_{x_2}e_1 + A_1(e_1)_{x_2} \\ &= (A_1)_{x_2}e_1 + A_1\frac{(A_2)_{x_1}}{A_1}e_2 = (A_1)_{x_2}e_1 + (A_2)_{x_1}e_2 \\ f_{x_2x_1} &= (A_2e_2)_{x_1} = (A_2)_{x_1}e_2 + A_2(e_2)_{x_1} \\ &= (A_2)_{x_1}e_2 + A_2\frac{(A_1)_{x_2}}{A_2}e_1 = (A_2)_{x_1}e_2 + (A_1)_{x_2}e_1 \end{aligned}$$

Thus, $f_{x_1x_2} = f_{x_2x_1}$ and the System (3.4.1) is solvable. We can solve the System 3.4.1 by integration. From Theorem 3.3.1, it follows that $(f, e_1, e_2, e_3)$ is a solution of the System (3.3.1) with initial data $(y_0, v_1, v_2, v_3)$, where $(f, e_1, e_2, e_3)(p_0) = (y_0, v_1, v_2, v_3)$. We need to check that $f$ is a parametrized surface (In other words, $f_{x_1}$ and $f_{x_2}$ are linearly independent). We have that $f_{x_1} = A_1 e_1$ and $f_{x_2} = A_2 e_2$, then $f_{x_1}$ and $f_{x_2}$ are linearly independent. So, $f$ is a parametrized surface. From Theorem 3.3.1, $e_3 = N$ and hence $g_{11} = A_1^2$ and $g_{22} = A_2^2$ (since $A_1 = \sqrt{g_{11}}$ and $A_2 = \sqrt{g_{22}}$). Also, $g_{12} = g_{21} = 0$ (because $f$ is parametrized by line of curvature). Thus,

$$I = A_1^2 dx_1^2 + A_2^2 dx_2^2.$$

We also have

$$\begin{aligned} l_{11} &= \langle f_{x_1 x_1}, e_3 \rangle = \langle (f_{x_1})_{x_1}, e_3 \rangle = \langle (A_1)_{x_1} e_1 + A_1 (e_1)_{x_1}, e_3 \rangle \\ &= \langle A_1 (e_1)_{x_1}, e_3 \rangle = \Big\langle A_1 \Big( \frac{-(A_1)_{x_2}}{A_2} e_2 + r_1 e_3 \Big), e_3 \Big\rangle = \langle A_1 r_1 e_3, e_3 \rangle = A_1 r_1. \end{aligned}$$

Similarly, $l_{22} = A_2 r_2$.Moreover,

$$l_{21} = l_{12} = \langle f_{x_1 x_2}, e_3 \rangle = \langle (A_1)_{x_2} e_1 + (A_2)_{x_1} e_2, e_3 \rangle = 0.$$

Finally, we get

$$II = A_1 r_1 dx_1^2 + A_2 r_2 dx_2^2.$$

$\square$

**Proposition 3.4.2.** *Suppose $f, g : O \longrightarrow \mathbb{R}^3$ are two surfaces parametrized by line of curvature coordinates. Assume that $f$ and $g$ have the same first and* *second fundamental forms given by*

$$\begin{aligned} I &= A_1^2 dx_1^2 + A_2^2 dx_2^2, \\ II &= A_1 r_1 dx_1^2 + A_2 r_2 dx_2^2. \end{aligned}$$

*Then, there exists a rigid motion $\phi$ of $\mathbb{R}^3$ such that $g = \phi \circ f$.*

*Proof.* Let $e_1 = \dfrac{f_{x_1}}{A_1}$, $e_2 = \dfrac{f_{x_2}}{A_2}$, $e_3 = \dfrac{f_{x_1} \times f_{x_2}}{\|f_{x_1} \times f_{x_2}\|}$, $\xi_1 = \dfrac{g_{x_1}}{A_1}$, $\xi_2 = \dfrac{g_{x_2}}{A_2}$, and $\xi_3 = \dfrac{g_{x_1} \times g_{x_2}}{\|g_{x_1} \times g_{x_2}\|}$. Fix $p_0 \in O$ and let $\phi(x) = Tx + b$ be the rigid motion such that $\phi(p_0) = g(p_0)$ and $T(e_i(p_0)) = \xi_i(p_0)$ for $1 \leq i \leq 3$. Then:

1. $\phi \circ f$ and $f$ have the same $I$ and $II$.

2. The orthonormal frame for $\phi \circ f$ is $(Te_1, Te_2, Te_3)$.

Therefore, $(\phi \circ f, Te_1, Te_2, Te_3)$ and $(g, \xi_1, \xi_2, \xi_3)$ are solutions of (3.3.1) with initial condition $(g(p_0), \xi_1(p_0), \xi_2(p_0), \xi_3(p_0))$. But, by Frobenius Theorem, Theorem 3.1.2, the solution is unique. Hence,

$$(\phi \circ f, Te_1, Te_2, Te_3) = (g, \xi_1, \xi_2, \xi_3),$$

since $Te_1 = \xi_1$, $Te_2 = \xi_2$, and $Te_3 = \xi_3$. This gives that $\phi \circ f = g$. $\square$

## 3.5 The Gauss Theorem in Line of Curvature Coordinates

Usually, we know that the Gaussian curvature $K$ depends on both $I$ and $II$. The Gauss Theorem states that $K$ can be calculated from $I$ alone. This will be shown first for surfaces parametrized by line of curvature coordinates.

**Theorem 3.5.1. (Gauss Theorem in Line of Curvature Coordinates).** *Let $f : O \longrightarrow \mathbb{R}^3$ be a surface parametrized by line of curvature coordinates and*

$$\begin{aligned} I &= A_1^2 dx_1^2 + A_2^2 dx_2^2 \\ II &= A_1 r_1 dx_1^2 + A_2 r_2 dx_2^2. \end{aligned}$$

*Then,*

$$K = -\frac{\left(\frac{(A_1)_{x_2}}{A_2}\right)_{x_2} + \left(\frac{(A_2)_{x_1}}{A_1}\right)_{x_1}}{A_1 A_2}$$

*Therefore, $K$ can be computed from $I$ alone.*

*Proof.* We have

$$K = \frac{\det(l_{ij})}{\det(g_{ij})} = \frac{l_{11} l_{22}}{g_{11} g_{22}} = \frac{r_1 A_1 r_2 A_2}{A_1^2 A_2^2} = \frac{r_1 r_2}{A_1 A_2}.$$

But from the Gauss-Codazzi equations, we have,

$$r_1 r_2 = -\left[\left(\frac{(A_1)_{x_2}}{A_2}\right)_{x_2} + \left(\frac{(A_2)_{x_1}}{A_1}\right)_{x_1}\right]$$

Thus,

$$K = -\frac{\left(\frac{(A_1)_{x_2}}{A_2}\right)_{x_2} + \left(\frac{(A_2)_{x_1}}{A_1}\right)_{x_1}}{A_1 A_2}.$$

□

## 3.6 The Gauss-Codazzi Equations in Local Coordinates

In this section, we will find the Gauss-Codazzi equations for any parametrized surface $f : O \longrightarrow U \subset \mathbb{R}^3$. We use the frame $(f_{x_1}, f_{x_2}, N)$, where $N = \dfrac{f_{x_1} \times f_{x_2}}{\|f_{x_1} \times f_{x_2}\|}$ is the unit normal vector. Since $(f_{x_1}, f_{x_2}, N)$ is a basis for $\mathbb{R}^3$, then the derivatives of $f_{x_1}$, $f_{x_2}$, and $N$ can be written as a linear combination of $f_{x_1}$, $f_{x_2}$, and $N$. Thus,

$$\begin{cases} (f_{x_1}, f_{x_2}, N)_{x_1} = (f_{x_1}, f_{x_2}, N)P, \\ (f_{x_1}, f_{x_2}, N)_{x_2} = (f_{x_1}, f_{x_2}, N)Q, \end{cases} \tag{3.6.1}$$

where $P = (p_{ij})$ and $Q = (q_{ij})$ are Gl(3) valued maps. We have

$$\begin{cases} (f_{x_1})_{x_1} = p_{11} f_{x_1} + p_{21} f_{x_2} + p_{31} N, \\ (f_{x_2})_{x_1} = p_{12} f_{x_1} + p_{22} f_{x_2} + p_{32} N, \\ N_{x_1} = p_{13} f_{x_1} + p_{23} f_{x_2} + p_{33} N, \\ (f_{x_1})_{x_2} = q_{11} f_{x_1} + q_{21} f_{x_2} + q_{31} N, \\ (f_{x_2})_{x_2} = q_{12} f_{x_1} + q_{22} f_{x_2} + q_{32} N, \\ N_{x_2} = q_{13} f_{x_1} + q_{23} f_{x_2} + q_{33} N, \end{cases}$$

and we know that

$$\begin{cases} g_{11} = \langle f_{x_1}, f_{x_1} \rangle, \\ g_{21} = g_{12} = \langle f_{x_1}, f_{x_2} \rangle, \\ g_{22} = \langle f_{x_2}, f_{x_2} \rangle, \\ l_{11} = \langle f_{x_1 x_1}, N \rangle, \\ l_{12} = l_{21} = \langle f_{x_1 x_2}, N \rangle, \\ l_{22} = \langle f_{x_1 x_2}, N \rangle. \end{cases}$$

The goal is to express $P$ and $Q$ in terms of $l_{ij}$ and $g_{ij}$. To do so, we need the following propositions.

**Proposition 3.6.1.** *Let $V$ be a vector space with an inner product $\langle ., . \rangle$. Let $\{v_1, v_2, .., v_n\}$ be a basis of $V$ and $g_{ij} = \langle v_i, v_j \rangle$. Let $\xi \in V$ such that $\xi = \sum_{i=1}^{n} x_i v_i$ and $\xi_i = \langle \xi, v_i \rangle$. Then,* $\begin{pmatrix} x_1 \\ x_2 \\ .. \\ .. \\ x_n \end{pmatrix} = G^{-1} \begin{pmatrix} \xi_1 \\ \xi_2 \\ .. \\ .. \\ \xi_n \end{pmatrix}$ *where $G = (g_{ij})$.*

*Proof.* We have

$$\xi_i = \langle \xi, v_i \rangle = \left\langle \sum_{j=1}^{n} x_j v_j, v_i \right\rangle = \sum_{j=1}^{n} x_j \langle v_j, v_i \rangle = \sum_{j=1}^{n} x_j g_{ij} = \sum_{j=1}^{n} g_{ij} x_j$$

Thus, $\begin{pmatrix} \xi_1 \\ \xi_2 \\ .. \\ \xi_n \end{pmatrix} = G \begin{pmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{pmatrix}$ and $\begin{pmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{pmatrix} = G^{-1} \begin{pmatrix} \xi_1 \\ \xi_2 \\ .. \\ \xi_n \end{pmatrix}$. □

**Proposition 3.6.2.** *The following statements are true:*

1. *$P = (p_{ij})$ and $Q = (q_{ij})$ from Proposition 3.6.1 can be written in terms of $l_{ij}$, $g_{ij}$, and the first partial derivatives of $g_{ij}$.*

2. *The entries $p_{ij}$ and $q_{ij}$, for $1 \leq i, j \leq 2$, can be calculated from $I$.*

*Proof.* 1. We know that $f_{x_1x_1} = p_{11}f_{x_1} + p_{21}f_{x_2} + p_{31}N$, where,

$$\begin{cases} p_{11} = \langle f_{x_1x_1}, f_{x_1} \rangle, \\ p_{21} = \langle f_{x_1x_1}, f_{x_2} \rangle, \\ p_{31} = \langle f_{x_1x_1}, N \rangle, \end{cases}$$

and so on. We claim that $\langle f_{x_ix_j}, f_{x_k} \rangle$, $\langle f_{x_ix_j}, N \rangle$, $\langle N_{x_i}, f_{x_j} \rangle$, and $\langle N_{x_i}, N \rangle$ can be expressed in terms of $g_{ij}$, $l_{ij}$, and first partial derivatives of $g_{ij}$. To prove this, we proceed as follows:

$$(g_{ii})_{x_i} = \langle f_{x_i}, f_{x_i} \rangle_{x_i} = \langle f_{x_ix_i}, f_{x_i} \rangle + \langle f_{x_i}, f_{x_ix_i} \rangle = 2\langle f_{x_ix_i}, f_{x_i} \rangle.$$

Hence $\langle f_{x_ix_i}, f_{x_i} \rangle = \frac{1}{2}(g_{ii})_{x_i}$. Similarly, $\dfrac{1}{2}(g_{ii})_{x_j} = \langle f_{x_ix_j}, f_{x_i} \rangle$. Now,

$\langle f_{x_i}, f_{x_j}\rangle = g_{ij}$, so we have

$$\begin{aligned}(g_{ij})_{x_i} &= \langle f_{x_i x_i}, f_{x_j}\rangle + \langle f_{x_i}, f_{x_j x_i}\rangle \\ \Longrightarrow (g_{ij})_{x_j} &= \langle f_{x_i x_j}, f_{x_j}\rangle + \langle f_{x_i}, f_{x_j x_j}\rangle\end{aligned}$$

Thus, we have

$$(g_{ij})_{x_i} - \frac{1}{2}(g_{ii})_{x_j} = \langle f_{x_i x_i}, f_{x_j}\rangle + \langle f_{x_i}, f_{x_j x_i}\rangle - \langle f_{x_i x_i}, f_{x_j}\rangle = \langle f_{x_i x_i}, f_{x_j}\rangle$$

By definition, $\langle f_{x_i x_j}, N\rangle = l_{ij}$ and $\langle N_{x_i}, f_{x_j}\rangle = -l_{ij}$. Also, clearly $\langle N_{x_i}, N\rangle = 0$. So, $\langle f_{x_i x_j}, f_{x_k}\rangle$, $\langle f_{x_i x_j}, N\rangle$, $\langle N_{x_i}, f_{x_j}\rangle$, and $\langle N_{x_i}, N\rangle$ can be written in terms of $g_{ij}$, $l_{ij}$, and the first partial derivatives of $g_{ij}$.

2. Now, let $G = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{12} & g_{22} & 0 \\ 0 & 0 & 1\end{pmatrix}$.

By Proposition 3.6.1, we have

$$\begin{cases} P = \begin{pmatrix} g^{11} & g^{12} & 0 \\ g^{12} & g^{22} & 0 \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} \frac{1}{2}(g_{11})_{x_1} & \frac{1}{2}(g_{11})_{x_2} & -l_{11} \\ (g_{12})_{x_1} - \frac{1}{2}(g_{11})_{x_2} & \frac{1}{2}(g_{22})_{x_1} & -l_{12} \\ l_{11} & l_{12} & 0\end{pmatrix} = G^{-1}A_1 \\ \\ Q = \begin{pmatrix} g^{11} & g^{12} & 0 \\ g^{12} & g^{22} & 0 \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} \frac{1}{2}(g_{11})_{x_2} & (g_{12})_{x_2} - \frac{1}{2}(g_{22})_{x_1} & -l_{12} \\ \frac{1}{2}(g_{22})_{x_1} & \frac{1}{2}(g_{22})_{x_2} & -l_{12} \\ l_{12} & l_{22} & 0\end{pmatrix} = G^{-1}A_2. \end{cases} \tag{3.6.2}$$

This proves the proposition.

□

The entries of $P$ and $Q$ in terms of $g_{ij}$ and $l_{ij}$ are related to the so-called Christofell symbols $\Gamma^k_{jk} = \frac{1}{2} g^{km}[ij, m]$, where:

- $(g^{ij})$ is the inverse matrix of $(g_{ij})$.
- $[ij, k] = g_{ki,j} + g_{jk,i} - g_{ij,k}$
- $g_{ij,k} = \dfrac{\partial g_{ij}}{\partial x_k}$

**Theorem 3.6.3.** *For* $1 \le i, j \le 2$, *we have:*

$$\begin{cases} p_{ji} = \Gamma^j_{i1} \\ g_{ji} = \Gamma^j_{i2} \end{cases} \tag{3.6.3}$$

*Proof.* From Proposition 3.6.2, we have

$$\begin{aligned}
p_{11} &= g^{11}\left(\frac{1}{2}(g_{11})_{x_1}\right) + g^{12}\left((g_{12})_{x_1} - \frac{1}{2}(g_{11})_{x_2}\right) \\
&= \frac{1}{2} g^{11} g_{11,1} + g^{12} g_{12,1} - \frac{1}{2} g^{12} g_{11,2} \\
\Gamma^1_{11} &= \frac{1}{2} g^{1m}[11, m] = \frac{1}{2} g^{11}[11, 1] + \frac{1}{2} g^{12}[11, 2] \\
&= \frac{1}{2} g^{11} g_{11,1} + \frac{1}{2} g^{12}\left(g_{21,1} + g_{12,1} - g_{11,2}\right) \\
&= \frac{1}{2} g^{11} g_{11,1} + g^{12} g_{12,1} - \frac{1}{2} g^{12} g_{11,2}.
\end{aligned}$$

Thus, $p_{11} = \Gamma^1_{11}$. In a similar way,

$$\begin{aligned}
p_{12} &= \frac{1}{2}g^{11}(g_{11})_{x_2} + \frac{1}{2}g^{12}(g_{22})_{x_1} = \frac{1}{2}g^{11}g_{11,2} + \frac{1}{2}g^{12}g_{22,1} \\
\Gamma^1_{21} &= \frac{1}{2}g^{11}[21,1] + \frac{1}{2}g^{12}[21,2] \\
&= \frac{1}{2}g^{11}(g_{12,1} + g_{11,2} - g_{21,1}) + \frac{1}{2}g^{12}(g_{22,1}) \\
&= \frac{1}{2}g^{11}g_{11,2} + \frac{1}{2}g^{12}g_{22,1}.
\end{aligned}$$

Thus, $p_{12} = \Gamma^1_{21}$. Now,

$$\begin{aligned}
p_{21} &= \frac{1}{2}g^{12}g_{11,1} + g^{22}g_{12,1} - frac12g^{22}g_{11,2} \\
\Gamma^2_{11} &= \frac{1}{2}g^{21}[11,1] + \frac{1}{2}g^{22}[11,2] \\
&= \frac{1}{2}g^{12}g_{11,1} + \frac{1}{2}g^{22}(g_{21,1} + g_{12,1} - g_{11,,2}) \\
&= \frac{1}{2}g^{12}g_{11,1} + g^{22}g_{12,1} - \frac{1}{2}g^{22}g_{11,2}.
\end{aligned}$$

Thus, $p_{21} = \Gamma^2_{11}$. Also,

$$\begin{aligned}
p_{22} &= \frac{1}{2}g^{12}g_{11,2} + \frac{1}{2}g^{22}g_{22,1} \\
\Gamma^2_{21} &= \frac{1}{2}g^{21}[21,1] + \frac{1}{2}g^{22}[21,2] = \frac{1}{2}g^{21}g_{11,2} + \frac{1}{2}g^{22}g_{22,1}.
\end{aligned}$$

Thus, $p_{22} = \Gamma^2_{21}$ and hence $p_{ji} = \Gamma^j_{i1}$ for $1 \leq i, j \leq 2$. Moreover,

$$\begin{aligned}
q_{11} &= \frac{1}{2}g^{11}g_{11,2} + \frac{1}{2}g^{12}g_{22,1} \\
\Gamma^1_{12} &= \frac{1}{2}g^{11}[12,1] + \frac{1}{2}g^{12}g^{12}[12,2] = \frac{1}{2}g^{11}g_{11,2} + \frac{1}{2}g^{12}g_{22,1}.
\end{aligned}$$

Thus, $q_{11} = \Gamma^1_{12}$. Also,

$$\begin{aligned}
q_{12} &= g^{11}g_{12,2} - \frac{1}{2}g^{11}g_{22,1} + \frac{1}{2}g^{12}g_{22,2} \\
\Gamma^1_{22} &= \frac{1}{2}g^{11}[22,1] + \frac{1}{2}g^{12}[22,2] \\
&= \frac{1}{2}g^{11}(g_{12,2} + g_{21,2} - g_{22,1}) + \frac{1}{2}g^{12}g_{22,2} \\
&= g^{11}g_{12,2} - \frac{1}{2}g^{11}g_{22,1} + \frac{1}{2}g^{12}g_{22,2}.
\end{aligned}$$

Thus, $q_{12} = \Gamma^1_{22}$. Now,

$$\begin{aligned}
q_{21} &= \frac{1}{2}g^{12}g_{11,2} + \frac{1}{2}g^{22}g_{22,1} \\
\Gamma^2_{12} &= \frac{1}{2}g^{21}[12,1] + \frac{1}{2}g^{22}[12,2] = \frac{1}{2}g^{12}g_{11,2} + \frac{1}{2}g^{22}g_{22,1}.
\end{aligned}$$

Thus, $q_{21} = \Gamma^2_{12}$. Also,

$$\begin{aligned}
q_{22} &= g^{12}g_{12,2} - \frac{1}{2}g^{12}g_{22,1} + \frac{1}{2}g^{22}g_{22,2} \\
\Gamma^2_{22} &= \frac{1}{2}g^{21}[22,1] + \frac{1}{2}g^{22}[22,2] \\
&= \frac{1}{2}g^{21}(g_{12,2} + g_{21,2} - g_{22,1}) + \frac{1}{2}g^{22}g_{22,2} \\
&= g^{12}g_{12,2} - \frac{1}{2}g^{12}g_{22,1} + \frac{1}{2}g^{22}g_{22,2}.
\end{aligned}$$

Thus, $q_{22} = \Gamma^2_{22}$ and hence $q_{ji} = \Gamma^j_{i2}$ for $1 \leq i, j \leq 2$. Note that: $q_{11} = p_{12}$ and $q_{21} = p_{22}$. □

**Theorem 3.6.4. (The Fundamental Theorem of Surfaces in $\mathbb{R}^3$**).

- *Let $f : O \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a parametrized surface with $g_{ij}$ and $l_{ij}$ being the coefficients of $I$ and $II$. Let $P$ and $Q$ be smooth* Gl(3)*-valued maps* *defined in terms of* $g_{ij}$ *and* $l_{ij}$ *as in Proposition 3.6.2. Then,* $P$ *and* $Q$ *satisfy*

$$P_{x_2} - Q_{x_1} = [P, Q] \tag{3.6.4}$$

*(Already proved in Proposition 3.1.4)*

- *Conversely, let* $O$ *be open in* $\mathbb{R}^2$*. Let* $g_{ij}, l_{ij} : O \longrightarrow \mathrm{Gl}(2)$ *be smooth maps such that* $(g_{ij})$ *is positive definite and* $(l_{ij})$ *is symmetric. Also,* $P, Q : U \longrightarrow Gl(3)$ *are the maps defined as in Proposition 3.6.2. Suppose* $P$ *and* $Q$ *satisfy* $P_{x_2} - Q_{x_1} = [P, Q] = PQ - QP$*. Let* $(x_1^0, x_2^0) \in O$ *and* $p_0 \in \mathbb{R}^3$ *with* $\{u_1, u_2, u_3\}$ *a basis for* $\mathbb{R}^3$ *such that* $\langle u_i, u_j \rangle = g_{ij}(x_1^0, x_2^0)$ *and* $\langle u_i, u_3 \rangle = 0$ *for* $1 \leq i, j \leq 2$*. Then, there exists* $O_0 \subseteq O$ *open of* $(x_1^0, x_2^0)$ *and a unique immersion* $f : O_0 \longrightarrow \mathbb{R}^3$ *such that* $f$ *maps* $O_0$ *homeomorphically to* $f(O_0)$ *and*

  1. $I$ *and* $II$ *of* $f(O_0)$ *are given by* $(g_{ij})$ *and* $(l_{ij})$ *respectively.*
  2. $f(x_1^0, x_2^0) = p_0$ *and* $f(x_1^0, x_2^0) = u_i$ *for* $i = 1, 2$*.*

*Proof.* Assume $P_{x_2} - Q_{x_1} = [P, Q]$. Then, by the Frobenius Theorem, Theorem 3.1.2, the system:

$$\begin{cases} (v_1, v_2, v_3)_{x_1} = (v_1, v_2, v_3)P \\ (v_1, v_2, v_3)_{x_2} = (v_1, v_2, v_3)Q \\ (v_1, v_2, v_3)(x_1^0, x_2^0) = (u_1, u_2, u_3) \end{cases} \tag{3.6.5}$$

has a unique local solution. We want to solve

$$\begin{cases} f_{x_1} = v_1, \\ f_{x_2} = v_2, \\ f(x_1^0, x_2^0) = p_0. \end{cases} \tag{3.6.6}$$

The compatibility relation is $f_{x_1 x_2} = f_{x_2 x_1}$. We have

$$\begin{aligned} (v_1)_{x_2} &= (v_2)_{x_1}, \\ (v_1)_{x_2} &= \sum_{j=1}^{3} q_{j1} v_j = q_{11} v_1 + q_{21} v_2 + q_{31} v_3, \\ (v_2)_{x_1} &= \sum_{j=1}^{3} p_{j2} v_j = p_{12} v_1 + p_{22} v_2 + p_{32} v_3. \end{aligned}$$

From Proposition 3.6.2, $q_{11} = p_{12}$, $q_{21} = p_{22}$, and $q_{31} = p_{32}$. Then, $(v_1)_{x_2} = (v_2)_{x_1}$. Therefore, there exists a unique solution $f$ for the System (3.6.6). We still need to show that $f$ is an immersion satisfying 1 and 2. We need to show that $v_3 \perp f$, $\|v_3\| = 1$, $\langle f_{x_i}, f_{x_j} \rangle = g_{ij}$, and $\langle (v_3)_{x_i}, f_{x_i} \rangle = -l_{ij}$. First, let's show that the $3 \times 3$ matrix $\phi = (\langle v_i, v_j \rangle)$ is equal to the matrix $G = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{12} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

We have

$$\begin{aligned}
\langle v_i, v_j\rangle_{x_1} &= \langle (v_i)_{x_1}, v_j\rangle + \langle v_i, (v_j)_{x_1}\rangle \\
&= \langle p_{1i}v_1 + p_{2i}v_2 + p_{3i}v_3, v_j\rangle + \langle v_i, p_{1j}v_1 + p_{2j}v_2 + p_{3j}v_3\rangle \\
&= p_{1i}g_{1j} + p_{2i}g_{2j} + p_{3i}g_{3j} + p_{3j}g_{i3} + p_{2j}g_{i2} + p_{1j}g_{i1} \\
&= (GP)_{ji} + (GP)_{ij} \\
&= (GP + (GP)^T)_{ij}.
\end{aligned}$$

Now, from Proposition 3.6.2, $P = G^{-1}A_1$, then $GP = G(G^{-1}A_1) = A_1$ and

$$\begin{aligned}
A_1 + A_1^T &= \begin{pmatrix} \frac{1}{2}(g_{11})_{x_1} & \frac{1}{2}(g_{11})_{x_2} & -l_{11} \\ (g_{12})_{x_1} - \frac{1}{2}(g_{11})_{x_2} & \frac{1}{2}(g_{22})_{x_1} & -l_{12} \\ l_{11} & l_{12} & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}(g_{11})_{x_1} & (g_{12})_{x_1} - \frac{1}{2}(g_{11})_{x_2} & l_{11} \\ \frac{1}{2}(g_{11})_{x_2} & \frac{1}{2}(g_{22})_{x_1} & l_{12} \\ -l_{11} & -l_{12} & 0 \end{pmatrix} \\
&= \begin{pmatrix} (g_{11})_{x_1} & (g_{12})_{x_1} & 0 \\ (g_{12})_{x_1} & (g_{22})_{x_1} & 0 \\ 0 & 0 & 0 \end{pmatrix} = G_{x_1}.
\end{aligned}$$

Thus, $GP + (GP)^T = G_{x_1}$ and so $\phi_{x_1} = G_{x_1}$. Similarly, $\phi_{x_2} = G_{x_2}$, but $\phi(x_1^0, x_2^0) = G(x_1^0, x_2^0)$, so $\phi = G$. In other words, $\langle f_{x_i}, f_{x_j}\rangle = g_{ij}$ and $\langle f_{x_i}, v_3\rangle = 0$. Thus,

1. $f_{x_1}$ and $f_{x_2}$ are linearly independent, and so $f$ is an immersion.

2. $v_3$ is the unit normal to $f$.

3. The first fundamental form $I$ of $f$ is $I = \sum g_{ij}dx_idx_j$.

Now, let's find the second fundamental form $II$ of $f$. We have

$$\begin{aligned}
\langle -(v_3)_{x_1}, v_j\rangle &= \langle -p_{13}v_1 - p_{23}v_2 - p_{33}v_3, v_j\rangle \\
&= (l_{11}g^{11} + l_{12}g^{12})\langle v_1, v_j\rangle + (l_{11}g^{12} + l_{12}g^{22})\langle v_2, v_j\rangle + 0 \\
&= (l_{11}g^{11} + l_{12}g^{12})g_{1j} + (l_{11}g^{12} + l_{12}g^{22})g_{2j} \\
&= l_{11}(g^{11}g_{1j} + g^{12}g_{2j}) + l_{12}(g^{12}g_{1j} + g^{22}g_{2j}) \\
&= l_{11}\delta_{ij} + l_{12}\delta_{2j},
\end{aligned}$$

where $\delta_{ij} = \begin{cases} 1 & for \;\; j = i \\ 0 & for \;\; j \neq i \end{cases}$. Then, $\langle -(v_3)_{x_1}, v_1\rangle = l_{11}$ and $\langle -(v_3)_{x_1}, v_2\rangle = l_{12}$. Similarly, $\langle -(v_2)_{x_2}, v_j\rangle = l_{2j}$. Therefore, $II = \sum l_{ij}dx_idx_j$. The System (3.6.4) with $P$ and $Q$ defined by (3.6.2) is called the system of Gauss-Codazzi equations for the surface $f(O)$, which is a second order PDE with 9 equations for 6 functions $g_{ij}$ and $l_{ij}$. Also, from Propositions (3.6.1), (3.6.2), and (3.6.3), we get the following

$$\begin{cases} f_{x_ix_1} = \sum_{j=1}^{2} p_{ji}f_{x_j} + l_{i1}N = \sum_{j=1}^{2} \Gamma^j_{i1}f_{x_j} + l_{i1}N, \\ f_{x_ix_2} = \sum_{j=1}^{2} q_{ji}f_{x_j} + l_{i2}N = \sum_{j=1}^{2} \Gamma^j_{i2}f_{x_j} + l_{i2}N. \end{cases}$$

So, we have,

$$f_{x_ix_j} = \Gamma^1_{ij}f_{x_1} + \Gamma^2_{ij}f_{x_2} + l_{ij}N \tag{3.6.7}$$

□

**Proposition 3.6.5.** *Let $f : O \longrightarrow \mathbb{R}^3$ be a local coordinate system of an embedded surface $M$ in $\mathbb{R}^3$, and $\alpha(t) = f(x_1(y), x_2(t))$. Then, $\alpha$ satisfies the geodesic Equation (3.6.7) if and only if $\alpha''(t)$ is normal to $M$ at $\alpha(t)$ for all $t$.*

*Proof.* Assume $\alpha$ satisfies Equation 3.6.7, then we have

$$\begin{aligned}
\alpha' &= \sum_{i=1}^{2} f_{x_i} x_i', \\
\alpha'' &= \sum_{i,j=1}^{2} f_{x_i x_j} x_i' x_j' + f_{x_i} x_i'' = \sum_{i,j,k=1}^{2} \Gamma_{ij}^k f_{x_k} x_i' x_j' + l_{ij} N + f_{x_i} x_i'' \\
&= \sum_{i,j=1}^{2} (\Gamma_{ij}^k x_i' x_j' + x_k'') f_{x_k} + l_{ij} N = 0 + l_{ij} N = l_{ij} N
\end{aligned}$$

Thus, $\alpha''$ and $N$ are collinear $\iff$ $\alpha''(t)$ is normal to $M$ at $\alpha(t)$ for all $t$. □

## 3.7 The Gauss Theorem

In this section, we will state and prove the Gauss Theorem which revolves around being able to find the Gaussian curvature using only the First Fundamental Form of a surface in $\mathbb{R}^3$. We recall that:

- $P_{x_2} - Q_{x_1} = PQ - QP$ is the Gauss Codazzi equation for a surface $M$ in $\mathbb{R}^3$.
- The Gaussian curvature is given by:

$$K = \frac{l_{11} l_{22} - l_{12}^2}{g_{11} g_{22} - g_{12}^2} = \frac{\det(l_{ij})}{\det(g_{ij})}$$

**Theorem 3.7.1. (The Gauss Theorem).** *The Gaussian curvature of a* *surface in $\mathbb{R}^3$ can be computed from the first fundamental form.*

*Proof.* Let's take the 1,2 entry of $P_{x_2} - Q_{x_1}$ and the 1,2 entry of $PQ - QP = P_{x_2} - Q_{x_1}$. The 1,2 entry of $PQ - QP$ is

$$p_{11}q_{12} + p_{12}q_{22} + p_{13}q_{32} - (q_{11}p_{12} + q_{12}p_{22} + q_{13}p_{32}) = \sum_{j=1}^{3} p_{1j}q_{j2} - \sum_{j=1}^{3} q_{1j}p_{j2}$$

The 1,2 entry of $P_{x_2} - Q_{x_1}$ is $(p_{12})_{x_2} - (q_{12})_{x_1}$. So we have,

$$(p_{12})_{x_2} - (q_{12})_{x_1} = \sum_{j=1}^{3} p_{1j}q_{j2} - \sum_{j=1}^{3} q_{1j} - p_{j2}.$$

It can be also written as

$$(p_{12})_{x_2} - (q_{12})_{x_1} - \sum_{j=1}^{2}(p_{1j}q_{j2} - q_{1j}p_{j2}) = p_{13}q_{32} - q_{13}p_{32}.$$

Now from Proposition 3.6.2, we have

$$\begin{cases} p_{13} = -g^{11}l_{11} - g^{12}l_{12} \\ q_{13} = -g^{11}l_{12} - g^{12}l_{22} \\ p_{32} = l_{12} \\ q_{32} = l_{22} \end{cases} .$$

Thus,

$$\begin{aligned}
& (p_{12})_{x_2} - (q_{12})_{x_1} - \sum_{j=1}^{2} (p_{1j}q_{j2} - q_{1j} - p_{j2}) \\
= \;& (-g^{11}l_{11} - g^{12}l_{12})l_{22} + (g^{11}l_{12} + g^{12}l_{22})l_{12} \\
= \;& -g^{11}l_{11}l_{22} + g^{11}l_{12}l_{12} = -g^{11}(l_{11}l_{22} - l_{12}^2) \\
= \;& -g^{11}(g_{11}g_{22} - g_{12}^2)K,
\end{aligned}$$

where $K = \dfrac{l_{11}l_{22} - l_{12}^2}{g_{11}g_{22} - g_{12}^2}$. Then, $K$ can be written purely in terms of $g_{ij}$ and its derivatives. In fact,

$$K = \frac{l_{11}l_{22} - l_{12}^2}{g_{11}g_{22} - g_{12}^2} = \frac{(p_{12})_{x_2} - (q_{12})_{x_1} - \sum_{j=1}^{2}(p_{1j}q_{j2} - q_{1j}p_{j2})}{-g^{11}(g_{11}g_{22} - g_{12}^2)}$$

□

The equation given by

$$(p_{12})_{x_2} - (q_{12})_{x_1} - \sum_{j=1}^{2}(p_{1j}q_{j2} - q_{1j}p_{j1}) = -g^{11}(g_{11}g_{22} - g_{12}^2)K$$

is called the Gauss equation.

**Remark** A geometric quantity on an embedded surface $M$ in $\mathbb{R}^3$ is called intrinsic if it only depends on the first fundamental form. Otherwise, it is called extrinsic (depending on both $I$ and $II$). The Gaussian curvature and geodesics are intrinsic, whereas the mean curvature is extrinsic.

**Remark** If $\phi : M_1 \longrightarrow M_2$ is a diffeomorphism and $f(x_1, x_2)$ is a local coordinate system on $M_1$, then $\phi \circ f(x_1, x_2)$ is a local coordinate system on $M_2$.

The diffeomorphism $\phi$ is an isometry if the first fundamental forms for $M_1$ and $M_2$ are the same written in terms of $dx_1$ and $dx_2$. In particular:

1. $\phi$ preserves arc length. In other words, the arc length of the curve $\phi(\alpha)$ is the same as that of the curve $\alpha$.

2. $\phi$ preserves angles. In other words, the angle between the curves $\phi(\alpha)$ and $\phi(\beta)$ is the same as the angle between the curves $\alpha$ and $\beta$.

3. $\phi$ maps geodesics to geodesics.

## 3.8 The Gauss-Codazzi Equations in Orthogonal Coordinates

If the local coordinates $x_1$ and $x_2$ are orthogonal (i.e, $g_{12} = 0$), then the Gauss-Codazzi equations $P_{x_2} - Q_{x_1} = [P, Q]$ becomes much simpler. Instead of putting $g_{12} = 0$ in $P_{x_2} - Q_{x_1} = [P, Q]$, we directly derive the Gauss Codazzi equation using an orthonormal moving frame. We write: $g_{11} = A_1^2$, $g_{22} = A_2^2$, and $g_{12} = 0$. Let $e_1 = \dfrac{f_{x_1}}{A_1}$, $e_2 = \dfrac{f_{x_2}}{A_2}$, and $e_3 = N$. Then, $(e_1, e_2, e_3)$ is an orthogonal moving frame on $M$. We have

$$\begin{cases} (e_1, e_2, e_3)_{x_1} = (e_1, e_2, e_3)\tilde{P} \\ (e_1, e_2, e_3)_{x_2} = (e_1, e_2, e_3)\tilde{Q} \end{cases} .$$

Since $(e_1, e_2, e_3)$ is orthogonal, then $\tilde{P}$ and $\tilde{Q}$ are skew symmetric. Also, $\tilde{p_{ij}} = \langle (e_j)_{x_1}, e_i \rangle$ and $\tilde{q_{ij}} = \langle (e_j)_{x_2}, e_i \rangle$. Now, we need to calculate the coefficients of

$\tilde{P}$ and $\tilde{Q}$:

$$\begin{aligned} p_{21} &= \langle (e_1)_{x_1}, e_2 \rangle = \left\langle (\frac{f_{x_1}}{A_1})_{x_1}, \frac{f_{x_2}}{A_2} \right\rangle = \left\langle \frac{f_{x_1 x_1}}{A_1}, \frac{f_{x_2}}{A_2} \right\rangle \\ &= \frac{\langle f_{x_1 x_1}, f_{x_2} \rangle}{A_1 A_2} = \frac{\langle f_{x_1}, f_{x_2} \rangle_{x_1} - \langle f_{x_1}, f_{x_2 x_2} \rangle}{A_1 A_2} \\ &= \frac{0 - \frac{1}{2}\langle f_{x_1}, f_{x_1} \rangle_{x_2}}{A_1 A_2} = \frac{-(\frac{1}{2}A_1^2)_{x_2}}{A_1 A_2} = -\frac{(A_1)_{x_2}}{A_2} \end{aligned}$$

Similarly, we get the coefficients $\tilde{p_{ij}}$ and $\tilde{q_{ij}}$.

$$\begin{aligned} \tilde{P} &= \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} = \begin{pmatrix} 0 & \frac{(A_1)_{x_2}}{A_2} & -\frac{l_{11}}{A_1} \\ -\frac{(A_1)_{x_2}}{A_2} & 0 & -\frac{l_{12}}{A_2} \\ \frac{l_{11}}{A_1} & \frac{l_{12}}{A_2} & 0 \end{pmatrix} \\ \tilde{Q} &= \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{(A_2)_{x_1}}{A_1} & -\frac{l_{12}}{A_1} \\ \frac{(A_2)_{x_1}}{A_1} & 0 & -\frac{l_{22}}{A_2} \\ \frac{l_{12}}{A_1} & \frac{l_{22}}{A_2} & 0. \end{pmatrix} \end{aligned}$$

To get the Gauss-Codazzi equations of the surface parametrized by orthogonal coordinates, we need to compute the $21^{\text{st}}$, $31^{\text{st}}$, and $32^{\text{th}}$ entry of

$$\tilde{P}_{x_2} - \tilde{Q}_{x_1} = [\tilde{P}, \tilde{Q}]$$

- The $21^{\text{st}}$ entry is given by

$$-\frac{(A_1)_{x_2}}{A_2} - \frac{(A_2)_{x_1}}{A_1} = \frac{-l_{12}^2 + l_{11} l_{22}}{A_1 A_2}$$

- The $31^{\text{st}}$ entry is given by

$$\left(\frac{l_{11}}{A_1}\right)_{x_2} - \left(\frac{l_{12}}{A_1}\right)_{x_1} = \frac{l_{12}(A_2)_{x_1}}{A_1A_2} + \frac{l_{22}(A_1)_{x_2}}{A_2^2}$$

- The $32^{\text{th}}$ entry:

$$\left(\frac{l_{12}}{A_2}\right)_{x_2} - \left(\frac{l_{22}}{A_2}\right)_{x_1} = -\frac{l_{11}(A_2)_{x_1}}{A_1^2} - \frac{l_{12}(A_1)_{x_2}}{A_1A_2}$$

The first equation is called the Gauss-Codazzi equation. The Gaussian curvature is given by

$$K = \frac{l_{11}l_{22} - l_{12}^2}{(A_1A_2)^2} = \frac{-\frac{(A_1)_{x_2}}{A_2} - \frac{(A_2)_{x_1}}{A_1}}{A_1A_2}.$$

Now, we see that the Gauss Codazzi equation becomes much simpler in orthogonal coordinates. Can we always find local orthogonal coordinates on a surface in $\mathbb{R}^3$ ? We answer this question by the following theorem which we state without a proof.

**Theorem 3.8.1.** *Suppose $f : O \longrightarrow \mathbb{R}^3$ is a surface. Let $x_0 \in O$ and $Y_1, Y_2 : O \longrightarrow \mathbb{R}^3$ be smooth maps such that $Y_1(x_0)$ and $Y_1(x_0)$ are linearly independent and tangent to $M = f(O)$ at $f(x_0)$. Then, there exist open subset $O_0$ of $O$ such that $x_0 \in O$, open subset $O_1$ of $\mathbb{R}^2$, and a diffeomorphism $h : O_1 \longrightarrow O_0$ such that $(f \circ h)_{y_1}$ and $(f \circ h)_{y_2}$ are parallel to $Y_1 \circ h$ and $Y_2 \circ h$ respectively.*

**Remark** The above theorem states that if we have two linearly independent vector fields $Y_1$ and $Y_2$ on a surface, then we can find a local coordinate system $\phi(y_1, y_2)$ such that $\phi_{y_1}$ and $\phi_{y_2}$ are parallel to $Y_1$ and $Y_2$ respectively. Given any local coordinate system $f(x_1, x_2)$ on $M$, we can apply the Gram Schmidt process to $f_{x_1}$ and $f_{x_2}$ to construct smooth orthonormal vector fields $e_1$ and $e_2$ where:

$$\begin{aligned} e_1 &= \frac{f_{x_1}}{\sqrt{g_{11}}}, \\ e_2 &= \frac{\sqrt{g_{11}}(f_{x_2} - \frac{g_{12}}{g_{11}} f_{x_1})}{\sqrt{g_{11}g_{22} - g_{12}^2}}. \end{aligned}$$

By Theorem 3.8.1, there exists new local coordinate system $\widetilde{f}(y_1, y_2)$ so that $\dfrac{\partial \widetilde{f}}{\partial x_1}$ and $\dfrac{\partial \tilde{f}}{\partial x_2}$ are parallel to $e_1$ and $e_2$. So, the first fundamental form written in this coordinate system has the form

$$\widetilde{g_{11}} dy_1^2 + \widetilde{g_{22}} dy_2^2.$$

However, in general we cannot find coordinate system $\widetilde{f}(y_1, y_2)$ so that $e_1$ and $e_2$ are parallel to coordinate vector fields $\frac{\partial \widetilde{f}}{dy_1}$ and $\frac{\partial \widetilde{f}}{dy_2}$ because if we can, then the first fundamental form of the surface is

$$I = dy_1^2 + dy_2^2,$$

which implies that the Gaussian curvature of the surface must be 0.

# Bibliography

[1] Abate M. and Tovena F. Curves and surfaces. Springer-Verlag Mailand, Edition 1(ISBN-13: 978-8847019409, ISBN-10: 8847019400), 2012.

[2] Do Carmo M.P. Differential geometry of curves and surfaces. *Prentice-Hall, Inc., Engle-wood Cliffs, N.J.*, 1976.

[3] Guerrero H.F. A proof of the fundamental theorem of curves in space and its applications. *arXiv: General Mathematics*, 2018.

[4] Terng C.L. Math 162a lecture notes on curves and surfaces. *Lecture notes, University of California at Irvine*, 2005.

[5] Shirfin T. Differential geometry: A first course in curves and surfaces. *Lecture notes, University of Georgia*, 2018.

[6] Orloff J. Mattuck A., Miller H. and Lewis J. 18.03sc differential equations. *Massachusetts Institute of Technology: MIT OpenCourseWare.*

[7] Knill O. Linear algebra and vector analysis. *Open Math Notes, Harvard University*, 2019.

[8] Andrews B. Lectures on differential geometry,. *Lecture notes, Australian National University.*