FUZZY MEASURE SPACES AND FUZZY INTEGRALS: AN OVERVIEW AND AN APPLICATION

A Thesis

presented to

the Faculty of Natural and Applied Sciences

at Notre Dame University-Louaize

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Mathematics

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by

FATEN MAHMOUD

JUNE 2021

Notre Dame University-Louaize,
Zouk Mosbeh, Lebanon

Department of Mathematics and Statistics

Faten Mahmoud

We, the thesis committee for the above candidate for the Master of Science degree, hereby recommend acceptance of this thesis.

Roger Nakad – Thesis Advisor
Department of Mathematics and Statistics

Ziad Rached – Reader
Department of Mathematics and Statistics

This thesis is accepted by the Faculty of Natural and Applied Sciences.

George Eid
Dean of the Faculty of Natural and
Applied Sciences

Abstract of the Thesis

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in

Mathematics

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2021

Fuzzy measure theory is a generalization of classical measure theory. It was first introduced by Lotfi Zadeh in 1965 in his famous paper ”Fuzzy Sets”. After more than 50 years of the existence and development of classical measure theory, mathematicians felt that the additivity property is, in some applications, too restrictive. It is also unrealistic under real and physical conditions where measurement errors are unavoidable. According to Sugeno, fuzzy measures are obtained by replacing the additivity condition of classical mea-

sures with weaker conditions of monotonicity and continuity.

Chapter 1 defines fuzzy measures, semi-continuous fuzzy measures, and λ-fuzzy measures. We show that a λ-fuzzy measure naturally exists on a finite set X. Then, we prove that a non-additive measure is induced from a classical measure by a transformation of range of the classical measure. It is called quassi-measure. Then, other non-additive measures are constructed in different ways. These measures include belief, plausibility, possibility, and necessity measures. We end Chapter 1 by giving some properties of finite fuzzy measures.

In Chapter 2, we define measurable functions on fuzzy measure spaces. Also, we explain what it means for a sequence of measurable functions to converge almost everywhere, pseudo-almost everywhere, almost uniformly, pseudo-almost uniformly, in measure or pseudo in measure to a function.

In Chapter 3, we define a fuzzy integral and give some of its properties. Moreover, we discuss several convergence theorems of fuzzy integral sequences, in addition to the transformation theorem of fuzzy integrals. We end the chapter by defining fuzzy measures using the fuzzy integral.

Finally, in Chapter 4, we give an application of fuzzy measure theory in real life. We apply this theory in areas where human decision-making plays an important role. Students' failure is one of the issues that all academic institutes face. For this problem, there are many interactive and interdependent criteria. As a result, the most important reasons for students' failure are given.

To my family.

Contents

Acknowledgements

No words can express my sincere gratitude to my advisor Dr. Roger Nakad for his valuable guidance, patience, and encouragement throughout my thesis journey. His motivation, vision, and sincerity have inspired me. It was an honor to work and study under his guidance.

I would also like to thank all the members of the Department of Mathematics and Statistics at Notre Dame University. In particular, I want to thank Dr. Ziad Rached for being a member in the jury of my thesis defense. Special thanks to Dr. Jessica Merhej who believed in me and helped me build my critical thinking and problem solving skills throughout the courses she gave me.

I am grateful to my parents for their unconditional love, prayers, and sacrifices.

A thank you deep from the heart goes to my husband Majd. Thank you for supporting me, encouraging me whenever I felt down, believing in me, and never leaving my side.

Chapter 1

Fuzzy Measures

In this chapter, we define fuzzy measures, semi-continuous fuzzy measures, and λ-fuzzy measures (see [1]). We also define quasi-measures that are induced from classical measures. Then, we introduce different non-additive measures such as belief, plausibility, possibility, and necessity measures. At the end of the chapter, we give some properties of finite fuzzy measures.

Throughout the whole thesis, we use the following conventions:

- $0 \times \infty = 0$
- $\frac{1}{\infty} = 0$
- $\infty - \infty = 0$
- $\sum_{i \in \emptyset} a_i = 0$, for every sequence of real numbers $\{a_i\}$
- $\sup_{x \in \emptyset} \{x, x \in [0, \infty]\} = 0$
- $\inf_{x \in \emptyset} \{x, x \in [0, 1]\} = 1$

Remark Let $(E, \mathcal{A})$ be a measurable space where E is a set and $\mathcal{A}$ is a σ-algebra on E. A **classical measure** μ on $(E, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and μ is **additive**, *i.e.*, if $\{A_n\}_n$ is a sequence of $\mathcal{A}$ that are pairwise disjoint, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

1.1 Fuzzy and Semi-Continuous Fuzzy Measures

Let X be a non-empty set and $\mathcal{C}$ a nonempty class of subsets of X. We consider $\mu : \mathcal{C} \longrightarrow [0, +\infty]$ a non-negative, extended real-valued set function defined on $\mathcal{C}$.

Definition 1.1.1. *The function μ is called a* **fuzzy measure** *on $(X, \mathcal{C})$ if and only if:*

1. *$\mu(\emptyset) = 0$ if $\emptyset \in \mathcal{C}$,*
2. **[Monotonicity].** *For every $E, F \in \mathcal{C}$ such that $E \subseteq F$, we have*

$$\mu(E) \leq \mu(F),$$

3. **[Continuity from below].** *Let $\{E_n\}_n$ be a family of elements in $\mathcal{C}$*

such that $E_1 \subseteq E_2 \subseteq \cdots$. *If* $\bigcup_{n=1}^{\infty} E_n \in C$, *then*

$$\lim_{n\to\infty} \mu(E_n) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right).$$

4. **[Continuity from above].**] *Let* $\{E_n\}_n$ *be a family of elements in* C *such that* $E_1 \supseteq E_2 \supseteq \cdots$ *and* $\mu(E_1) < \infty$. *If* $\bigcap_{n=1}^{\infty} E_n \in C$, *then*

$$\lim_{n\to\infty} \mu(E_n) = \mu\left(\bigcap_{n=1}^{\infty} E_n\right).$$

Definition 1.1.2. *The function* μ *is called a* **lower semi-continuous fuzzy measure** *on* (X, C) *if and only if it satisfies conditions* (1), (2) *and* (3) *of Definition 1.1.1 and it is called an* **upper semi-continuous fuzzy measure** *on* (X, C) *if and only if it satisfies conditions* (1), (2) *and* (4) *of Definition 1.1.1. Both of them are simply called* **semi-continuous fuzzy measures**. *Furthermore, we say that a fuzzy measure or a semi-continuous fuzzy measure is* **regular** *if and only if* $X \in C$ *and* $\mu(X) = 1$.

Remark 1. The class C of subsets of X, where μ is defined, is in general a monotone class, semi-ring, ring, algebra, δ-algebra, or a power set of C.

2. The space $(X, \mathcal{F}, \mu)$ is called a fuzzy measure space (or semi-continuous fuzzy measure space) if μ is a fuzzy measure (or semi-continuous fuzzy

measure) on a measurable space $(X, \mathcal{F})$. In this thesis, $(X, \mathcal{F})$ will always represent a measurable space and $X \in \mathcal{F}$.

3. On a semi-ring, the fuzzy measure (or semi-continuous fuzzy measure) abandons the additivity, but reserves the monotonicity, the continuity (or the partial continuity), and vanishing on the empty set.

4. Fuzzy measures and semi-continuous fuzzy measures are not additive in general.

5. As for classical measures, the same concepts of finiteness and δ-finiteness can be defined for fuzzy measures and semi-continuous fuzzy measures. In fact, let μ be a fuzzy measure. We say that μ is finite if and only if $\mu(X) < \infty$ (finiteness). Also, μ is δ- finite if and only if $X = \bigcup_{n=1}^{\infty} E_n$ where $E_n \in C$ and $\mu(E_n)$ finite(δ-finiteness).

Proposition 1.1.3. *On a semi-ring, any classical measure is a fuzzy measure.*

Proof. Let X be a set and $\mu : C \to \mathbb{R}$ is a classical measure (here C is a σ-algebra). Let's prove μ a fuzzy measure. Of course, we have $\mu(\emptyset) = 0$. Now, let $E, F \in C$ such that $E \subseteq F$. We know that $F = E \cup (F \setminus E)$. Thus,

$$\mu(F) = \mu(E) + \mu(F \setminus E) \geq \mu(E) + 0 = \mu(E).$$

Hence, $\mu(E) \leq \mu(F)$. Take now a family $\{E_n\}_n$ of elements in C such that $E_1 \subseteq E_2 \subseteq \cdots$ and $\bigcup_{n=1}^{\infty} E_n \in C$. Let's prove that $\lim_{n\to\infty} \mu(E_n) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right)$. We consider a new family $\{B_n\}_n$ of elements in C defined by $B_1 = E_1$ and for

$n \geq 2$, $B_n = E_n \setminus \bigcup_{k=1}^{n-1} E_k$. One can easily check that

$$\begin{cases} B_n \in \mathcal{C} \text{ and } \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} E_n, \\ \bigcup_{n=1}^{N} B_n = \bigcup_{n=1}^{N} E_n, \\ \{B_n\}_n \text{ are pairwise disjoint.} \end{cases}$$

Since $\{E_n\}_n$ is an increasing sequence, we have $B_n = E_n \setminus E_{n-1}$ and so $\bigcup_{n=1}^{N} B_n = \bigcup_{n=1}^{N} E_n = E_N$. Thus,

$$\mu\left(\bigcup_{n=1}^{N} E_n\right) = \mu\left(\bigcup_{n=1}^{N} B_n\right) = \mu(E_N).$$

We then have

$$\begin{aligned} \lim_{N\to\infty} \mu(E_N) &= \lim_{N\to\infty} \mu\left(\bigcup_{n=1}^{N} E_n\right) = \lim_{N\to\infty} \mu\left(\bigcup_{n=1}^{N} B_n\right) \\ &= \lim_{N\to\infty} \left(\sum_{n=1}^{N} \mu(B_n)\right) = \sum_{n=1}^{\infty} \mu(B_n) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right). \end{aligned}$$

In a similar way, one can prove the continuity from above. □

Example 1.1.4. *Let μ be the Dirac measure on $(X, \mathcal{P}(X))$, i.e., for any $E \in \mathcal{P}(X)$, we have*

$$\mu(E) = \begin{cases} 1 & \text{if } x_0 \in E, \\ 0 & \text{if } x_0 \notin E. \end{cases}$$

where x_0 is a fixed point on X. This set function μ is a regular fuzzy measure. In fact,

1. $\mu(\emptyset) = 0$ because $x_0 \notin \emptyset$.

2. **[Monotonicity].** Assume that $E \subseteq F$. If $x_0 \in E$, then $x_0 \in F$ and we have $\mu(E) = 1 = \mu(F)$. If $x_0 \notin E$, we have $\mu(E) = 0$. In this case, we have $\mu(F) = 0$ or 1 depending if x_0 belongs or not to F. In both cases, we have $\mu(E) = 0 \leq \mu(F)$.

3. **[Continuity from below].** Consider a family of elements $\{E_n\}_n$ in $\mathcal{F}$ such that $E_1 \subseteq E_2 \subseteq \cdots$ and $\bigcup_{n=1}^{\infty} E_n \in \mathcal{P}(X)$.

   Case 1. If $x_0 \in \bigcup_{n=1}^{\infty} E_n$, then $\mu(\bigcup_{n=1}^{\infty} E_n) = 1$ and $\exists\, n_1$ such that $x_0 \in E_{n_1}$. Since $\{E_n\}_n$ is increasing, it means that $x_0 \in E_n$, for every $n \geq n_1$. Thus, $\forall n \geq n_1$, we have $\mu(E_n) = 1$. This implies that $\lim_{n\to\infty} \mu(E_n) = 1$ and so $\lim_{n\to\infty} \mu(E_n) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = 1$.

   Case 2. If $x_0 \notin \bigcup_{n=1}^{\infty} E_n$, then $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = 0$ and $\forall n \geq 1$, we have $x_0 \notin E_n$. Thus, $\mu(E_n) = 0$, $\forall n \geq 1$. This implies that $\lim_{n\to\infty} \mu(E_n) = 0$ and so $\lim_{n\to\infty} \mu(E_n) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = 0$.

4. **[Continuity from above].** Consider a family of elements $\{E_n\}_n$ in $\mathcal{P}(X)$ such that $E_1 \supseteq E_2 \supseteq \cdots$ such that $\mu(E_1) < \infty$ and $\bigcap_{n=1}^{\infty} E_n \in \mathcal{P}(X)$.

Case 1. *If* $x_0 \in \bigcap_{n=1}^{\infty} E_n$, *then* $x_0 \in E_n$ *for every* n *and*

$$\mu\Big(\bigcap_{n=1}^{\infty} E_n\Big) = \lim_n \mu(E_n) = 1.$$

Case 2. *If* $x_0 \notin \bigcap_{n=1}^{\infty} E_n$, *then* $\mu\Big(\bigcap_{n=1}^{\infty} E_n\Big) = 0$. *Since* $x_0 \notin \bigcap_{n=1}^{\infty} E_n$, *then* $\exists n_1$ *such that* $x_0 \notin E_{n_1}$. *Because* $\{E_n\}_n$ *is decreasing, it means that* $x_0 \notin E_n$ *for every* $n \geq n_1$. *Thus,* $\mu(E_n) = 0$ *for every* $n \geq n_1$ *and so* $\lim_{n\to\infty} \mu(E_n) = 0$. *This implies that*

$$\mu\Big(\bigcap_{n=1}^{\infty} E_n\Big) = \lim_{n\to\infty} \mu(E_n) = 0.$$

Because $\mu(X) = 1$, μ *is regular.*

Theorem 1.1.5. *Let* X *be a non-empty* **finite** *set and* C *be a nonempty class of subsets of* X. *We consider* $\mu : C \longrightarrow [0, +\infty]$ *a non-negative, extended real-valued set function defined on* C. *Then the* **continuity from above** *and the* **continuity from below** *are automatically satisfied.*

Proof. Let $X = \{a_0, \cdots, a_n\}$ be a finite set. Take $E_1 \subseteq E_2 \subseteq \cdots$ such that $\bigcup_{n=1}^{\infty} E_n \in C$. Since X is finite, the sequence is stationary, *i.e.,* we have $E_n = E_\alpha$, for some α and for all $n \geq \alpha$. Thus,

$$\lim_{n\to\infty} \mu(E_n) = \sup \mu(E_n) = \mu(E_\alpha) = \mu\Big(\bigcup_{j=1}^{\alpha} E_j\Big) = \mu\Big(\bigcup_{j=1}^{\infty} E_j\Big).$$

Therefore, μ is continuous from below. Now take $E_1 \supseteq E_2 \supseteq \cdots$ such that

$\bigcap_{n=1}^{\infty} E_n \in \mathcal{C}$. Again, the sequence is stationary, *i.e.,* we have $E_n = E_\beta$, for some β and for all $n \geq \beta$. Thus,

$$\lim_{n\to\infty} \mu(E_n) = \inf \mu(E_n) = \mu(E_\beta) = \mu\left(\bigcap_{j=1}^{\beta} E_j\right) = \mu\left(\bigcap_{j=1}^{\infty} E_j\right).$$

Therefore, μ is continuous from above. □

Example 1.1.6. *Let $X = \{1, 2, \cdots, n\}$ and $\mathcal{C} = \mathcal{P}(X)$. For $E \in \mathcal{C}$, we define μ by*

$$\mu(E) = \left(\frac{\text{Card}(E)}{n}\right)^2,$$

where Card(E) *is the cardinality of E. Then μ is a regular fuzzy measure. In fact,*

1. *$\emptyset \in \mathcal{C}$ and we have $\mu(\emptyset) = \left(\frac{\text{Card}(\emptyset)}{n}\right)^2 = 0$*
2. *Let $E \in \mathcal{C}$, $F \in \mathcal{C}$ such that $E \subseteq F$. Of course, we have* Card$(E) \leq$ Card(F) *which gives that $\mu(E) \leq \mu(F)$.*
3. *By Theorem 1.1.5 and since X is finite, the continuity from above and below are satisfied.*

   Clearly, μ is regular.

Example 1.1.7. *Let f be a non-negative, extended, real- valued function defined on $X = (-\infty, +\infty)$. We define for every $E \in \mathcal{P}(X)$,*

$$\mu(E) = \sup_{x\in E} f(x).$$

We have that μ is a fuzzy measure. In fact,

1. $\mu(\emptyset) = \sup_{x\in\emptyset} \boldsymbol{f}(x) = 0$

2. *Let* $E \in \mathcal{P}(X), F \in \mathcal{P}(X)$ *such that* $E \subseteq F$. *We have*

$$\mu(E) = \sup_{x\in E} \boldsymbol{f}(x) \leq \sup_{x\in F} \boldsymbol{f}(x) = \mu(F)$$

3. *Let* $\{E_n\} \subseteq \mathcal{P}(X)$ *such that* $E_1 \subseteq E_2 \subseteq \cdots$ *and* $\bigcup_{n=1}^{\infty} E_n \in \mathcal{P}(X)$. *We have*

$$\lim_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} \sup_{x\in E_n} \boldsymbol{f}(x) = \sup_{x\in E_{\boldsymbol{i}}} \boldsymbol{f}(x)$$

and

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sup_{x\in \bigcup_{n=1}^{\infty} E_n} \boldsymbol{f}(x) = \sup_{x\in E_{\boldsymbol{i}}} \boldsymbol{f}(x)$$

where $\boldsymbol{i} = 1, 2, \cdots,$ *or n.*

Proposition 1.1.8. *Let* μ *be a regular fuzzy measure (respectively a regular upper semi-continuous fuzzy measure or a regular lower semi-continuous fuzzy measure) on* $(X, \boldsymbol{R})$ *where* $\boldsymbol{R}$ *is an algebra of subsets of* X. *For any* $E \in \boldsymbol{R}$, *we define a set function* v *on* $(X, \boldsymbol{R})$ *by*

$$v(E) = 1 - \mu(\overline{E}),$$

where $\overline{E}$ *denotes the complement of* E *in* X. *Clearly,* v *is also a regular fuzzy measure (respectively a regular upper semi-continuous fuzzy measure or regular lower semi-continuous fuzzy measure). The fuzzy measure* v *is called a dual fuzzy measure (respectively a dual upper semi-continuous fuzzy measure or a* *dual lower semi-continuous fuzzy measure semi-continuous fuzzy measure).*

Proof. It is obvious to prove that $v(E)$ is a regular fuzzy measure. □

1.2 λ-Fuzzy Measures

Definition 1.2.1. *Let μ be a fuzzy measure on $(X, \mathcal{C})$.*

- *We say that μ satisfies the λ-rule on $\mathcal{C}$ if and only if there exists $\lambda \in \left(\frac{-1}{\sup \mu}, \infty\right) \cup \{0\}$, where $\sup \mu = \sup_{E \in \mathcal{C}} \mu(E)$, such that*

  $$\mu(E \cup F) = \mu(E) + \mu(F) + \lambda\mu(E)\mu(F),$$

  whenever $E \in \mathcal{C}$, $F \in \mathcal{C}$, $E \cup F \in \mathcal{C}$ and $E \cap F = \emptyset$.

- *We say that μ satisfies the finite λ-rule on $\mathcal{C}$ if and only if there exists λ such that*

  $$\mu\left(\bigcup_{i=1}^{n} E_i\right) = \begin{cases} \frac{1}{\lambda}\left[\prod_{i=1}^{n}\left(1 + \lambda\mu(E_i)\right) - 1\right] & \text{if } \lambda \neq 0, \\ \sum_{i=1}^{n} \mu(E_i), & \text{if } \lambda = 0, \end{cases}$$

  for any finite disjoint class $\{E_1, \cdots, E_n\}$ of sets in $\mathcal{C}$ whose union is also in $\mathcal{C}$.

- *We say that μ satisfies the $\delta - \lambda$ rule on $\mathcal{C}$ if and only if there exists λ*

such that

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \begin{cases} \frac{1}{\lambda}\left(\prod_{i=1}^{\infty}(1+\lambda\mu(E_i)) - 1\right) & \text{if} \quad \lambda = 0 \\ \sum_{i=1}^{\infty} \mu(E_i) & \text{if} \quad \lambda = 0, \end{cases}$$

for any finite disjoint class $\{E_1, \cdots, E_n\}$ *of sets in* C *whose union is also in* C.

Remark When $\lambda = 0$, the λ-rule is just the additivity rule, the finite λ-rule is the finite additivity rule and the $\delta - \lambda$ rule is just the δ-additivity rule.

Theorem 1.2.2. *If* $C = \boldsymbol{R}$ *is a ring and* μ *satisfies the* λ*-rule, then* μ *satisfies the finite* λ*-rule.*

Proof. When $\lambda = 0$, we have from the previous remark that the finite λ-rule is just the finite additivity. So, μ satisfies the finite λ-rule. When $\lambda = 0$ and $\{E_1, E_2, \cdots, E_n\}$ is a disjoint class of sets in $\boldsymbol{R}$, we will use the mathematical induction to prove that

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) = \frac{1}{\lambda}\left(\prod_{i=1}^{n}[1+\lambda\mu(E_i)] - 1\right). \tag{1.2.1}$$

From the definition of the λ-rule, we know that Equation (1.2.1) is true when $n = 2$. Suppose that Equation (1.2.1) is true for $n = k - 1$ and let's prove it for $n = k$. We have

$$
\begin{aligned}
\mu\left(\bigcup_{i=1}^{n} E_i\right) &= \mu\left(\bigcup_{i=1}^{k-1} E_i \cup E_k\right) \\
&= \mu\left(\bigcup_{i=1}^{k-1} E_i\right)(1 + \lambda\mu(E_k)) + \mu(E_k) \\
&= \frac{1}{\lambda}\left[\prod_{i=1}^{k}(1 + \lambda\mu(E_i)) - 1\right](1 + \lambda\mu(E_k)) + \mu(E_k) \\
&= \frac{1}{\lambda}\left[\prod_{i=1}^{k}(1 + \lambda\mu(E_i)) - (1 + \lambda\mu(E_k))\right] + \mu(E_k) \\
&= \frac{1}{\lambda}\left[\prod_{i=1}^{k}(1 + \lambda\mu(E_i)) - (1 + \lambda\mu(E_k)) + \lambda\mu(E_k)\right] \\
&= \frac{1}{\lambda}\left[\prod_{i=1}^{k}(1 + \lambda\mu(E_i)) - 1\right].
\end{aligned}
$$

So Equation (1.2.1) is true for $n = k$. □

Remark Theorem 1.2.2 is valid also when C is a semi-ring.

Example 1.2.3. *Let $X = \{a, b\}$ and $C = \mathcal{P}(X)$. We define μ as follows:*

$$
\mu(E) = \begin{cases} 0, & \text{if } E = \emptyset \\ 0.2, & \text{if } E = \{a\} \\ 0.4, & \text{if } E = \{b\} \\ 1, & \text{if } E = X \end{cases}
$$

We have $1 = 0.2 + 0.4 + \lambda 0.08$*. Hence,* $\lambda = 5$ *and*

$$
5 \in \left(\frac{-1}{\sup \mu}, \infty\right) \cup \{0\} = (-1, \infty) \cup \{0\}.
$$

Thus μ satisfies the λ-rule with λ = 5. Since C is a finite ring, by Theorem 1.2.2, μ also satisfies the finite λ-rule and the δ − λ rule.

Definition 1.2.4. *μ is called a λ-fuzzy measure on C if and only if it satisfies the δ −λ rule on C and there exists at least one set $E \in C$ such that $\mu(E) < \infty$. The λ-fuzzy measure is denoted by g_λ. When C is a δ-algebra and $g_\lambda(X) = 1$, the λ-fuzzy measure g_λ is also called a* **Sugeno Measure**.

Example 1.2.5. *Let $X = \{x_1, x_2, \cdots\}$ be a countable set, C be the semi-ring consisting of all singletons of X and the empty set. Consider $\{a_i\}$ a sequence of non-negative real numbers. Define μ as follows:*

$$\begin{cases} \mu(\{x_i\}) = a_i \quad \text{for} \quad i = 1, 2, \cdots, \\ \mu(\emptyset) = 0. \end{cases}$$

Then μ is a λ-fuzzy measure for any $\lambda \in \left(\frac{-1}{\sup \mu}, \infty\right) \cup \{0\}$.

Theorem 1.2.6. *If g_λ is a λ-fuzzy measure on a class C containing the empty set ∅, then $g_\lambda(\emptyset) = 0$ and g_λ satisfies the finite λ-rule.*

Proof. From Definition 1.2.4, we know that $\exists\, E \in C$ such that $g_\lambda(E) < \infty$.

Case 1: Assume that $\lambda = 0$. We have that g_λ is a classical measure (from Definition 1.2.1). So $g_\lambda(\emptyset) = 0$.

Case 2: Assume that $\lambda = 0$. Since $\{E, E_2, \cdots\}$, where $E_2 = E_3 = \cdots = \emptyset$, is a disjoint sequence of sets in C, whose union is E, we have

$$g_\lambda(E) = \frac{1}{\lambda}\left[\prod_{i=2}^{\infty}(1 + \lambda g_\lambda(E_i))(1 + \lambda g_\lambda(E)) - 1\right].$$

That is,

$$1+\lambda g_\lambda(E) = (1+\lambda g_\lambda(E)) \left(\prod_{i=2}^{\infty}(1+\lambda g_\lambda(E_i))\right).$$

Note that $\lambda \in \left(\frac{-1}{\sup \mu}, \infty\right)$ and $g_\lambda(E) < \infty$, so

$$0 < 1+\lambda g_\lambda(E) < \infty.$$

Thus we have, $\prod_{i=2}^{\infty}(1+\lambda g_\lambda(E_i)) = 1$ and so

$$1+\lambda g_\lambda(\emptyset) = 1.$$

Hence, $g_\lambda(\emptyset) = 0$. By the above result, the second conclusion is clear. □

Theorem 1.2.7. *If g_λ is a λ-fuzzy measure on a semi-ring $\mathcal{L}$, then g_λ is monotone.*

Proof. If $\lambda = 0$, g_λ is a classical measure, so g_λ is monotone. Assume now that $\lambda \neq 0$. Let $E \in \mathcal{L}$, $F \in \mathcal{L}$ and $E \subseteq F$. Since $\mathcal{L}$ is a semi-ring, $F - E = \bigcup_{i=1}^{n} D_i$, where $\{D_i\}$ is a finite disjoint class of sets in $\mathcal{L}$. We have

$$\frac{1}{\lambda}\left(\prod_{i=1}^{n}(1+\lambda g_\lambda(D_i)) - 1\right) \geq 0,$$

in both cases when $\lambda > 0$ and $\lambda < 0$. By Theorem 1.2.6, g_λ satisfies the finite

λ-rule and we have

$$\begin{aligned} g_\lambda(F) &= g_\lambda(E \cup D_1 \cup \cdots \cup D_n) \\ &= \frac{1}{\lambda}\left[\prod_{i=1}^{n}(1+\lambda g_\lambda(D_i))(1+\lambda g_\lambda(E)) - 1\right] \\ &= g_\lambda(E) + \frac{1}{\lambda}\left[\prod_{i=1}^{n}(1+\lambda g_\lambda(D_i)) - 1\right](1+\lambda g_\lambda(E)) \\ &\geq g_\lambda(E). \end{aligned}$$

□

Remark Any λ-fuzzy measure on a semi-ring possesses the continuity. On a semi-ring, any λ-fuzzy measure is a fuzzy measure.

Definition 1.2.8. *1. We say that μ is subadditive if and only if*

$$\mu(E) \leq \mu(E_1) + \mu(E_2),$$

whenever $E \in C$, $E_1 \in C$, $E_2 \in C$ and $E = E_1 \cup E_2$.

2. We say that μ is superadditive if and only if

$$\mu(E) \geq \mu(E_1) + \mu(E_2),$$

whenever $E \in C$, $E_1 \in C$, $E_2 \in C$, $E_1 \cap E_2 = \emptyset$ and $E = E_1 \cup E_2$

Theorem 1.2.9. *Let g_λ be a λ-fuzzy measure on a semi-ring $\mathbf{L}$. Then, it is subadditive when $\lambda < 0$, superadditive when $\lambda > 0$ and additive when $\lambda = 0$.*

Proof. From Theorems 1.2.6 and 1.2.7, we know that μ satisfies the λ-rule and is monotone. If $\lambda < 0$, we have

$$\mu(E) - \lambda\mu(E_1)\mu(E_2) = \mu(E_1) + \mu(E_2).$$

Since $\lambda\mu(E_1)\mu(E_2)$ is negative, we get

$$\mu(E) \leq \mu(E_1) + \mu(E_2),$$

and so g_λ is subadditive. If $\lambda > 0$, we have

$$\mu(E) - \lambda\mu(E_1)\mu(E_2) = \mu(E_1) + \mu(E_2).$$

Since $\lambda\mu(E_1)\mu(E_2)$ is positive, we get

$$\mu(E) \geq \mu(E_1) + \mu(E_2),$$

so g_λ is superadditive. If $\lambda = 0$, we have $\mu(E) = \mu(E_1) + \mu(E_2)$ and g_λ is additive. □

Theorem 1.2.10. *Let g_λ be a λ-fuzzy measure on a ring R. Then, for any $E \in R$ and $F \in R$, we have*

1. $g_\lambda(E - F) = \dfrac{g_\lambda(E) - g_\lambda(E \cap F)}{1 + \lambda g_\lambda(E \cap F)}$
2. $g_\lambda(E \cup F) = \dfrac{g_\lambda(E) + g_\lambda(F) - g_\lambda(E \cap F) + \lambda g_\lambda(E) g_\lambda(F)}{1 + \lambda g_\lambda(E \cap F)}$

3. *If R is an algebra and g_λ is regular, then we have*

$$g_\lambda(\overline{E}) = \frac{1 - g_\lambda(E)}{1 + \lambda g_\lambda(E)}.$$

Proof. 1. We have

$$\begin{aligned} g_\lambda(E) &= g_\lambda\big((E \cap F) \cup (E \setminus F)\big) \\ &= g_\lambda(E \cap F) + g_\lambda(E \setminus F)\big(1 + \lambda g_\lambda(E \cap F)\big), \end{aligned}$$

so we get $g_\lambda(E \setminus F) = \dfrac{g_\lambda(E) - g_\lambda(E \cap F)}{1 + \lambda g_\lambda(E \cap F)}$.

2. We have

$$\begin{aligned} g_\lambda(E \cup F) &= g_\lambda\big(E \cup (F \setminus (E \cap F))\big) \\ &= g_\lambda(E) + g_\lambda\big(F \setminus (E \cap F)\big)(1 + \lambda g_\lambda(E)) \\ &= g_\lambda(E) + \frac{g_\lambda(F) - g_\lambda(E \cap F)}{1 + \lambda g_\lambda(E \cap F)}(1 + \lambda g_\lambda(E)) \\ &= \frac{g_\lambda(E) + g_\lambda(F) - g_\lambda(E \cap F) + \lambda g_\lambda(E) g_\lambda(F)}{1 + \lambda g_\lambda(E \cap F)} \end{aligned}$$

3. We have

$$g_\lambda(\overline{E}) = g_\lambda(X \setminus E) = \frac{g_\lambda(X) - g_\lambda(X \cap E)}{1 + \lambda g_\lambda(X \cap E)} = \frac{1 - g_\lambda(E)}{1 + \lambda g_\lambda(E)}.$$

□

Lemma 1.2.11. *Let X be a finite set. A λ- fuzzy measure is a nonnegative* *set function* $g_\lambda : P(X) \to [0,1]$ *satisfying:*

$$g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A) g_\lambda(B),$$

for all $A, B \in X$ *whenever* $A \cap B = \emptyset$ *where* $\lambda \in (-1, \infty)$. *Moreover, if* X *finite, i.e.,* $X = \{x_1, x_2, ..., x_n\}$, *then* $g_\lambda(X)$ *can be formulated as follows:*

$$\begin{aligned} g_\lambda(X) &= g_\lambda(x_1, x_2, ..., x_n) \\ &= \sum_{i=1}^{n} g_\lambda(\{x_i\}) + \lambda \sum_{i=1}^{n-1} \sum_{j=2}^{n} g_\lambda(\{x_i\}) g_\lambda(\{x_j\}) \\ &\quad + \lambda^2 \sum_{i=1}^{n-2} \sum_{j=2}^{n-1} \sum_{k=3}^{n} g_\lambda(\{x_i\}) g_\lambda(\{x_j\}) g_\lambda(\{x_k\}) + \cdots \\ &\quad + \lambda^{n-1} g_\lambda(\{x_1\}) g_\lambda(\{x_2\}) \cdots g_\lambda(\{x_n\}) \end{aligned}$$

Example 1.2.12. *We can always construct a* λ*-fuzzy measure on any finite set* X. *In fact, let* $X = \{x_1, \cdots, x_n\}$ *is a finite set and* C *consists of* X *and all singletons of* X. *We define* μ *on* C *as* $\mu(\{x_i\}) < \mu(X) < \infty$ *for* $i = 1, 2, \cdots, n$. *There are at least* 2 *points* x_{i_1} *and* x_{i_2} *satisfying* $\mu(\{x_{i_j}\}) > 0$, *for* $j = 1, 2$. *So, the set function* μ *is always a fuzzy measure on* C *for some parameter* λ.

Remark When $\mu(X) = \sum_{i=1}^{n} \mu(\{x_i\})$ holds, the parameter λ is 0. Otherwise, λ can be determined by the equation

$$\mu(X) = \frac{1}{\lambda} \left(\prod_{i=1}^{n} \left(1 + \lambda \mu(\{x_i\})\right) - 1 \right) \qquad (1.2.2)$$

Theorem 1.2.13. *The equation*

$$1 + \lambda\mu(X) = \left(\prod_{i=1}^{n} (1 + \lambda\mu(\{x_i\})) - 1\right)$$

determines the parameter λ uniquely. In fact,

1. $\lambda > 0$ *when* $\sum_{i=1}^{n} \mu(\{x_i\}) < \mu(X)$
2. $\lambda = 0$ *when* $\sum_{i=1}^{n} \mu(\{x_i\}) = \mu(X)$
3. $\frac{-1}{\mu(X)} < \lambda < 0$ *when* $\sum_{i=1}^{n} \mu(\{x_i\}) > \mu(X)$

Proof. Denote $\mu(X) = a$, $\mu(\{x_i\}) = a_i$ for $i = 1, 2, \cdots, n$ and let

$$f_k(\lambda) = \prod_{i=1}^{k} (1 + a_i\lambda) \quad \text{for} \quad k = 2, \cdots, n.$$

There is no loss of generality in assuming $a_1 > 0, a_2 > 0$. We know that $(1+a_k\lambda) > 0$ for $k = 1, \cdots, n$ and $\lambda \in (\frac{-1}{a}, \infty)$. Since $f_k(\lambda) = (1+a_k\lambda)f_{k-1}(\lambda)$, we get

$$f_k(\lambda) = a_k f_{k-1}(\lambda) + (1 + a_k\lambda)f_{k-1}(\lambda),$$

$$f_k(\lambda) = 2a_k f_{k-1}(\lambda) + (1 + a_k\lambda)f_{k-1}(\lambda),$$

for any $k = 2, \cdots, n$ and any $\lambda \in (\frac{-1}{a}, \infty)$. If $f_{k-1}(\lambda) > 0$ and $f_{k-1}(\lambda) > 0$, then $f_k(\lambda) > 0$ and $f_k(\lambda) > 0$. Since

$$f_2(\lambda) = a_1(1 + a_2\lambda) + a_2(1 + a_1\lambda) > 0,$$

and $f_2(\lambda) = 2a_1a_2 > 0$, then $f_k(\lambda) > 0$. So, $f_n(\lambda)$ is concave in $(\frac{-1}{a}, \infty)$. From the derivation of $f_n(\lambda)$, we have $f_n(0) = \sum_{i=1}^{n} a_i$. Note also that $\lim_{\lambda\to\infty} f_n(\lambda) = \infty$.

Case 1: If $\sum_{i=1}^{n} a_i < a$, then $f_n(0) < g(0)$. So the curve of $f_n(\lambda)$ has a unique intersection point with the line $f(\lambda) = 1 + a\lambda = g(\lambda)$ on some $\lambda > 0$.

Case 2: If $\sum_{i=1}^{n} a_i = a$, means $f_n(0) = g(0)$, so the line $f(\lambda) = 1 + a\lambda$ is just a tangent of $f_n(\lambda)$ at point $\lambda = 0$. Hence, there is only one point of intersection between the curve of $f_n(\lambda)$ and the line $f(\lambda) = 1 + a\lambda$.

Case 3: If $\sum_{i=1}^{n} a_i > a$, then $f_n(0) > g(0)$. Since $f_n(\lambda) > 0$ and $f(\lambda) = 1 + a\lambda \leq 0$ when $\lambda \leq \frac{-1}{a}$, the curve of $f_n(\lambda)$ must have a unique intersection point with the line $f(\lambda) = 1 + a\lambda$ for some $\lambda \in \left(\frac{-1}{a}, 0\right)$. □

Remark
- If there is some x_i such that $\mu(\{x_i\}) = \mu(X)$, then Equation (1.2.2) has infinitely many solutions. This means that μ is a λ-fuzzy measure for any $\lambda \in \left(\frac{-1}{\mu(X)}, \infty\right)$ only when $\mu(\{x_j\}) = 0$ for all $j \neq i$. Otherwise, it has no solution in $\left(\frac{-1}{\mu(X)}, \infty\right)$.
- After determining the value of λ, it is not difficult to extend this λ-fuzzy measure from C onto the power set $\mathcal{P}(X)$ by using the finite λ-rule.

Example 1.2.14. *Let $X = \{a, b, c\}$ and define μ as*

$$\mu(X) = 1, \ \mu(\{a\}) = \mu(\{b\}) = 0.2, \ \ \mu(\{c\}) = 0.1$$

According to Theorem 1.2.13, μ is a λ-fuzzy measure. To calculate λ, using Equation (1.2.2), we have

$$1 = \frac{(1 + 0.2\lambda)(1 + 0.2\lambda)(1 + 0.1\lambda) - 1}{\lambda},$$

that is $0.004(\lambda^2) + 0.08\lambda - 0.5 = 0$. *So we have*

$$\lambda_1 = \frac{-0.08 - \sqrt{0.0064 + 0.008}}{0.008} = -25 \quad or \quad \lambda_2 = \frac{-0.08 + \sqrt{0.0064 + 0.008}}{0.008} = 5.$$

According to Theorem 1.2.13, and since $0.2 + 0.2 + 0.1 < 1$, *we have* $\lambda = 5$.

1.3 Quasi-Measures

Definition 1.3.1. *Let* $a \in (0, \infty]$. *An extended real function* $\theta : [0, a] \to [0, \infty]$ *is called a* **T-function** *if and only if* θ *is continuous, strictly increasing, such that* $\theta(0) = 0$ *and* $\theta^{-1}(\{\infty\}) = \emptyset$ *or* $\{\infty\}$, *depending on the point a being finite or not.*

Definition 1.3.2. 1. *μ is called* **quasi-additive** *if and only if there exists a T-function* θ *whose domain of definition contains the range of* μ, *such that the set function* $\theta \circ \mu$ *defined on* C *by:*

$$(\theta \circ \mu)(E) = \theta(\mu(E)), \text{for any } E \in C,$$

is additive.

2. *μ is called* **quasi-measure** *if and only if there exists a T-function* θ *such that* $\theta \circ \mu$ *is a classical measure on* C. *The T-function* θ *is called the* **proper T-function** *of μ.*

3. *A regular quasi-measure is called a* **quassi-probability**. *It is clear that any classical measure is a quasi-measure with the identity function as its proper T- function.*

Example 1.3.3. *The fuzzy measure, given in Example 1.1.6, is a quasi measure. Its proper T-function is* $\theta(y) = \sqrt{y}, y \in [0, 1]$. *This is because* $\mu(E) = (\frac{\text{Card}E}{n})^2$, *and so* $(\theta \circ \mu)(E) = \frac{\text{Card}E}{n}$ *is a classical measure.*

Theorem 1.3.4. *Any quasi measure on a semi-ring is a quasi-additive fuzzy measure.*

Proof. Let μ be a quasi-measure on a semi-ring $\mathcal{L}$ and θ be its proper T-function. Since any classical measure on a semi-ring is additive, μ is quasi-additive. Furthermore, θ^{-1} exists and it is continuous, strictly increasing, and $\theta^{-1}(0) = 0$. So $\mu = \theta^{-1} \circ (\theta \circ \mu)$ is continuous. And since $\theta \circ \mu$ is monotone, we have

$$\theta \circ \mu(A) \leq \theta \circ \mu(B),$$

so $\theta^{-1} \circ \theta \circ \mu(A) \leq \theta^{-1} \circ \theta \circ \mu(B)$. Thus μ is monotone and $\mu(\emptyset) = 0$. Therefore, μ is fuzzy measure □

Theorem 1.3.5. *If* μ *is a classical measure, then, for any T-function* θ *whose range contains the range of* μ, *we have that* $\theta^{-1} \circ \mu$ *is a quasi-measure with proper T-function given by* θ.

Proof. To prove that $\theta^{-1} \circ \mu$ is a quasi-measure, according to Definition 1.3.2, we need to find a T-function θ_1 such that $\theta_1 \circ (\theta^{-1} \circ \mu)$ is a classical measure. Take $\theta_1 = \theta$. So, $\theta \circ (\theta^{-1} \circ \mu) = \mu$ which is a classical measure. Therefore $\theta^{-1} \circ \mu$ is a quasi-measure with proper T-function given by θ. □

The following theorem gives the relation between quasi-additive and quasi-measure.

Theorem 1.3.6. *Let μ be quasi-additive on a ring R with $\mu(\emptyset) = 0$. If μ is either continuous from below on R, or continuous from above at $\emptyset$ and finite, then μ is a quasi-measure on R.*

Proof. Since μ is quasi-additive, there exists a T-function θ such that $\theta \circ \mu$ is additive on R. The composition $\theta \circ \mu$ is either continuous from below on R or continuous from above at $\emptyset$ and finite. So, $\theta \circ \mu$ is a measure on R. Therefore, by Theorem 1.3.5, μ is a quasi-measure on R. □

Corollary 1.3.7. *Any quasi-additive fuzzy measure on a ring is a quasi-measure.*

Proof. Let μ be a quasi-additive fuzzy measure. So, there exists a T-function θ such that $\theta \circ \mu$ defined by $(\theta \circ \mu)(E) = \theta(\mu(E))$, for any $E \in C$, is additive. Let's prove that $\theta \circ \mu$ is a classical measure. First, $\forall E \in C$, $\theta \circ \mu(E) \geq 0$. Let's prove that $\theta \circ \mu(\emptyset) = 0$. We know that $\mu(\emptyset) = 0$ since μ is a fuzzy measure and $\theta(0) = 0$. So,

$$\theta \circ \mu(\emptyset) = \theta(\mu(\emptyset)) = \theta(0) = 0.$$

Now, take $\{E_k\}_{k=1}^{\infty}$ pairwise disjoint sets in C. Given that $\theta \circ \mu$ is additive, we get

$$(\theta \circ \mu)\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \theta \circ \mu(E_k).$$

Therefore, $\theta \circ \mu$ is a classical measure. □

The following theorem states the relation between λ-fuzzy measures, quasi-measures, and classical measures.

Theorem 1.3.8. *Let $\lambda = 0$. Any λ-fuzzy measure g_λ is a quasi-measure with*

$$\theta_\lambda(y) = \frac{\ln(1 + \lambda y)}{k\lambda},$$

as its proper T-function, where $y \in [0, \sup g_\lambda]$ and k is an arbitrary finite positive real number. Conversely, if μ is a classical measure, then $\theta_\lambda^{-1} \circ \mu$ is a λ-fuzzy measure where

$$\theta_\lambda^{-1}(x) = \frac{e^{k\lambda x - 1}}{\lambda},$$

for $x \in [0, \infty]$ and k is an arbitrary finite positive real number.

Proof. θ_λ is a T-function. Let $\{E_n\}$ be a disjoint sequence of sets in C, whose union $\bigcup_{n=1}^{\infty} E_n$ is also in C. If g_λ is a λ-fuzzy measure on C, then it satisfies the δ-λ rule and $\exists E_0 \in C$ such that $g_\lambda(E_0) < \infty$. Therefore, we have

$$\begin{aligned}
(\theta_\lambda \circ g_\lambda)\left(\bigcup_{n=1}^{\infty} E_n\right) &= \frac{1}{k\lambda} \ln\left(1 + \lambda g_\lambda\left(\bigcup_{n=1}^{\infty} E_n\right)\right) \\
&= \frac{1}{k\lambda} \ln\left(1 + \prod_{n=1}^{\infty}(1 + \lambda g_\lambda(E_n)) - 1\right) \\
&= \frac{1}{k\lambda} \sum_{n=1}^{\infty} \ln(1 + g_\lambda(E_n)) \\
&= \sum_{n=1}^{\infty} \frac{\ln(1 + \lambda g_\lambda(E_n))}{k\lambda} \\
&= \sum_{n=1}^{\infty} (\theta_\lambda \circ g_\lambda)(E_n),
\end{aligned}$$

and $(\theta_\lambda \circ g_\lambda)(E_0) = \theta_\lambda(g_\lambda(E_0)) < \infty$. So, $\theta_\lambda \circ g_\lambda$ is a classical measure on C. Conversely, if μ is a classical measure on C, then it is δ-additive, and $\exists E_0 \in C$

such that $\mu(E_0) < \infty$. Therefore, we have

$$
\begin{aligned}
(\theta_\lambda^{-1} \circ \mu)\left(\bigcup_{n=1}^{\infty} E_n\right) &= \theta_\lambda^{-1}\left(\sum_{n=1}^{\infty} \mu(E_n)\right) \\
&= \frac{e^{k\lambda \left(\sum_{n=1}^{\infty} \mu(E_n)\right)} - 1}{\lambda} \\
&= \frac{\prod_{n=1}^{\infty} e^{k\lambda\mu(E_n)} - 1}{\lambda} \\
&= \frac{1}{\lambda}\left(\prod_{n=1}^{\infty} (1 + \lambda\theta_\lambda^{-1}(\mu(E_n))) - 1\right) \\
&= \frac{1}{\lambda}\left(\prod_{n=1}^{\infty} (1 + \lambda(\theta_\lambda^{-1} \circ \mu)(E_n)) - 1\right).
\end{aligned}
$$

That is $\theta_\lambda^{-1} \circ \mu$ satisfies the $\delta-\lambda$-rule. Note that $(\theta_\lambda^{-1} \circ \mu)(E_0) = (\theta_\lambda^{-1}(\mu(E_0)) < \infty$, so we know that $\theta_\lambda^{-1} \circ \mu$ is a λ-fuzzy measure on C. □

Example 1.3.9. *Let $X = \{a, b\}$, $F = P(X)$ and define g_λ by:*

$$
g_\lambda(E) = \begin{cases} 0 & \text{if } E = \varphi \\ 0.2 & \text{if } E = \{a\} \\ 0.4 & \text{if } E = \{b\} \\ 1 & \text{if } E = X \end{cases}
$$

Let's calculate λ *using Equation (1.2.2). We have*

$$\mu\left(\bigcup_{i=1}^{4} E_i\right) = \frac{1}{\lambda}((1+0.2\lambda)(1+0.4\lambda) - 1)$$

$$\Rightarrow \; 1 = \frac{1}{\lambda}((1+0.2\lambda)(1+0.4\lambda) - 1)$$

$$\Rightarrow \; 1 = \frac{1}{\lambda}(0.6\lambda + 0.08\lambda^2)$$

$$\Rightarrow \; 1 = 0.6 + 0.08\lambda$$

$$\Rightarrow \; \lambda = 5.$$

So, g_λ *is a* λ*-fuzzy measure with* $\lambda = 5$ *and* $\theta_\lambda(y) = \dfrac{\ln(1+\lambda y)}{k\lambda)} = \dfrac{\ln(1+5y)}{5k}$. *By taking* $k = \frac{\ln 6}{5}$, *we get* $\theta_\lambda(y) = \frac{\ln(1+5y)}{\ln 6}$. *So, we have*

$$(\theta_\lambda \circ g_\lambda)(E) = \begin{cases} 0 & \text{if } E = \varphi \\ 0.387 & \text{if } E = \{a\} \\ 0.613 & \text{if } E = \{b\} \\ 1 & \text{if } E = X \end{cases}$$

Hence, $\theta_\lambda \circ g_\lambda$ *is a probability measure.*

Example 1.3.10. *Let* $X = \{a, b\}$ *and* $\mathcal{F} = \mathcal{P}(X)$. *Define the* λ*-fuzzy measure* g_λ *by:*

$$g_\lambda(E) = \begin{cases} 0 & \text{if } E = \varphi \\ 0.5 & \text{if } E = \{a\} \\ 0.8 & \text{if } E = \{b\} \\ 1 & \text{if } E = X \end{cases}$$

Let's calculate λ using Equation (1.2.2). We have

$$\begin{aligned}
\mu(\cup E_i) &= \frac{1}{\lambda}((1+0.5\lambda)(1+0.8\lambda)-1) \\
\Rightarrow\ 1 &= \frac{1}{\lambda}((1+0.5\lambda)(1+0.8\lambda)-1) \\
\Rightarrow\ 1 &= \frac{1}{\lambda}(1.3\lambda+0.4\lambda^2) \\
\Rightarrow\ 1 &= 1.3+0.4\lambda \\
\Rightarrow\ \lambda &= -0.75,
\end{aligned}$$

So $\theta_\lambda(y) = \frac{\ln(1+\lambda y)}{k\lambda)} = \frac{\ln(1-0.75y)}{-0.75k}$. *By taking* $k = \frac{\ln 0.25}{-0.75}$, *we have* $\theta_\lambda(y) = \frac{\ln(1-0.75y)}{\ln 0.25}$ *and*

$$(\theta_\lambda \circ g_\lambda)(E) = \begin{cases} 0 & \text{if } E = \emptyset \\ 0.34 & \text{if } E = \{a\} \\ 0.66 & \text{if } E = \{b\} \\ 1 & \text{if } E = X \end{cases}$$

Hence, $\theta_\lambda \circ g_\lambda$ *is a probability measure.*

Remark Under the mapping θ_λ, the λ-rule and the finite λ-rule become the additivity and the finite additivity, respectively. Under the mapping θ_λ^{-1}, the additivity and finite additivity become the λ-rule and the finite λ-rule, respectively.

Corollary 1.3.11. *On a semi-ring, the λ-rule is equivalent to the finite λ-rule.*

Proof. $\Rightarrow$) On a semi-ring, the σ-additivity is satisfied. So, the λ-rule is satisfied which means that $\lambda = 0$. Then, $\mu(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} \mu(E_i)$. Therefore, the finite λ-rule is satisfied.

$\Leftarrow$) The finite λ-rule is satisfied. On a semi-ring, the σ-additivity is satisfied. So, we take the case $\lambda = 0$. Then, $\mu(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} \mu(E_i)$. Therefore the λ-rule is satisfied. □

Corollary 1.3.12. *Any λ-fuzzy measure on a semi-ring is continuous.*

Corollary 1.3.13. *On a ring, the λ-rule with the continuity, together, are equivalent to the δ-λ rule. Thus, on a ring, any fuzzy measure that satisfies the λ-rule is a λ-fuzzy measure.*

Remark A fuzzy measure on a semi-ring that satisfies the λ-rule may not satisfy the $\delta - \lambda$ rule. A fuzzy measure on a semi-ring that is quasi-additive may not be a quasi-measure.

Corollary 1.3.14. *If g_λ is a regular λ-fuzzy measure on an algebra R, then its dual-fuzzy measure μ, defined by $\mu(E) = 1 - g_\lambda(\overline{E})$, for any $E \in R$, is also a regular λ-fuzzy measure on R, and the corresponding parameter is given by $\lambda = \frac{-\lambda}{\lambda+1}$.*

Proof. Let $E \in R$, $F \in R$, and $E \cap F = \emptyset$. Using Theorem 1.2.10, we have:

$$
\begin{aligned}
& \mu(E) + \mu(F) - \frac{\lambda}{\lambda+1}\mu(E)\mu(F) \\
= \; & 1 - g_\lambda(\overline{E}) + 1 - g_\lambda(\overline{F}) - \frac{\lambda}{\lambda+1}(1 - g_\lambda)(\overline{F}) \\
= \; & \frac{(\lambda+1)g_\lambda(E)}{1+\lambda g_\lambda(E)} + \frac{(\lambda+1)g_\lambda(F)}{1+\lambda g_\lambda(F)} - \lambda\frac{(\lambda+1)g_\lambda(E)g_\lambda(F)}{(1+\lambda g_\lambda(E))(1+\lambda g_\lambda(F))} \\
= \; & \frac{(\lambda+1)(g_\lambda(E) + g_\lambda(F) + \lambda g_\lambda(E)g_\lambda(F))}{(1+\lambda g_\lambda(E))(1+\lambda g_\lambda(F))} \\
= \; & \frac{(\lambda+1)g_\lambda(E \cup F)}{1+\lambda g_\lambda(E \cup F))} \\
= \; & 1 - g_\lambda(\overline{E \cup F})) = \mu(E \cup F).
\end{aligned}
$$

Since μ is continuous, by Corollary 1.3.13, we get that μ satisfies the $\delta - \lambda$ rule with a parameter $\lambda = \frac{-\lambda}{\lambda+1}$. Since $\mu(X) = 1 - g_\lambda(\emptyset) = 1$, μ is a regular λ-fuzzy measure on R with parameter $\lambda = \frac{-\lambda}{\lambda+1}$. □

1.4 Belief Measures and Plausibility Measures

In the previous section, we induced a non-additive measure from a classical measure by a transformation of range of the classical measure. In this section, we attempt to construct a non-additive measure in another way.

Definition 1.4.1. *Let $\mathcal{P}(\mathcal{P}(X))$ be the power set of $\mathcal{P}(X)$. If p is discrete probability measure on $(\mathcal{P}(X), \mathcal{P}(\mathcal{P}(X)))$ with $p(\{\emptyset\}) = 0$, then the set function $m : \mathcal{P}(X) \to [0, 1]$ determined by $m(E) = p(\{E\})$ for any $E \in \mathcal{P}(X)$ is called a basic probability assignment on $\mathcal{P}(X)$.*

Theorem 1.4.2. *A set function $m : \mathcal{P}(X) \to [0, 1]$ is a basic probability* *assignment if and only if*

$$\begin{cases} m(\emptyset) = 0 \\ \sum_{E \in P(X)} m(E) = 1 \end{cases}$$

Proof. $\Rightarrow$) Let $m : P(X) \to [0, 1]$ be a basic probability assignment. So, $m(\emptyset) = p(\{\emptyset\}) = 0$ and

$$\sum_{E \in P(X)} m(E) = \sum_{E \in P(X)} p(\{E\}) = 1.$$

$\Leftarrow$) Consider $D_n = \{E / \frac{1}{n+1} < m(E) \leq \frac{1}{n}\}$, for $n = 1, 2, \cdots$. Every D_n is a finite class so countable. Then

$$D = \bigcup_{n=1}^{\infty} D_n = \{E / m(E) > 0\}$$

is a countable class. We have $\mathcal{E} = \{\{E\} / E \in P(X)\} \cup \{\emptyset\}$ is a semiring and we define, for any $E \in P(X)$

$$p(\{E\}) = \begin{cases} m(E) & \text{for } E \in D \\ 0, & \text{else} \end{cases}$$

and $p(\{\emptyset\}) = 0$. Then p is a probability measure on $\mathcal{E}$ with $p(\{\emptyset\}) = 0$. Moreover, p can be extended uniquely to a discrete probability measure on

$(\mathcal{P}(X), \mathcal{P}(\mathcal{P}(X)))$ by

$$p(\mathcal{E}) = \sum_{E \in \mathcal{E}} p(\{E\}),$$

for any $\mathcal{E} \in \mathcal{P}(\mathcal{P}(X))$. □

Definition 1.4.3. *If m is a basic probability assignment on $\mathcal{P}(X)$, then the set function* Bel : $\mathcal{P}(X) \to [0, 1]$ *determined by*

$$\text{Bel}(E) = \sum_{F \subseteq E} m(F) \quad \textit{for any } E \in \mathcal{P}(X), \tag{1.4.1}$$

*is called a **belief measure** on $(X, \mathcal{P}(X))$ or more exactly a **belief measure induced from m**.*

Lemma 1.4.4. *If E is a nonempty finite set, then*

$$\sum_{F \subseteq E} (-1)^{\text{Card}(F)} = 0$$

Proof. Let $E = \{x_1, x_2, \cdots, x_n\}$. Then we have

$$\{\text{Card}(F), F \subseteq E\} = \{0, 1, \cdots, n\},$$

and $\text{Card}\left(\{F, \text{Card}(F) = i\}\right) = \binom{n}{i}$, for $i = 0, 1, \cdots, n$. So, we have

$$\sum_{F \subseteq E} (-1)^{\text{Card}(F)} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} = (1-1)^n = 0,$$

where i is the cardinality of the set and $\binom{n}{i}$ is the number of sets whose cardinality is i. □

Lemma 1.4.5. *If E is a finite set, $F \subsetneq E$, then*

$$\sum_{G/F \subseteq G \subseteq E} (-1)^{\mathrm{Card}(G)} = 0$$

Proof. $E \setminus F$ is a non-empty finite set. Using Lemma 1.4.4, we have

$$\sum_{G/F \subseteq G \subseteq E} (-1)^{\mathrm{Card}(G)} = \sum_{D \subseteq E \setminus F} (-1)^{\mathrm{Card}(F \cup D)} = (-1)^{F} \sum_{D \subseteq E \setminus F} (-1)^{\mathrm{Card}(D)} = 0$$

□

The following lemma gives the relation between 2 finite set functions.

Lemma 1.4.6. *Let X be finite, and λ and v be two finite set functions defined on $\mathcal{P}(X)$. Then, for any $E \in \mathcal{P}(X)$, we have:*

$$\lambda(E) = \sum_{F \subseteq E} v(F) \iff v(E) = \sum_{F \subseteq E} (-1)^{\mathrm{Card}(E \setminus F)} \lambda(F)$$

Proof. If $\lambda(E) = \sum_{F \subseteq E} v(F)$ for any $E \in \mathcal{P}(X)$, then

$$
\begin{aligned}
\sum_{F \subseteq E} (-1)^{\mathrm{Card}(E \setminus F)} \lambda(F) &= (-1)^{\mathrm{Card}(E)} \sum_{F \subseteq E} (-1)^{\mathrm{Card}(F)} \lambda(F) \\
&= (-1)^{\mathrm{Card}(E)} \sum_{F \subseteq E} (-1)^{\mathrm{Card}(F)} \sum_{G \subseteq F} v(G) \\
&= (-1)^{\mathrm{Card}(E)} \sum_{F, G / G \subseteq F \subseteq E} (-1)^{\mathrm{Card}(F)} v(G) \\
&= (-1)^{\mathrm{Card}(E)} \sum_{G \subseteq E} \left(\sum_{F / G \subseteq F \subseteq E} (-1)^{\mathrm{Card}(F)} v(G) \right) \\
&= (-1)^{\mathrm{Card}(E)} \sum_{G \subseteq E} v(G) \left(\sum_{F / G \subseteq F \subseteq E} (-1)^{\mathrm{Card}(F)} \right) \\
&= (-1)^{\mathrm{Card}(E)} v(E) (-1)^{\mathrm{Card}(E)} \\
&= v(E).
\end{aligned}
$$

Conversely, if $v(E) = \sum_{F \subseteq E} (-1)^{\mathrm{Card}(E \setminus F)} \lambda(F)$ for any $E \in \mathcal{P}(X)$, then

$$
\begin{aligned}
\sum_{F \subseteq E} v(F) = \sum_{F \subseteq E} \sum_{G \subseteq F} (-1)^{\mathrm{Card}(F \setminus G)} \lambda(G) &= \sum_{G \subseteq E} \left((-1)^{\mathrm{Card}\, G} \lambda(G) \sum_{G \subseteq F \subseteq E} (-1)^{\mathrm{Card}\, F} \right) \\
&= (-1)^{\mathrm{Card}\, E} \lambda(E) (-1)^{\mathrm{Card}\, E} \\
&= \lambda(E).
\end{aligned}
$$

□

Theorem 1.4.7. *If* Bel *is a belief measure on* $(X, \mathcal{P}(X))$, *then*

1. $\mathrm{Bel}(\emptyset) = 0$.

2. $\mathrm{Bel}(X) = 1$.

3. If $\{E_1, \cdots, E_n\}$ *is a finite subclass of* $\mathcal{P}(X)$*, then*

$$\mathrm{Bel}\left(\bigcup_{i=1}^{n} E_i\right) \geq \sum_{I \subseteq \{1,\cdots,n\}, I \neq \varphi} (-1)^{\mathrm{Card}(I)+1} \mathrm{Bel}\left(\bigcap_{i \in I} E_i\right)$$

4. Bel *is continuous from above.*

Proof. 1. $\mathrm{Bel}(\emptyset) = \sum_{F \subseteq \varphi} m(F) = m(\emptyset) = 0$

2. $\mathrm{Bel}(X) = \sum_{F \subseteq X} m(F) = 1$

3. Let $\{E_1, E_2, \cdots, E_n\}$ be a finite subclass and let

$$I(F) = \{i / 1 \leq i \leq n, F \subseteq E_i\},$$

for any $F \in \mathcal{P}(X)$. Using Lemma 1.4.4, we have:

$$\begin{aligned}
& \sum_{I \subseteq \{1,\cdots,n\}, I \neq \emptyset} (-1)^{\mathrm{Card}(I)+1} \mathrm{Bel}\Big(\bigcap_{i \in I} E_i\Big) \\
= & \sum_{I \subseteq \{1,\ldots,n\}, I \neq \varphi} (-1)^{\mathrm{Card}(I)+1} \sum_{F \subseteq \bigcap_{i \in I} E_i} m(F) \\
= & \sum_{F / I(F) \neq \varphi} m(F) \sum_{I \subseteq I(F), I \neq \varphi} (-1)^{\mathrm{Card}(I)+1} \\
= & \sum_{F / I(F) \neq \varphi} m(F) \Big(1 - \sum_{I \subseteq I(F)} (-1)^{\mathrm{Card}(I)}\Big) \\
= & \sum_{F / I(F) \neq \emptyset} m(F) \\
= & \sum_{F \subseteq E_i \text{ for some } i} m(F) \\
\leq & \sum_{F \subseteq \cup_{i=1}^{n} E_i} m(F) \\
= & \mathrm{Bel}\Big(\bigcup_{i=1}^{n} E_i\Big).
\end{aligned}$$

4. Let $\{E_i\}$ be a decreasing sequence of sets in $\mathcal{P}(X)$ and $\bigcap_{i=1}^{\infty} E_i = E$. From Theorem 1.4.2, we know there exists a countable class $\{D_n\} \subseteq \mathcal{P}(X)$ such that $m(F) = 0$ whenever $F \notin \{D_n\}$, and for any $c > 0$, $\exists n_0$ such that $\sum_{n>n_0} m(Dn) < c$. For each D_n, where $n \leq n_0$, if $D_n \not\subseteq E$ (that is $D_n \cap E = \varphi$), then $\exists i(n) / D_n \not\subseteq E_{i(n)}$. Taking $i_0 = \max(i(1), \ldots, i(n_0))$, then if $D_n \not\subseteq E$, we have $D_n \not\subseteq E_{i_0}$ for any $n \leq n_0$.

So, we have:

$$
\begin{aligned}
\text{Bel}(E) &= \sum_{F \subseteq E} m(F) \\
&= \sum_{D_n \subset E} m(D_n) \\
&\geq \sum_{D_n \subseteq E, n \leq n_0} m(D_n) \\
&\geq \sum_{D_n \subseteq E_{i_0}, n \leq n_0} m(D_n) \\
&\geq \sum_{D_n \subseteq E_{i_0}} m(D_n) - \sum_{n > n_0} m(D_n) \\
&> \sum_{F \subseteq E_{i_0}} m(F) - c \\
&= \text{Bel}(E_{i_0}) - c.
\end{aligned}
$$

Noting that $\text{Bel}(E) \leq \text{Bel}(E_i)$ for $i = 1, 2, \cdots$ and $\{\text{Bel}(E_i)\}$ is decreasing with respect to i, we have $\text{Bel}(E) = \lim_i \text{Bel}(E_i)$.

□

Theorem 1.4.8. *Any belief measure is monotone and superadditive.*

Proof. Let $E_1 \subseteq X$, $E_2 \subseteq X$, and $E_1 \cap E_2 = \emptyset$. We have

$$
\begin{aligned}
&\text{Bel}(E_1 \cup E_2) \\
\geq\ & \text{Bel}(E_1) + \text{Bel}(E_2) - \text{Bel}(E_1 \cap E_2) \\
=\ & \text{Bel}(E_1) + \text{Bel}(E_2) \\
\geq\ & \text{Bel}(E_1).
\end{aligned}
$$

Thus, $\text{Bel}(E_1 \cup E_2) \geq \text{Bel}(E_1) + \text{Bel}(E_2)$. Therefore, Bel is superadditive. Now, we have $E_1 \subseteq E_1 \cup E_2$, So $\mathrm{Bel}(E_1) \leq \mathrm{Bel}(E_1 \cup E_2)$. So, Bel is monotone. □

Remark From Theorems 1.4.7 and 1.4.8, we know that the belief measure is an upper semi-continuous fuzzy measure since $\mathrm{Bel}(\emptyset) = 0$, Bel is monotone, and continuous from above.

The following theorem shows that on a finite space, we can express a basic probability assignment by the belief measure induced by it.

Theorem 1.4.9. *Let X be finite. If a set function $\mu : \mathcal{P}(X) \to [0, 1]$ satisfies the conditions:*

1. $\mu(\emptyset) = 0$
2. $\mu(X) = 1$
3. *If $\{E_1, \cdots, E_n\}$ is a finite subclass of $\mathcal{P}(X)$, we have*

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) \geq \sum_{I \subseteq \{1,\cdots,n\}, I \neq \emptyset} (-1)^{\mathrm{Card}(I)+1} \mu\left(\bigcap_{i \in I} E_i\right)$$

Then, the set function m determined by

$$m(E) = \sum_{F \subseteq E} (-1)^{\mathrm{Card}(E \setminus F)} \mu(F), \quad \text{for any } E \in \mathcal{P}(X), \qquad (1.4.2)$$

is a basic probability assignment, and μ is just the belief measure induced from m, that is: $\mu(E) = \mathrm{Bel}(E) = \sum_{F \subseteq E} m(F)$

Proof. We have $m(\emptyset) = \sum_{F \subseteq \emptyset} (-1)^{\mathrm{Card}(\emptyset \setminus F)} \mu(F) = \mu(\emptyset) = 0$. From Equation

(1.4.2) and Lemma 1.4.6, we have

$$\sum_{E \subseteq X} m(E) = \mu(X) = 1.$$

We already have that $m(\emptyset) = 0$ and $\sum m(E) = 1$. We still have to prove that $m(E) \geq 0$ for any $E \subseteq X$. Since X is finite, E is also finite and we can write $E = \{x_1, \cdots, x_n\}$. If we denote $E_i = E \setminus \{x_i\}$, then $E = \bigcup_{i=1}^{n} E_i$ and

$$\begin{aligned} m(E) &= \sum_{F \subseteq E} (-1)^{\mathrm{Card}(E \setminus F)} \mu(F) \\ &= \mu(E) - \sum_{I \subseteq \{1,\cdots,n\}, I \neq \emptyset} (-1)^{\mathrm{Card}(I)+1} \mu\Big(\bigcap_{i \in I} E_i\Big) \\ &= \mu\Big(\bigcup_{i=1}^{n} E_i\Big) - \sum_{I \subseteq \{1,\cdots,n\}, I \neq \emptyset} (-1)^{\mathrm{Card}(I)+1} \mu\Big(\bigcap_{i \in I} E_i\Big) \geq 0. \end{aligned}$$

Therefore, m is a basic probability assignment and μ is the belief measure induced from m. □

Definition 1.4.10. *If m is a basic probability assignment on $\mathcal{P}(X)$, then the set function $Pl : \mathcal{P}(X) \to [0, 1]$ determined by*

$$\mathrm{Pl}(E) = \sum_{F \cap E \neq \emptyset} m(F), \quad \textit{for any } E \in \mathcal{P}(X) \qquad (1.4.3)$$

is called **plausibility measure on** $(X, \mathcal{P}(X))$ **or a plausibility measure induced from m.**

The following theorem gives the relation between belief measure and plausibility measure.

Theorem 1.4.11. *If* Bel *and* Pl *are the belief measure and plausibility measure induced from the same basic probability assignment respectively, then*

$$\mathrm{Bel}(E) = 1 - \mathrm{Pl}(\overline{E}) \quad \textit{and} \quad \mathrm{Bel}(E) \leq \mathrm{Pl}(E),$$

for any $E \subseteq X$.

Proof. We have

$$\begin{aligned}
\mathrm{Bel}(E) &= \sum_{F \subseteq E} m(F) \\
&= \sum_{F \subseteq X} m(F) - \sum_{F \not\subseteq E} m(F) \\
&= 1 - \sum_{F \cap \overline{E} = \emptyset} m(F) \\
&= 1 - \mathrm{Pl}(\overline{E}).
\end{aligned}$$

Since $\mathrm{Bel}(E) = \sum_{F \subseteq E} m(F)$ and $\mathrm{Pl}(E) = \sum_{F \cap E = \emptyset} m(F)$, so we have, for any $E \subseteq X$,

$$\mathrm{Bel}(E) \leq \mathrm{Pl}(E)$$

□

Theorem 1.4.12. *If* $\mathrm{Pl}(E)$ *is a plausibility measure on* $(X, \mathcal{P}(X))$*, then:*

1. $\mathrm{Pl}(\emptyset) = 0$
2. $\mathrm{Pl}(X) = 1$

3. *If $\{E_1, \cdots, E_n\}$ is a finite subclass of $\mathcal{P}(X)$, then*

$$\mathrm{Pl}\left(\bigcap_{i=1}^{n} E_i\right) \leq \sum_{I\subseteq\{1,\cdots,n\}, I\neq\emptyset} (-1)^{\mathrm{Card}(I)+1}\,\mathrm{Pl}\left(\bigcup_{i\in I} E_i\right)$$

4. Pl *is continuous from below.*

Proof. 1. $\mathrm{Bel}(X) = 1 = 1 - \mathrm{Pl}(\emptyset)$, so $\mathrm{Pl}(\emptyset) = 0$

2. $\mathrm{Bel}(\emptyset) = 0 = 1 - \mathrm{Pl}(X)$, so $\mathrm{Pl}(X) = 1$

3. Using Lemma 1.4.4, we have:

$$\begin{aligned}
&\mathrm{Pl}\left(\bigcap_{i=1}^{n} E_i\right) \\
&= 1 - \mathrm{Bel}\left(\overline{\bigcap_{i=1}^{n} E_i}\right) \\
&= 1 - \mathrm{Bel}\left(\bigcup_{i=1}^{n} \overline{E_i}\right) \\
&\leq 1 - \sum_{I\subseteq\{1,\cdots,n\}, I\neq\emptyset} (-1)^{\mathrm{Card}(I)+1}\,\mathrm{Bel}\left(\bigcap_{i\in I} \overline{E_i}\right) \\
&= \sum_{I\subseteq\{1,\cdots,n\}, I\neq\emptyset} (-1)^{\mathrm{Card}(I)+1}\left(1 - \mathrm{Bel}\left(\bigcap_{i\in I} \overline{E_i}\right)\right) \\
&= \sum_{I\subseteq\{1,\cdots,n\}, I\neq\emptyset} (-1)^{\mathrm{Card}(I)+1}\left(1 - \mathrm{Bel}\left(\overline{\bigcup_{i\in I} E_i}\right)\right) \\
&= \sum_{I\subseteq\{1,\cdots,n\}, I\neq\emptyset} (-1)^{\mathrm{Card}(I)+1}\left(\mathrm{Pl}\left(\bigcup_{i\in I} E_i\right)\right).
\end{aligned}$$

4. Clearly that Pl is continuous from below.

□

Theorem 1.4.13. *Any plausibility measure is monotone and subadditive.*

Proof. If $E \subseteq F \subseteq X$, then $\overline{F} \subseteq \overline{E} \subseteq X$. From Theorems 1.4.8 and 1.4.11, we have that a belief measure is monotone, so

$$\mathrm{Bel}(\overline{F}) \leq \mathrm{Bel}(\overline{E}).$$

Take $E \subseteq F$, then

$$\mathrm{Pl}(E) = 1 - \mathrm{Bel}(\overline{E}) \leq 1 - \mathrm{Bel}(\overline{F}) = \mathrm{Pl}(F).$$

Thus, Pl is monotone. For subadditivity, if $E_1 \subseteq X$ and $E_2 \subseteq X$, then

$$0 \leq \mathrm{Pl}(E_1 \cap E_2) \leq \mathrm{Pl}(E_1) + \mathrm{Pl}(E_2) - \mathrm{Pl}(E_1 \cup E_2).$$

Thus, $\mathrm{Pl}(E_1 \cup E_2) \leq \mathrm{Pl}(E_1) + \mathrm{Pl}(E_2)$ and therefore, Pl is subadditive. □

Remark According to Theorem 1.4.12 and Theorem1.4.13 , we conclude that Pl measure is a lower semi-continuous fuzzy measure.

Theorem 1.4.14. *Any discrete probability measure p on $(X, \mathcal{P}(X))$ is both a belief measure and a plausibility measure. The corresponding basic probability assignment focuses on the singetons of $(X, \mathcal{P}(X))$. Conversely, if m is a basic probability assignment focusing on the singletons of $\mathcal{P}(X)$, then the belief measure and the plausibility measure induced from m coincide, resulting in a discrete probability measure on $(X, \mathcal{P}(X))$.*

Proof. Since p is a discrete probability measure, there exists a countable set

$\{x_1, x_2, \cdots\} \subseteq X$ such that $\sum_{i=1}^{\infty} p(\{x_i\}) = 1$. Define

$$m(E) = \begin{cases} p(E) & E = \{x_i\} \\ 0 & \text{otherwise} \end{cases}$$

for any $E \in \mathcal{P}(X)$. Note that m is a basic probability assignment. We have

$$p(E) = \sum_{x_i \in E} p(\{x_i\}) = \sum_{F \subseteq E} m(F) = \sum_{F \cap E = \emptyset} m(F).$$

Therefore, p is both a belief measure and plausibility measure. Conversely, if a basic probability assignment m focuses only on the singletons of $\mathcal{P}(X)$, then, for any $E \in \mathcal{P}(X)$,

$$\mathrm{Bel}(E) = \sum_{F \subseteq E} m(F) = \sum_{x \in E} m(\{x\}) = \sum_{F \cap E = \emptyset} m(F) = \mathrm{Pl}(E).$$

So, Pl and Bel coincide and they are δ -additive. Therefore, the belief and possibility measures are discrete probability measures on $(X, \mathcal{P}(X))$ □

Theorem 1.4.15. *Let* Pl *and* Bel *be the plausibility measure and the belief measure, respectively, induced from a basic probability assignment m. If* Bel *coincide with* Pl*, then m focuses only on singletons.*

Proof. If there exists $E \in \mathcal{P}(X)$), which is not a singleton of $\mathcal{P}(X)$ such that

$m(E) > 0$, then, for any $x \in E$, we have

$$
\begin{aligned}
\mathrm{Bel}(\{x\}) &= m(\{x\}) \\
&< m(\{x\}) + m(E) \\
&\leq \sum_{F \cap \{x\} \neq \emptyset} m(F) \\
&= \mathrm{Pl}(\{x\}).
\end{aligned}
$$

This is a contradiction with the coincidence of Bel and Pl □

Remark The Sugeno measures defined on the power set $\mathcal{P}(X)$ are the special examples of belief measures and plausibility measures when X is countable.

The following theorem gives the relation between the λ-fuzzy measure g_λ, belief measure and plausibility measure.

Theorem 1.4.16. *Let X be countable, and g_λ, $\lambda = 0$ be a Sugeno measure on $(X, \mathcal{P}(X))$. Then, when $\lambda > 0$, g_λ is a belief measure, and when $\lambda < 0$, it is a plausibility measure.*

Proof. Let $X = \{x_1, x_2, \cdots\}$. When $\lambda > 0$, we define $m : \mathcal{P}(X) \to [0, 1]$ by

$$
m(E) = \begin{cases} \lambda^{\mathrm{Card}(E)-1} \prod_{x_i \in E} g_\lambda(\{x_i\}) & \text{for } E = \emptyset \\ 0 & \text{for } E = \emptyset. \end{cases}
$$

Obviously, $m(E) \geq 0$ for any $E \in \mathcal{P}(X)$. From Definition 1.2.1, we have

$$\begin{aligned} g_\lambda(E) &= \frac{1}{\lambda}\Big(\prod_{x_i \in E}(1+\lambda g_\lambda(\{x_i\})) - 1\Big) \\ &= \frac{1}{\lambda}\sum_{F \subseteq E, F \neq \emptyset}\Big(\lambda^{\mathrm{Card}(F)} \prod_{x_i \in F} g_\lambda(\{x_i\})\Big) \\ &= \sum_{F \subseteq E, F \neq \emptyset}\Big(\lambda^{\mathrm{Card}(F)-1} \prod_{x_i \in F} g_\lambda(\{x_i\})\Big) \\ &= \sum_{F \subseteq E} m(F). \end{aligned}$$

Since $g_\lambda(X) = 1$, we have $\sum_{F \subseteq X} m(F) = 1$. Therefore, m is a basic probability assignment, and thus, g_λ is the belief measure induced from m. When $\lambda < 0$, we have $\lambda = \frac{-\lambda}{(\lambda+1)} > 0$. Using Corollary 1.3.14 and Theorem 1.4.11, we conclude that g_λ is a plausibility measure. □

1.5 Possibility and Necessity Measures

Definition 1.5.1. *μ is fuzzy additive (or f-additive) on C if and only if*

$$\mu\Big(\bigcup_{t \in T} E_t\Big) = \sup_{t \in T} \mu(E_t), \tag{1.5.1}$$

for any subclass $\{E_t / t \in T\}$ of C whose union is in C and where T is an arbitrary index set. If C is a finite class, then the f-additivity of μ on C is equivalent to the simpler requirement that

$$\mu(E_1 \cup E_2) = \mu(E_1) \vee \mu(E_2) \tag{1.5.2}$$

whenever E_1, E_2, and $E_1 \cup E_2 \in C$. Here, we denote by $mu(E_1) \vee mu(E_2)$ the supremum of $\mu(E_1)$ and $\mu(E_2)$.

Definition 1.5.2. *μ is called a* **generalized possibility measure** *on C if and only if it is f-additive on C and $\exists E \in C / \mu(E) < \infty$. Usually a generalized possibility measure is denoted by π.*

Definition 1.5.3. *If π is a generalized possibility measure defined on $\mathcal{P}(X)$, then the function $\boldsymbol{f}$ defined on X by $\boldsymbol{f}(x) = \pi(\{x\})$ for any $x \in X$ is called* **possibility density function**.

Theorem 1.5.4. *Any generalized possibility measure π on C is a lower semi-continuous fuzzy measure on C.*

Proof. According to the convention, when $T = \emptyset$, we have $\bigcup_{t \in T} E_t = \emptyset$ and $\sup_{t \in T} \mu(E_t) = 0$. So if $\emptyset \in C$, then $\pi(\varphi) = 0$ (vanishing at φ). Furthermore, if $E \in C$, $F \in C$, and $E \subseteq F$, then by using f-additivity, we have

$$\pi(F) = \pi(E \cup F) = \pi(E) \wedge \pi(F) \geq \pi(E).$$

At last, π is continuous from below. In fact, if $\{E_n\}$ is an increasing sequence of sets in C whose union E is also in C, we have from the definition of the supremum that for any $c > 0$,

$$\exists n_0 / \pi(E_{n_0}) \geq \sup_n \pi(E_n) - c = \pi(E) - c.$$

Noting that π is monotone, we know $\lim_n \pi(E_n) = \pi(E)$. □

Definition 1.5.5. *A regular generalized possibility measure* π *defined on* $\mathcal{P}(X)$ *is called a* **possibility measure**.

The following example shows that a possibility measure is not necessarily continuous from above.

Example 1.5.6. *Let* $X = (-\infty, \infty)$ *and* $\pi : \mathcal{P}(X) \to [0, 1]$ *the set function defined by*

$$\pi(E) = \begin{cases} 1 & E = \emptyset \\ 0 & E = \emptyset \end{cases}$$

Let's prove that π *is f-additive. If for all* t, $E_t = \emptyset$, *then* $\mu(\bigcup E_t) = 1$ *and* $\sup(\mu(E_t)) = 1$. *If all* $E_t = \emptyset$, *then* $\mu(\bigcup E_t) = 0$ *and* $\sup(\mu(E_t)) = 0$. *If* $\exists t_i$ *such that* $E_{t_i} = \emptyset$, *so* $\mu(\bigcup E_t) = 1$ *and* $\sup(\mu(E_t)) = 1$. *In all cases,* π *is f-additive and* $\pi(x) = 1$ *since* $X = \emptyset$. *Therefore, it is a possibility measure on* $\mathcal{P}(X)$. *However, it is not continuous from above. In fact, if we take* $E_n = \left(0, \frac{1}{n}\right)$, *then* $\{E_n\}$ *is decreasing, and* $\bigcap_{n=1}^{\infty} E_n = \emptyset$. *We have* $\pi(E_n) = 1$, $\forall n = 1, 2, \cdots$ *but* $\pi(\emptyset) = 0$. *So,*

$$\lim_n \pi(E_n) = \pi\left(\bigcap_{n=1} \sim E_n\right).$$

Therefore, π *is a possibility measure but not continuous from above.*

Theorem 1.5.7. *If* f *is the density function of a possibility measure* π, *then* $\sup_{x \in X} f(x) = 1$. *Conversely, if a function* $f : X \to [0, 1]$ *satisfies* $\sup_{x \in X} f(x) = 1$, *then* f *can determine a possibility measure* π *uniquely and* f *is the density function of* π.

Proof. From Equation (1.5.1), we have

$$\sup_{x\in X} f(x) = \sup_{x\in X} \pi(\{x\}) = \pi(\cup_{x\in X}\{x\}) = \pi(X) = 1.$$

Conversely, set $\pi(E) = \sup_{x\in E} f(x)$ for any $E \in \mathcal{P}(X)$, then π is a possibility measure and $\pi(\{x\}) = \sup_{x\in\{x\}} f(x) = f(x)$. □

Remark Any function $f : X \to [0, \infty]$ can uniquely determine a generalized possibility measure π on $\mathcal{P}(X)$ by $\pi(E) = \sup_{x\in E} f(x)$ for any $E \in \mathcal{P}(X)$.

Definition 1.5.8. *A basic probability assignment is called* **consonant** *if and only if it focuses on a nest (A nest is a class fully ordered by the inclusion relation of sets).*

The following theorem gives the relation between the possibility measure and the plausibility measure.

Theorem 1.5.9. *Let X be finite. Then any possibility measure is a plausibility measure, and the corresponding basic probability assignment is consonant. Conversely, the plausibility measure induced by a consonant basic probability assignment is a possibility measure.*

Proof. Let $X = \{x_1, x_2, \cdots, x_n\}$ and π be a possibility measure. There is no loss of generality in assuming

$$1 = \pi(\{x_1\}) \geq \pi(\{x_2\}) \geq \cdots \geq \pi(\{x_n\}).$$

Define a set function m on $\mathcal{P}(X)$ by

$$m(E) = \begin{cases} \pi(\{x_i\}) - \pi(\{x_{i+1}\}) & E = F_i, \, i = 1, \cdots, n-1 \\ \pi(\{x_n\}) & E = F_n \\ 0 & \text{else,} \end{cases}$$

where $F_i = \{x_1, \cdots, x_i\}, i = 1, \cdots, n$. Then m is a basic probability assignment focusing on $\{F_1, \cdots, F_n\}$, which is a nest. The plausibility measure induced from this basic probability assignment m is just π. Conversely, let m be a basic probability assignment focusing on a nest $\{F_1, \cdots, F_k\}$ which satisfies $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k$ and Pl be the plausibility measure induced by m for any $E_1 \in \mathcal{P}(X)$. Denote by $j_0 = \min\{j / F_j \cap (E_1 \cup E_2) = \emptyset\}$ and $j_{0,i} = \min\{j / F_j \cap E_i = \emptyset\}$, for $i = 1, 2, \cdots$. Then we have

$$\begin{aligned}
& \text{Pl}(E_1 \cup E_2) \\
= & \sum_{F_j \cap (E_1 \cup E_2) = \emptyset} m(F_j) \\
= & \sum_{j \geq j_0} m(F_j) \\
= & \sum_{j \geq j_0} m(F_j) \\
= & \sum_{j \geq j_{0,1}} m(F_j) \vee \sum_{j \geq j_{0,2}} m(F_j) \\
= & \sum_{F_j \cap E_1 = \emptyset} m(F_j) \vee \sum_{F_j \cap E_2 = \emptyset} m(F_j) \\
= & \text{Pl}(E_1) \vee \text{Pl}(E_2).
\end{aligned}$$

That is Pl satisfies Equation (1.5.2) on $\mathcal{P}(X)$. So, Pl is a possibility measure.

□

Example 1.5.10. *Let* $X = \{x_1, x_2, x_3, x_4, x_5\}$ *and* π *be a possibility measure on* $(X, \mathcal{P}(X))$ *with a density function* $f(x) = \pi(\{x\}), x = x_1, \cdots, x_5$ *defined as follows:*

$$f(x_1) = 1, f(x_2) = 0.9, f(x_3) = 0.5, f(x_4) = 0.5, f(x_5) = 0.3$$

The corresponding basic probability assignment m *focuses on* 4 *subsets of* X, *which are*

$$F_1 = \{x_1\}, F_2 = \{x_1, x_2\}, F_4 = \{x_1, x_2, x_3, x_4\}, F_5 = X,$$

with

$$m(F_1) = f(x_1) - f(x_2) = 1 - 0.9 = 0.1, m(F_2) = f(x_2) - f(x_3) = 0.9 - 0.5 = 0.4,$$

$$m(F_3) = f(x_3) - f(x_4) = 0.5 - 0.5 = 0, m(F_4) = f(x_4) - f(x_5) = 0.5 - 0.3 = 0.2,$$

$$m(F_5) = f(x_5) = 0.3.$$

Notice that $\{F_1, F_2, F_4, F_5\}$ *forms a nest.*

Remark When X is not finite, a possibility measure on $\mathcal{P}(X)$ may not be a plausibility measure even when X is countable.

Example 1.5.11. *Let* X *be the set of all rational numbers in* $[0, 1]$ *and* $f(x) = x$ *for all* $x \in X$. *The set* X *is a countable set. Define a set function* π *on*

$\mathcal{P}(X)$ as follows:

$$\pi(E) = \sup_{x\in E} f(x), \quad \forall E \in \mathcal{P}(X).$$

Let's prove that π is a possibility measure on $\mathcal{P}(X)$. We have

$$\pi(E_1 \cup E_2) = \sup_{x\in E_1 \cup E_2} f(x) = \sup_{x\in E_1} f(x) \vee \sup_{x\in E_2} f(x),$$

so π is f-additive. Let $E = \emptyset, \pi(\emptyset) = \sup_{x\in\emptyset} f(x) = 0 < \infty$. Therefore, π is a possibility measure on $\mathcal{P}(X)$ but it is not a plausibility measure. Assume π is a plausibility measure . If we take $E = X$, then

$$\mathrm{Pl}(X) = \sum_{F\cap X=\emptyset} m(F) = 0.$$

However, $\pi(X) = \sup_{x\in X} f(x) = 1$. So, $0 = 1$ which leads to a contradiction.

Definition 1.5.12. *If π is a possibility measure on $\mathcal{P}(X)$, then its dual set function v ,which is defined by $v(E) = 1 = \pi(\overline{E})$ for any $E \in \mathcal{P}(X)$ is called a* **necessity measure or consonant belief function** *on $\mathcal{P}(X)$.*

Theorem 1.5.13. *A set function $v : \mathcal{P}(X) \to [0, 1]$ is a necessity measure* ***if*** *and only if it satisfies*

$$v\left(\bigcap_{t\in T} E_t\right) = \inf_{t\in T} v(E_t)$$

for any subclass $\{E_t/t \in T\}$ of $\mathcal{P}(X)$, where T is an index set, and $v(\emptyset) = 0$.

Proof. According to Defintions 1.5.5 and 1.5.12, we have

$$v(\cap E_t) = 1 - \pi(\cup \overline{E}_t) = 1 - \sup \pi(\overline{E_t}) = \inf(1 - \pi(E_t)) = \inf_{t\in T} v(E_t).$$

□

The following theorem shows the relation between necessity measure and belief measure.

Theorem 1.5.14. *Any necessity measure is an upper semi-continuous fuzzy measure. Moreover, if X is finite, then any necessity measure is a special example of belief measure and the corresponding basic probability assignment is consonant.*

Proof. Let v be the necessity measure. Let's prove that v is an upper semi-continuous fuzzy measure. We have $v(\emptyset) = 1 - \pi(\overline{\emptyset}) = 1 - \pi(X) = 0$. Let $E \in \mathcal{P}(X), F \in \mathcal{P}(X)/E \subseteq F$. Let's prove $v(E) \leq v(F)$, $v(E) = 1 - \pi(\overline{E})$ and $v(F) = 1 - \pi(\overline{F})$. Since π is a possibility measure then π is a lower semi-continuous fuzzy measure, by Theorem 1.5.4. So, we have

$$\begin{aligned} & \overline{F} \subseteq \overline{E} \\ \Rightarrow\ & \pi(\overline{F}) \leq \pi(\overline{E}) \\ \Rightarrow\ & -\pi(\overline{E}) \leq -\pi(\overline{F}) \\ \Rightarrow\ & 1 - \pi(\overline{E}) \leq 1 - \pi(\overline{F}) \\ \Rightarrow\ & v(E) \leq v(F) \end{aligned}$$

Now, $v(E) = 1 - \pi(\overline{E})$, and π is continuous from below, so v is continuous from above. Therefore, v is an upper semi-continuous fuzzy measure. Also, $v(E) = 1 - \pi(\overline{E})$, but π is possibility measure and according to Theorem 1.5.9, when X is finite, a possibility measure is a plausibility measure but we have $\mathrm{Bel}(E) = 1 - \mathrm{Pl}(\overline{E})$. Therefore v is a special case of belief measure. □

1.6 Properties of Finite Fuzzy Measures

In this section, we take a δ-ring $\mathcal{F}$ as the class $\mathcal{C}$.

Theorem 1.6.1. *If* μ *is* **finite fuzzy measure**, *then we have*

$$\lim_{n\to\infty} \mu(E_n) = \mu(\lim_{n\to\infty} E_n)$$

for any sequence $\{E_n\} \subseteq \mathcal{F}$ *whose limit exists.*

Proof. Let $\{E_n\}$ be a sequence of sets in $\mathcal{F}$ whose limit exists. Write

$$E = \lim_{n\to\infty} E_n = \lim_{n\to\infty} \sup_n E_n = \lim_{n\to\infty} \inf_n E_n.$$

By applying the finiteness of μ, we have:

$$\begin{aligned}
&\mu(E)\\
=\ &\mu(\limsup_n E_n) = \lim_n \mu\Big(\bigcup_{i=n}^{\infty} E_i\Big)\\
=\ &\limsup_n \mu\Big(\bigcup_{i=n}^{\infty} E_i\Big) \geq \limsup_n \mu(E_n)\\
\geq\ &\liminf_n \mu(E_n) \geq \liminf_n \mu(E_n)\\
\geq\ &\liminf_n \mu\Big(\bigcap_{i=n}^{\infty} E_i\Big) = \lim_n \mu\Big(\bigcap_{i=n}^{\infty} E_i\Big)\\
=\ &\mu(\liminf_n E_n)\\
=\ &\mu(E).
\end{aligned}$$

Therefore, $\lim_{n\to\infty} \mu(E_n)$ exists and $\lim_{n\to\infty} \mu(E_n) = \mu(E)$. □

Definition 1.6.2. *We say that* μ *is exhaustive if and only if* $\lim_{n\to\infty} \mu(E_n) = 0$

for any disjoint sequence $\{E_n\}$ *of sets in* $\mathcal{F}$.

Theorem 1.6.3. *If* μ *is a finite upper semi-continuous fuzzy measure, then it is exhaustive.*

Proof. Let $\{E_n\}$ be a disjoint sequence of sets in $\mathcal{F}$. If we write $F_n = \bigcup_{i=n}^{\infty} E_i$, then $\{F_n\}$ is a decreasing sequence of sets in $\mathcal{F}$, and

$$\lim_n F_n = \bigcap_{n=1}^{\infty} F_n = \lim_n \sup E_n = \emptyset.$$

Since μ is a finite upper semi-continuous fuzzy measure, by using the finiteness and the continuity from above of μ, we have:

$$\lim_n \mu(F_n) = \mu(\lim_n F_n) = \mu(\emptyset) = 0.$$

Note that $0 \le \mu(E_n) \le \mu(F_n)$, thus we obtain $\lim_{n\to\infty} \mu(E_n) = 0$ and so μ is exhaustive. □

Corollary 1.6.4. *Any finite fuzzy measure on a measurable space is exhaustive.*

Proof. Let μ be a finite fuzzy measure. If μ is upper semi-continuous, then, by Theorem 1.6.3, it is exhaustive. If μ is lower semi-continuous, consider $\{E_n\} \subseteq C$ such that $E_1 \subseteq E_2 \subseteq \cdots$ and $\bigcup E_n \in C$, so we get

$$\lim_{n\to\infty} \mu(E_n) = \mu\left(\bigcup E_n\right) = \mu(\lim_{n\to\infty} (E_n))$$

□

Chapter 2

Measurable Functions on Fuzzy Measure Spaces

In this chapter, we define $\mathcal{F}-$measurable functions and Borel functions on fuzzy measure spaces, and we study the relation between them. Also, we explain what it means for a sequence of measurable functions to converge almost everywhere, pseudo-almost everywhere, almost uniformly, pseudo-almost uniformly, in measure or pseudo in measure to a function.

2.1 Measurable Functions

Definition 2.1.1. *A measurable space is a couple* $(X, \mathcal{F})$ *where* X *is a set and* $\mathcal{F}$ *is a σ-algebra on* X*. Elements of* $\mathcal{F}$ *are called measurable sets.*

Definition 2.1.2. *Let* $\mathcal{B}$ *be a collection of subsets of a topological space* X*.* $\mathcal{B}$ *is called a Borel field if and only if:*

1. *If* $(A_i)_i$ *is a family of elements of* $\mathcal{B}$, *then* $\bigcap_{i=1}^{\infty} A_i \in \mathcal{B}$

2. *If* $A \in \mathcal{B}$, *then* $X \setminus A \in \mathcal{B}$

Definition 2.1.3. *Let* $\mathcal{A}$ *collection of subsets of* X. *Let* $\mathcal{B}(\mathcal{A})$ *the intersection of all Borel fields containing* $\mathcal{A}$. *Then every set* $B \in \mathcal{B}(\mathcal{A})$ *is called a Borel set.*

Let $(X, \mathcal{F})$ be a measurable space, $\mu : \mathcal{F} \to [0, \infty]$ be a fuzzy measure (or semi-continuous fuzzy measure), and $\mathcal{B}$ the Borel field on $(-\infty, \infty)$.

Definition 2.1.4. *A real-valued function* $f : X \to (-\infty, \infty)$ *on* X *is* $\mathcal{F}$-*measurable (or measurable) if and only if* $f^{-1}(B) = \{x/f(x) \in B\} \in \mathcal{F}$ *for any Borel set* $B \in \mathcal{B}$

Theorem 2.1.5. *If* $f : X \to (-\infty, \infty)$ *is a real-valued function, then the following statements are equivalent:*

1. f *is measurable*

2. $\{x/f(x) \geq \alpha\} \in \mathcal{F}, \forall \alpha \in (-\infty, \infty)$

3. $\{x/f(x) > \alpha\} \in \mathcal{F}, \forall \alpha \in (-\infty, \infty)$

4. $\{x/f(x) \leq \alpha\} \in \mathcal{F}, \forall \alpha \in (-\infty, \infty)$

5. $\{x/f(x) < \alpha\} \in \mathcal{F}, \forall \alpha \in (-\infty, \infty)$

Proof. (1) $\Rightarrow$ (2): We have $\{x/f(x) \geq \alpha\} = f^{-1}([\alpha, \infty))$. Since f is measurable and $[\alpha, \infty)$ is a Borel set, we get

$$f^{-1}([\alpha, \infty) \in \mathcal{F}.$$

(2) $\Rightarrow$ (1): Since $\{x/f(x) \geq \alpha\} \in \mathcal{F}$, $\forall \alpha \in (-\infty, \infty)$, we have $f^{-1}(B) \in \mathcal{F}$ for any $B \in [\alpha, \infty)/\alpha \in (-\infty, \infty)$. Consider the sets $A = \{B/f^{-1}(B) \in \mathcal{F}\}$ and $C = [\alpha, \infty)/\alpha \in (-\infty, \infty)$. We have $C \subseteq A$. Given any $B \in A$, we have $f^{-1}(\overline{B}) = \overline{f^{-1}(B)} \in \mathcal{F}$, and so $\overline{B} \in A$, that is A is closed under the formation of complements. Now, let $\{B_n\} \subseteq A$. We have,

$$f^{-1}\left(\bigcup_{n=1}^{\infty} B_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(B_n) \in \mathcal{F}.$$

So $\bigcup_{n=1}^{\infty} B_n \in \mathcal{F}$ and hence A is closed under the formation of countable unions. Thus, A is a σ- algebra and we have $\mathcal{B} = \mathcal{F}(C) \subseteq A$. Therefore, f is a measurable function.

(2) $\Rightarrow$ (3) : We have $\{x/f(x) \geq \alpha\} = \{x/f(x) > \alpha\} \cup \{x/f(x) = \alpha\}$. Since $\{x/f(x) \geq \alpha\} \in \mathcal{F}$ and $\{x/f(x) = \alpha\} \in \mathcal{F}$ and since $\mathcal{F}$ is closed under complement, then $\{x/f(x) > \alpha\} \in \mathcal{F}$.

(3) $\Rightarrow$ (4) : We have that $\{x/f(x) > \alpha\} \in \mathcal{F}$, $\forall \alpha \in (-\infty, \infty)$. Let's prove that $\{x/f(x) \leq \alpha\} \in \mathcal{F}$. We have $\{x/f(x) \leq \alpha\} = \overline{\{x/f(x) > \alpha\}}$. Since $\mathcal{F}$ is closed under complement, we get $\{x/f(x) \leq \alpha\} \in \mathcal{F}$.

(4) $\Rightarrow$ (5) : We have that $\{x/f(x) \leq \alpha\} \in \mathcal{F}$, $\forall \alpha \in (-\infty, \infty)$. Let's prove that $\{x/f(x) < \alpha\} \in \mathcal{F}$, $\forall \alpha \in (-\infty, \infty)$. We have

$$\{x/f(x) \leq \alpha\} = \{x/f(x) < \alpha\} \cup \{x/f(x) = \alpha\}.$$

Since $\{x/f(x) \leq \alpha\} \in \mathcal{F}$ and $\{x/f(x) = \alpha\} \in \mathcal{F}$, then $\{x/f(x) < \alpha\} \in \mathcal{F}$

(5) $\Rightarrow$ (1) : We have that $\{x/f(x) < \alpha\} \in \mathcal{F}$, $\forall \alpha \in (-\infty, \infty)$. Let's prove that f is measurable. Since $\{x/f(x) < \alpha\} \in \mathcal{F}$, $\forall \alpha \in (-\infty, \infty)$, we get

that $f^{-1}(B) \in \mathcal{F}$ for any $B \in \{(-\infty, \alpha)/\alpha \in (-\infty, \infty)\}$. Consider the sets $\mathcal{A} = \{B/f^{-1}(B) \in \mathcal{F}\}$ and $\mathcal{C} = \{(-\infty, \alpha)/\alpha \in (-\infty, \infty)\}$. We have that $\mathcal{C} \subseteq \mathcal{A}$. Given any $B \in \mathcal{A}$, we have $\overline{B} \in \mathcal{A}$ because $f^{-1}(\overline{B}) = \overline{f^{-1}(B)} \in \mathcal{F}$. Thus, $\mathcal{A}$ is closed under complements. Let $\{B_n\} \subseteq \mathcal{A}$, so

$$f^{-1}\left(\bigcup_{n=1}^{\infty} B_n\right) = \bigcup f^{-1}(B_n) \in \mathcal{F}.$$

Then, $\bigcup_{n=1}^{\infty} B_n \in \mathcal{F}$. So, $\mathcal{A}$ is closed under formation of countable union. Therefore, $\mathcal{A}$ is a σ-algebra. Hence, $\mathcal{B} = \mathcal{F}(\mathcal{C}) \subseteq \mathcal{A}$ and f is a measurable function. □

Corollary 2.1.6. *If f is a measurable function, then $\{x/f(x) = \alpha\} \in \mathcal{F}$, $\forall \alpha \in (-\infty, \infty)$*

Proof. We know that $\{x/f(x) = \alpha\} = f^{-1}(\{\alpha\})$. Since $(-\infty, \alpha]$ and $[\alpha, \infty)$ are Borel sets, we get $(-\infty, \alpha] \cap [\alpha, \infty) = \{\alpha\}$ is also a Borel set. Because f measurable, we obtain $f^{-1}(\{\alpha\}) = \{x/f(x) = \alpha\} \in \mathcal{F}$, $\forall \alpha \in (-\infty, \infty)$ □

Definition 2.1.7. *Let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ be the n-dimensional product space. Denote*

$$\mathcal{L}^n = \left\{\prod_{i=1}^{n} [a_i, b_i) / -\infty < a_i \leq b_i < \infty, i = 1, 2, \cdots, n\right\}.$$

The σ-algebra $\mathcal{B}^{(n)} = \mathcal{F}(\mathcal{L}^{(n)})$ is called the Borel field on $\mathbb{R}^n$ and the sets in $\mathcal{B}^{(n)}$ are called n-dimensional Borel sets. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called an (n-ary) Borel function if and only if it is a measurable function on the measurable space $\left(\mathbb{R}^n, \mathcal{B}^{(n)}\right)$

Theorem 2.1.8. *Let $f_1, \cdots, f_n$ be measurable functions. If $g : \mathbb{R}^n \to \mathbb{R}$ is a Borel function, then $g(f_1, \cdots, f_n)$ is a measurable function.*

Proof. For any Borel set $B \subseteq (-\infty, \infty)$, we have:

$$\begin{aligned} & \left(g(f_1, \cdots, f_n)\right)^{-1}(B) \\ = \ & \{x / g\left(f_1(x), \cdots, f_n(x)\right) \in B\} \\ = \ & \{x / (f_1(x), \cdots, f_n(x)) \in g^{-1}(B)\} \end{aligned}$$

For any $E = \bigcap_{i=1}^{n} [a_i, b_i) \in \mathcal{L}^{(n)}$, we have

$$\{x / (f_1(x), \cdots, f_n(x)) \in E\} = \bigcap_{i=1}^{n} \{x / f_i(x) \in [a_i, b_i)\} \in \mathcal{F}.$$

Thus, $\{x / (f_1(x), \cdots, f_n(x)) \in F\} = \bigcap_{i=1}^{n} f_i^{-1}([a_i, b_i))$. Since $[a_i, b_i)$ is a Borel set, $f_i^{-1}([a_i, b_i)) \in \mathcal{F}$. Then,

$$\bigcap_{i=1}^{n} f_i^{-1}([a_i, b_i)) \in \mathcal{F}.$$

So, $\{x / (f_1(x), \cdots f_n(x)) \in F\} \in \mathcal{F}$ for any $F \in \mathcal{B}^{(n)}$. As g is a Borel function, $g^{-1}(B) \in \mathcal{B}^{(n)}$ for any Borel set $B \subset (-\infty, \infty)$. Thus, $\{x / (f_1(x), \cdots f_n(x)) \in g^{-1}(B)\} \in \mathcal{F}$ for any Borel set $B \subset (-\infty, \infty)$. Therefore, $g(f_1, \cdots, f_n)$ is measurable. □

Remark [A special case of Theorem 2.1.8] If f_1, f_2 are measurable, $\alpha \in (-\infty, \infty)$ is a constant, then the functions

$$\alpha f_1, f_1 + f_2, f_1 - f_2, |f_1|, f_1 \times f_2, |f_1|^{\alpha}, f_1 \vee f_2, f_1 \wedge f_2,$$

and the constant α are all measurable. Furthermore, $\{x/f_1(x) = f_2(x)\} = \{x/f_1(x) - f_2(x) = 0\} \in \mathcal{F}$.

Theorem 2.1.9. *If $\{f_n\}$ is a sequence of measurable functions, $h(x) = \sup_n \{f_n(x)\}$, and $g(x) = \inf_n \{f_n(x)\}$ for any $x \in X$. Then h and g are measurable.*

Proof. By Theorem 2.1.5, for any $\alpha \in (-\infty, \infty)$, we have

$$\{x/h(x) > \alpha\} = \{x/\sup_n \{f_n(x)\} > \alpha\} = \bigcup_{n=1}^{\infty} \{x/f_n(x) > \alpha\} \in \mathcal{F}$$

and

$$\{x/g(x) \geq \alpha\} = \{x/\inf_n \{f_n(x)\}\} = \bigcap_{n=1}^{\infty} \{x/f_n(x) \geq \alpha\} \in \mathcal{F}.$$

Therefore, h and g are measurable. □

Corollary 2.1.10. *Let $\{f_n\}$ be a sequence of measurable functions. Denote by $\overline{f}(x) = \overline{\lim}_n f_n(x)$ and $\underline{f}(x) = \underline{\lim}_n f_n(x)$. Then, $\overline{f}$ and $\underline{f}$ are measurable. Furthermore, if $\lim_n f_n$ exists , then it is also measurable.*

Proof. We have

$$\overline{f}(x) = \inf_m \sup_{n \geq m} f_n(x) \quad \text{and} \quad \underline{f}(x) = \sup_m \inf_{n \geq m} \{f_n(x)\}.$$

By Theorem 2.1.9, $\overline{f}$ and $\underline{f}$ are measurable. Assume that $\lim_n f_n$ exists and $f_n \to f$. Then, $\overline{\lim}_n f_n = \underline{\lim}_n f_n = f$. But $\overline{\lim}_n f_n$ and $\underline{\lim}_n f_n$ are measurable, hence $\lim_n f_n = f$ is measurable □

Remark • In the next sections, we will consider only measurable functions that are non-negative.

- $f, f_1, \cdots, f_n$ symbolize non-negative measurable functions.
- The class of all non-negative measurable functions is denoted by F.

2.2 "Almost" and "Pseudo Almost" Propositions

Definition 2.2.1. *Let $A \in \mathcal{F}$ and P be a proposition w.r.t points in A. If there exists $E \in \mathcal{F}$ with $\mu(E) = 0$ such that P is true on $A \setminus E$, then we say* ***"P is almost everywhere true on A"****. If there exists $F \in \mathcal{F}$ with $\mu(A \setminus F) = \mu(A)$ such that P is true on $A \setminus F$, then we say* ***"P is pseudo-almost everywhere true on A"***

Remark From now on, we denote almost everywhere by a.e and pseudo almost everywhere by p.a.e. Moreover, $\{f_n\}$ converges to f a.e will be denoted by $f_n \xrightarrow{\text{a.e}} f$, and $\{f_n\}$ converges to f p.a.e by $f_n \xrightarrow{\text{p.a.e}} f$

Example 2.2.2. *Let $X = \{0, 1\}, \mathcal{F} = \mathcal{P}(X)$ and*

$$\mu(E) = \begin{cases} 1 & E = \emptyset, \\ 0 & E = \emptyset, \end{cases}$$

for any $E \in \mathcal{F}$. Define a measurable function sequence on $(X, \mathcal{F}, \mu)$ as follows:

$$f_n(x) = \begin{cases} 1\text{-}1/n & x = 1, \\ 1/n & x = 0, \end{cases}$$

$n = 1, 2, \cdots$. Let's prove that $f_n \xrightarrow{p.a.e} 0$. Let $F = \{1\}$, then $X \setminus F = \{0\}$ and

$$\mu(X \setminus F) = \mu(\{0\}) = \mu(X) = 1.$$

We have $f_n(x) = \frac{1}{n} \to 0$. So, $f_n \to 0$ on $X \setminus F$. Therefore, $f_n \xrightarrow{p.a.e} 0$. Let's prove that $f_n \xrightarrow{p.a.e} 1$. Let $E = \{0\}$, then $X \setminus E = \{1\}$ and

$$\mu(X \setminus E) = \mu(X) = 1.$$

We have $f_n(x) = 1 - \frac{1}{n} \to 1$. So, $f_n \to 1$ on $X \setminus E$. Therefore, $f_n \xrightarrow{p.a.e} 1$. Let's check if $f_n \xrightarrow{a.e} 0$. Take $F = \emptyset$ so $\mu(F) = 0$ and $X \setminus F = X = \{0, 1\}$. So, in case $x = 0$, we have $f_n(x) = \frac{1}{n}$, so $f_n \to 0$ but in case $x = 1$, we have $f_n(x) = 1 - \frac{1}{n}$, so $f_n \to 1$. Therefore, $f_n \overset{a.e}{\not\longrightarrow} 0$. Let's check if $f_n \xrightarrow{a.e} 1$. Let $F = \emptyset$, then $\mu(F) = 0$ and $X \setminus F = X = \{0, 1\}$. So, in case $x = 1$, we have $f_n(x) = 1 - \frac{1}{n}$, so $f_n \to 1$, but in case $x = 0$, we have $f_n(x) = \frac{1}{n}$, so $f_n \to 0$. Therefore, $f_n \overset{a.e}{\not\longrightarrow} 1$.

Example 2.2.3. Let $X = \{0, 1\}$ and $\mathcal{F} = \mathcal{P}(X)$. We define

$$\mu(E) = \begin{cases} 1 & E = X, \\ 0 & E = X, \end{cases}$$

for any $E \in \mathcal{F}$ and

$$f_n(x) = \begin{cases} 1\text{-}1/n & x = 1, \\ 1/n & x = 0. \end{cases}$$

Let's prove that $f_n \xrightarrow{a.e} 0$. Take $F = \{1\}$. So, $F = X$ and $\mu(F) = 0$. We have

$f_n(x) = \frac{1}{n}$, so $f_n \to 0$. Therefore, $f_n \xrightarrow{a.e} 0$. Let's prove that $f_n \xrightarrow{a.e} 1$. Let $E = \{0\}$, so $E = X$ and $\mu(E) = 0$. So, $f_n(x) = 1 - \frac{1}{n}$ and hence $f_n \to 1$. Therefore, $f_n \xrightarrow{a.e} 0$. Let's check if $f_n \xrightarrow{p.a.e} 0$. Take $F = \emptyset$, $X \setminus F = X$ and $\mu(X \setminus F) = \mu(X) = 1$. So, in case $x = 0$, $f_n(x) = \frac{1}{n} \to 0$, but in case $x = 1$, $f_n(x) = 1 - \frac{1}{n} \not\to 0$. Therefore, $f_n \overset{p.a.e}{\not\to} 0$. Let's check if $f_n \xrightarrow{p.a.e} 1$. Let $F = \emptyset$, $X \setminus F = X$ and $\mu(X \setminus F) = \mu(X) = 1$. So, in case $x = 1$, we have $f_n(x) = 1 - \frac{1}{n} \to 1$, but in case $x = 0$, we have $f_n(x) = \frac{1}{n} \not\to 0$. Therefore, $f_n \overset{p.a.e}{\not\to} 1$.

Remark 1. In classical measure theory if both $f_n \xrightarrow{a.e} f$ and $f_n \xrightarrow{a.e} g$, then $f = g$.

2. If proposition P is true *a.e* on $A \in \mathcal{F}$, then P is also true *a.e* on any subset of A that belongs to $\mathcal{F}$. However if we replace *a.e* with *p.a.e*, this statement will no longer be true.

Example 2.2.4. *Let $X = \{a, b, c\}$ and $\mathcal{F} = \mathcal{P}(X)$. We define*

$$\mu(E) = \begin{cases} \text{Card}(E) & E = \{a, b\} \\ 0 & E = \{a, b\} \end{cases}$$

for any $E \in \mathcal{F}$ and the function f by

$$f(x) = \begin{cases} 0 & x \in \{a, b\}, \\ 1 & x = c. \end{cases}$$

Let's prove that μ is a fuzzy measure.

1. $\mu(\emptyset) = \text{Card}(\emptyset) = 0$

2. Let $E, F \in \mathcal{F}$ such that $E \subseteq F$. If $E \subseteq F$, then E has equal or less cardinality than F. Hence $\text{Card}(E) \leq \text{Card}(F)$. Therefore, $\mu(E) \leq \mu(F)$.

3. Let $\{E_n\} \subseteq \mathcal{F}$ such that $E_1 \subseteq E_2 \subseteq \cdots E_n$ and $\cup E_n \in \mathcal{F}$. We have

$$\mu(\cup E_n) = \mu(\mathcal{P}(X)) = 2^3 = 8.$$

   Thus, $\lim_n \mu(E_n) = 8$. So continuity from below is satisfied.

4. Consider $\{E_n\} \subseteq \mathcal{F}$ such that $E_1 \supseteq E_2 \supseteq \cdots E_n$ and $\cap E_n \in \mathcal{F}$. We have

$$\mu(\cap E_n) = \mu(\emptyset) = 0.$$

   Thus, $\lim_n \mu(E_n) = 0$. So continuity from above is satisfied.

Therefore, μ is a fuzzy measure. Let's check if $f = 0$ on X, p.a.e. Let $F = \{c\}$. We have

$$\mu(X \setminus F) = \mu(\{a, b\}) = 3,$$

and $\mu(X) = \text{Card}(X) = 3$. On $X \setminus F$, we have $f(x) = 0$. Therefore, $f = 0$ on X, p.a.e. Let's prove that "$f = 0$ on $\{a, c\}$ p.a.e" is not true. By contradiction, assume $f = 0$ on $\{a, c\}$ p.a.e, so $\exists E \in \mathcal{F}$ such that

$$\mu(\{a, c\} \setminus E) = \mu(\{a, c\})$$

and $f = 0$ *on* $\{a, c\} \setminus E$. *If we take* $E = \{a\}$, *we have*

$$\mu(E) = \mu\left(\{x / f(x) = 0, x \in \{a, c\}\}\right) = 1 = \mu(\{a, c\}) = 2,$$

which is a contradiction. Therefore $f = 0$ *on* $\{a, c\}$ *p.a.e is not true.*

Definition 2.2.5. *Let* $A \in \mathcal{F}, f \in F$ *and* $\{f_n\} \subseteq F$. *If* $\exists \{E_k\} \subseteq \mathcal{F}$ *with* $\lim_k \mu(E_k) = 0$ *such that* $\{f_n\}$ *converges to* f *on* $A \setminus E_k$ *uniformly for any fixed* $k = 1, 2, \cdots$, *then, we say that* $\{f_n\}$ *converges to* f *on* A ***almost uniformly*** *and it is denoted by* $f_n \xrightarrow{a.u} f$. *If* $\exists \{F_k\} \subseteq \mathcal{F}$ *with* $\lim_k \mu(A \setminus F_k) = \mu(A)$ *such that* $\{f_n\}$ *converges to* f *on* $A \setminus F_k$ *uniformly for any fixed* $k = 1, 2, \cdots$, *then, we say that* $\{f_n\}$ *converges to* f *on* A ***pseudo- almost uniformly*** *and it is denoted by* $f_n \xrightarrow{p.a.u} f$.

Definition 2.2.6. *Let* $A \in \mathcal{F}, f \in F, \{f_n\} \subseteq F$. *If*

$$\lim_n \mu\left(\{x / |f_n(x) - f(x)| \geq c\} \cap A\right) = 0$$

for any $c > 0$, *then we say that* $\{f_n\}$ ***converges in*** μ *(or converges in measure) to* f *on* A *and it is denoted by* $f_n \xrightarrow{\mu} f$ *on* A. *If*

$$\lim_n \mu(\{x / |f_n(x) - f(x)| < c\} \cap A) = \mu(A),$$

for any $c > 0$, *then we say that* $\{f_n\}$ ***converges-pseudo in*** μ *(or converges pseudo in measure) to* f *on* A *and denote it by* $f_n \xrightarrow{p.\mu} f$ *on* A.

Remark In Definitions 2.2.1, 2.2.5, 2.2.6, when $A = X$, we omit “on A” from the statements.

Example 2.2.7. *Let* $X = [0, \infty)$, $\mathcal{F} = \mathcal{B}_+$, *and* μ *be the Lebesgue measure. Here* $\mathcal{B}_+$ *is the class of all Borel sets in* $[0, +\infty)$.

- *Let's check if* $f_n \xrightarrow{p.a.u} f$. *Let* $f_n(x) = \frac{x}{n}$, $n = 1, 2, \cdots$ *and let* $f(x) = 0$. *We have*

$$\lim_n \mu\left(X \setminus \left\{\frac{x}{n}\right\}\right) = \mu(X)$$

  and $\frac{x}{n} \to 0$ *on* $X \setminus \{\frac{x}{n}\}$ *uniformly. Hence* $f_n \xrightarrow{p.a.u} f$

- *Let's check if* $f_n \xrightarrow{p.u} f$. *We want to prove that*

$$\lim_n \mu\left(\{x / |f_n(x) - f(x)| < c\} \cap X\right) = \mu(X)$$

  for any $c \geq 0$. *We have*

$$\begin{aligned} & \lim_n \mu\left(\left\{x / \left|\frac{x}{n} - 0\right| < c\right\} \cap [0, \infty)\right) \\ = \; & \lim_n \mu\left(\left\{x / \left|\frac{x}{n}\right| < c\right\} \cap [0, \infty)\right) \\ = \; & \lim_n \mu\left(\{x / -nc < x < nc\} \cap [0, \infty)\right) \\ = \; & \lim_n \mu([0, nc)) = \infty = \mu([0, \infty)). \end{aligned}$$

- *Let's check if* $f_n \xrightarrow{\mu} f$ *on* X. *We want to prove that*

$$\lim_n \mu\left(\{x / |f_n(x) - f(x)| \geq c\} \cap X\right) = \mu(X)$$

for any $c > 0$. *We have*

$$
\begin{aligned}
& \lim_n \mu\left(\{x/\ \frac{x}{n} - 0 \ \geq c\} \cap [0, \infty)\right) \\
= \ & \lim_n \mu\left(\{x/\ \frac{x}{n} \ \geq c\} \cap [0, \infty)\right) \\
= \ & \lim_n \mu\left(\{x/ - nc \leq x \leq nc\} \cap [0, \infty)\right) \\
= \ & \lim_n \mu([nc, \infty)) = 0.
\end{aligned}
$$

Chapter 3

Fuzzy Integrals

In this chapter, we will define the fuzzy integral and study its properties. We will also give several convergence theorems of fuzzy integral sequence. Moreover, we will discuss the transformation theorem for fuzzy integrals and explain how to define a fuzzy measure using a fuzzy integral.

3.1 Introduction

In this chapter, we assume that $(X, \mathcal{F})$ is a measurable space where $X \in \mathcal{F}$, $\mu : \mathcal{F} \to [0, \infty]$ is a fuzzy measure (or a nonnegative monotone set function for section 1.6) and F is the class of all finite nonnegative measurable functions defined on $(X, \mathcal{F})$. For any given $f \in F$, we write

$$F_\alpha = \{x / f(x) \geq \alpha\},$$

$$F_{\alpha+} = \{x / f(x) > \alpha\},$$

where $\alpha \in [0, \infty]$. We call F_α and $F_{\alpha+}$ the $\alpha-$ cut and the strict $\alpha-$ cut of f, respectively. We will take the convention:

$$\inf_{x \in \emptyset} f(x) = \infty,$$

since the range of the function f that we consider in this chapter is $[0, \infty)$.

Definition 3.1.1. *Let $A \in \mathcal{F}, f \in F$. The fuzzy integral of f (also called Sugeno Integral) on A with respect to μ, denoted by $⨍_A f d\mu$ is defined by*

$$⨍_A f d\mu = \sup_{\alpha \in [0, \infty]} [\alpha \wedge \mu(A \cap F_\alpha)].$$

When $A = X$, the fuzzy integral may also be denoted by $⨍ f d\mu$. When we write $⨍_A f d\mu$, it directly implies that $A \in \mathcal{F}$ and $f \in F$.

The geometric significance of $⨍_A f d\mu$. If $X = (-\infty, \infty)$, $\mathcal{F}$ is the Borel field $\mathcal{B}$, μ is the Lebesgue measure and $f : X \to [0, \infty)$ is a unimodel continous function, then $⨍_A f d\mu$ is the edge's length of the largest square between the curve of $f(x)$ and the $x-$ axis.

Lemma 3.1.2. *1. Both F_α and $F_{\alpha+}$ are nonincreasing with respect to α and $F_{\alpha+} \supseteq F_\beta$ when $\alpha < \beta$.*

2. We have

$$\lim_{\beta \to \alpha^-} F_\beta = \lim_{\beta \to \alpha^-} F_{\beta+} = F_\alpha \supset F_{\alpha+} = \lim_{\beta \to \alpha^+} F_\beta = \lim_{\beta \to \alpha^+} F_{\beta+}$$

Proof. 1. Let $\alpha_1, \alpha_2 \in [0, \infty]$ such that $\alpha_1 < \alpha_2$. Take $x \in F_{\alpha_2}$. Since $\alpha_1 < \alpha_2 \le f(x)$, we have $\{x/f(x) \ge \alpha_2\} \subseteq \{x/f(x) \ge \alpha_1\}$. Then,

$F_{\alpha_2} \subseteq F_{\alpha_1}$. This means that F_α is non-increasing. Now, let $\alpha, \beta \in [0, \infty]$ such that $\alpha < \beta$. Take $x \in F_\beta$. Since $\alpha < \beta \le f(x)$, we get $\{x/f(x) \ge \beta\} \subseteq \{x/f(x) > \alpha\}$. Hence, $F_\beta \subseteq F_{\alpha+}$.

2. Since F_β and $F_{\beta+}$ are nonincreasing with respect to α, we have

$$\lim_{\beta \to \alpha^-} F_\beta = \bigcap_{\beta<\alpha} \{x,\ f(x) \ge \beta\}$$

$$\lim_{\beta \to \alpha^-} F_\beta = \bigcap_{\beta<\alpha} \{x/f(x) > \beta\}$$

Thus,

$$\begin{aligned}
\bigcap_{\beta<\alpha} \{x/f(x) \ge \beta\} &= \bigcap_{\beta<\alpha} \{x/f(x) > \beta\} \\
&= \{x/f(x) \ge \alpha\} \supseteq \{x/f(x) > \alpha\} \\
&= \underset{\beta>\alpha}{} \{x,\ f(x) \ge \beta\} \\
&= \underset{\beta>\alpha}{} \{x,\ f(x) > \beta\}.
\end{aligned}$$

□

Theorem 3.1.3. *We have*

$$\begin{aligned}
\underset{A}{-} f d\mu &= \sup_{\alpha \in [0,\infty)} \alpha \wedge \mu(A \cap F_\alpha) \\
&= \sup_{\alpha \in [0,\infty]} \alpha \wedge \mu(A \cap F_{\alpha^+}) \\
&= \sup_{\alpha \in [0,\infty)} \alpha \wedge \mu(A \cap F_{\alpha^+}) \\
&= \sup_{E \in F(f)} \left(\inf_{x \in E} f(x)\right) \wedge \mu(A \cap E) \\
&= \sup_{E \in F} \left(\inf_{x \in E} f(x)\right) \wedge \mu(A \cap E)
\end{aligned}$$

where $\mathcal{F}(f)$ is the δ-algebra genrated by f, the smallest δ-algebra such that f is measurable.

Proof. When $\alpha = +\infty$, we have $F_\alpha = F_{\alpha^+} = \emptyset$ and so

$$-\!\!\!\!\!\int_A f d\mu = \sup_{\alpha\in[0,+\infty]} [\alpha \wedge \mu(A \cap F_\alpha)] = \sup_{\alpha\in[0,+\infty)} [\alpha \wedge \mu(A \cap F_\alpha)].$$

Similarly, $\sup_{\alpha\in[0,+\infty]} [\alpha \wedge \mu(A \cap F_{\alpha^+})] = \sup_{\alpha\in[0,+\infty)} [\alpha \wedge \mu(A \cap F_{\alpha^+})]$. Let's prove that

$$\sup_{x\in[0,\infty)} [\alpha \wedge \mu(A \cap F_\alpha)] = \sup_{x\in[0,\infty)} [\alpha \wedge \mu(A \cap F_{\alpha^+})].$$

By Lemma 3.1.2, $F_{\alpha^+} \subseteq F_\alpha$. So, $A \cap F_{\alpha^+} \subseteq A \cap F_\alpha$. Since μ is monotone, $\mu(A \cap F_{\alpha^+}) \leq \mu(A \cap F_\alpha)$ for any given $\alpha \in [0, \infty)$. Then,

$$\sup_{x\in[0,\infty)} [\alpha \wedge \mu(A \cap F_{\alpha^+})] \leq \sup_{x\in[0,\infty)} [\alpha \wedge \mu(A \cap F_\alpha)].$$

Let $c > 0, \alpha \in (0, \infty)$ and let $\alpha \in ((\alpha - c) \vee \alpha)$. We have $\alpha < \alpha$, so $F_\alpha \subseteq F_{\alpha^{t+}}$(by Lemma 3.1.2). Thus, $A \cap F_\alpha \subseteq A \cap F_{\alpha^{t+}}$. Then, $\mu(A \cap F_\alpha) \leq \mu(A \cap F_{\alpha^{t+}})$. It means that $\alpha \wedge \mu(A \cap F_\alpha) \leq (\alpha^{t+} c) \wedge \mu(A \cap F_\alpha^{t+})$. Hence, we have

$$\begin{aligned}
\sup_{\alpha\in[0,\infty)} \alpha \wedge \mu\big(A \cap F_\alpha \quad &= \sup_{\alpha\in(0,\infty)} \alpha \wedge \mu\big(A \cap F_\alpha \\
&\leq \sup_{\alpha^t\in(0,\infty)} \big(\alpha + c \;\; \wedge \mu\big(A \cap F_\alpha^{t+} \\
&\leq \sup_{\alpha^t\in(0,\infty)} [\alpha \;\wedge \mu\big(A \cap F_\alpha^{+} \quad + c \\
&= \sup_{\alpha\in[0,\infty)} \alpha \wedge \mu\big(A \cap F_\alpha^{+} \quad + c.
\end{aligned}$$

As c tends to 0, we get

$$\sup_{\alpha\in[0,+\infty)} [\alpha \wedge \mu(A \cap F_{\alpha}) \leq \sup_{\alpha\in[0,\infty)} [\alpha \wedge \mu(A \cap F_{\alpha^+}).$$

Therefore, $\sup_{\alpha\in[0,+\infty)} [\alpha \wedge \mu(A \cap F_{\alpha}) = \sup_{\alpha\in[0,\infty)} [\alpha \wedge \mu(A \cap F_{\alpha^+})$. We still have to prove that

$$\text{--}_{A} \boldsymbol{f} d\mu = \sup_{E\in \mathcal{F}(\boldsymbol{f})} [(\inf_{x\in E} \boldsymbol{f}(x)) \wedge \mu(A \cap E)] = \sup_{E\in \mathcal{F}} [(\inf_{x\in E} \boldsymbol{f}(x)) \wedge \mu(A \cap E)].$$

For any $\alpha \in [0, +\infty]$, since $\inf_{x\in F_{\alpha}} \boldsymbol{f}(x) \geq \alpha$ and denoting $F_{\alpha} \in \mathcal{F}(\boldsymbol{f})$, we have

$$[\alpha \wedge \mu(A \cap F_{\alpha})] \leq \sup_{E\in \mathcal{F}(\boldsymbol{f})} [(\inf_{x\in \mathcal{F}(\boldsymbol{f})}) \wedge \mu(A \cap E)].$$

Next, since $\boldsymbol{f}$ is $\mathcal{F}$ measurable, we have $\mathcal{F}(\boldsymbol{f}) \subseteq \mathcal{F}$, and therefore,

$$\sup_{E\in \mathcal{F}(\boldsymbol{f})} [(\inf_{x\in E} \boldsymbol{f}(x)) \wedge \mu(A \cap E)] \leq \sup_{E\in \mathcal{F}} [(\inf_{x\in E} \boldsymbol{f}(x)) \wedge \mu(A \cap E)].$$

Finally, for any given $E \in \mathcal{F}$, take $\alpha = \inf_{x\in E} \boldsymbol{f}(x)$, then $E \subseteq F_{\alpha^t}$. Hence, $\mu(A \cap E) \leq \mu(A \cap F_{\alpha^t})$ (by the monotonicity of μ). Now, we have

$$[\inf_{x\in E} \boldsymbol{f}(x)] \wedge \mu(A \cap E) \leq \alpha \wedge \mu(A \cap F_{\alpha}) \leq \sup_{\alpha\in[0,\infty]} [\alpha \wedge \mu(A \cap F_{\alpha})] = \quad \boldsymbol{f} d\mu,$$

for any $E \in \mathcal{F}$. So,

$$\sup_{E\in \mathcal{F}} [(\inf_{x\in E} \boldsymbol{f}(x)) \wedge \mu(A \cap E)] \leq \text{--}_{A} \boldsymbol{f} d\mu.$$

Therefore, $\text{--}_{A} \boldsymbol{f} d\mu = \sup_{E\in \mathcal{F}} [\inf_{x\in E} \boldsymbol{f}(x) \wedge \mu(A \cap E)]$. □

Remark To simplify the calculation of the fuzzy integral, for a given $(X, \mathcal{F}, \mu)$, $f \in F$, and $A \in \mathcal{F}$, we write

$$T = \left\{ \alpha/\alpha \in [0, \infty], \ \mu(A \cap F_\alpha) > \mu(A \cap F_\beta) \quad \text{for any given } \beta > \alpha \right\}.$$

So, we get

$$\fint_A f d\mu = \sup_{\alpha \in [0, \infty]} [\alpha \wedge \mu(A \cap F_\alpha)] = \sup_{\alpha \in T} [\alpha \wedge \mu(A \cap F_\alpha)]$$

Example 3.1.4. *Consider the fuzzy measure space given in Example 2.2.4. We recall that* $X = \{a, b, c\}$, $\mathcal{F} = \mathcal{P}(X)$, $F_\alpha = \{x/f(x) \geq \alpha\}$,

$$\mu(E) = \begin{cases} \text{Card}(E) & E = \{a, b\} \\ 3 & E = \{a, b\} \text{ for any } E \in \mathcal{F} \end{cases}$$

and

$$f(x) = \begin{cases} 3 & \text{if } x = a \\ 2.5 & \text{if } x = b \\ 2 & \text{if } x = c \end{cases}$$

We have $\fint_A f d\mu = [3 \wedge \mu(\{a\})] \vee [\mu(\{a, b\})] \vee [2 \wedge \mu(X)] = 1 \vee 2.5 \vee 2 = 2.5$

Example 3.1.5. *Let* $X = [0, 1]$, $\mathcal{F}$ *be the class of all Borel sets in* X, $\mu = m^2$ *where* m *is the Lebesgue measure and* $f(x) = \frac{x}{2}$. *We have*

$$F_\alpha = \{x/f(x) \geq \alpha\} = [2\alpha, 1].$$

Since $T = [0, \frac{1}{2})$*, we only need to consider* $\alpha \in [0, \frac{1}{2})$*. So we have*

$$\fint_A f d\mu = \sup_{\alpha \in [0, \frac{1}{2})} [\alpha \wedge \mu(F_\alpha)] = \sup_{\alpha \in [0, \frac{1}{2})} [\alpha \wedge (1 - 2\alpha)^2].$$

In this expression, $(1 - 2\alpha)^2$ *is a decreasing continuous function of* α *when* $\alpha \in [0, \frac{1}{2})$*. Hence, the supremum will be attained at the point which is one of the solutions of the equation* $\alpha = (1 - 2\alpha)^2$*. That is, at* $\alpha = \frac{1}{4}$ *so* $\fint_A f d\mu = \frac{1}{4}$*.*

3.2 Properties of the Fuzzy Integral

The following theorem gives the most elementary properties of the fuzzy integral.

Theorem 3.2.1. *1. If* $\mu(A) = 0$*, then* $\fint_A f d\mu = 0$ *for any* $f \in F$*.*

2. *If* $\fint_A f d\mu = 0$*, then* $\mu(A \cap \{x / f(x) > 0\}) = 0$*.*

3. *If* $f_1 \leq f_2$*, then* $\fint_A f_1 d\mu \leq \fint_A f_2 d\mu$*.*

4. $\fint_A f d\mu = \fint f \chi_A d\mu$ *, where* χ_A *is the characteristic function of* A*.*

5. $\fint_A a d\mu = a \wedge \mu(A)$ *for any constant* $a \in [0, \infty)$*.*

6. $\fint_A (f + a) d\mu \leq \fint_A f d\mu + \fint_A a d\mu$ *for any constant* $a \in [0, \infty)$

Proof. 1. If $\mu(A) = 0$, then

$$\fint_A f d\mu = \sup_{\alpha \in [0, \infty]} [\alpha \wedge \mu(A \cap F_\alpha)] = \sup_{\alpha \in [0, \infty]} [\alpha \wedge 0] = \sup_{\alpha \in [0, \infty]} (0) = 0.$$

2. Assume that $\mu(A \cap \{x/f(x) > 0\}) = c > 0$. We have

$$A \cap \{x/f(x) \geq \frac{1}{n}\} \quad A \cap \{x/f(x) > 0\}.$$

By using the continuity from below of μ, we have $\lim_n \mu(A \cap \{x/f(x) \geq \frac{1}{n}\}) = c$. Take $c = \frac{c}{2}$, then, $\exists n_0$ such that

$$\mu(A \cap F_{\frac{1}{n_0}}) = \mu \; A \cap \{x/f(x) \geq \frac{1}{n_0}\} \; \geq \frac{c}{2}.$$

Consequently, we have:

$$-_A f d\mu = \sup_{\alpha \in [0,\infty]} [\alpha \wedge \mu(A \cap F_\alpha)] \geq \frac{1}{n_0} \wedge \frac{c}{2} > 0.$$

This contradicts $-_A f d\mu = 0$.

3. Assume that $f_1 \leq f_2$. Let's prove $-_A f_1 d\mu \leq -_A f_2 d\mu$. We recall that

$$-_A f_1 d\mu = \sup_{\alpha \in [0,\infty]} [\alpha \wedge \mu(A \cap F^1_\alpha)],$$

$$-_A f_2 d\mu = \sup_{\alpha \in [0,\infty]} [\alpha \wedge \mu(A \cap F^2_\alpha)],$$

where $F^2_\alpha = \{x/f_2(x) \geq \alpha\}$ and $F^1_\alpha = \{x/f_1(x) \geq \alpha\}$. But $\alpha \leq f_1(x) \leq f_2(x)$. So, $F_{1\alpha} \subseteq F_{2\alpha}$.Then, $A \cap F_{1\alpha} \subseteq A \cap F_{2\alpha}$. By the monotonicity of μ, we get

$$\mu(A \cap F_{1\alpha)} \leq \mu(A \cap F_{2\alpha}).$$

So, $\alpha \wedge \mu(A \cap F_{1\alpha}) \leq \alpha \wedge \mu(A \cap F^2_\alpha)$ and hence

$$\sup_{\alpha\in[0,\infty]} [\alpha \wedge \mu(A \cap F^1_\alpha)] \leq \sup_{\alpha\in[0,\infty]} [\alpha \wedge \mu(A \cap F^2_\alpha).$$

Thus, $⨍_A f_1 d\mu \leq ⨍_A f_2 d\mu$

4. We have $⨍_A f d\mu = \sup(\alpha \wedge \mu(A \cap F_\alpha))$ where $F_\alpha = \{x/f(x) \geq \alpha\}$. Thus,

$$⨍ f.\chi_A d\mu = \sup[\alpha \wedge \mu(X \cap F^1_\alpha)],$$

where $F^1_\alpha = \{x/f.\chi_A(x) \geq \alpha\} = \{x \in A/f(x).1 \geq \alpha\} = F_\alpha$. Therefore, $⨍_A f d\mu = ⨍ f.\chi_A d\mu$.

5. Let's calculate $⨍ a d\mu$. We have $F_\alpha = \{x/f(x) \geq \alpha\} = \{x/a \geq \alpha\}$. Then,

$$⨍_A a d\mu = \sup_{\alpha\in[0,\infty]} [\alpha \wedge \mu(A \cap F_\alpha)].$$

Now F_α is either $\emptyset$ or X. So, $F_\alpha \cap A$ is either $\emptyset$ or A and hence $\mu(A \cap F_\alpha)$ is either 0 or $\mu(A)$. So, $⨍_A a d\mu = a \wedge \mu(A)$.

6. We have

$$\begin{aligned}
⨍_A (f + a) d\mu &= \sup_{E\in F} \left(\inf_{x\in E} f(x) + a \wedge \mu\left(A \cap E\right)\right) \\
&\leq \sup_{E\in F} \left(\left(\inf_{x\in E} f(x) \wedge \mu\left(A \cap E\right)\right) + \left(a \wedge \mu\left(A \cap E\right)\right)\right) \\
&\leq \sup_{E\in F} \left(\left(\inf_{x\in E} f(x) \wedge \mu\left(A \cap E\right)\right) + \left(a \wedge \mu\left(A\right)\right)\right) \\
&= \sup_{E\in F} \left(\inf_{x\in E} f(x) \wedge \mu\left(A \cap E\right)\right) + \left(a \wedge \mu\left(A\right)\right) \\
&= ⨍_A f d\mu + ⨍_A a d\mu
\end{aligned}$$

$\square$

Corollary 3.2.2. *1. If $A \supseteq B$, then $-_A \boldsymbol{f} d\mu \geq -_B \boldsymbol{f} d\mu$*

2. $-_A(\boldsymbol{f}_1 \vee \boldsymbol{f}_2)d\mu \geq -_A \boldsymbol{f}_1 d\mu \vee -_A \boldsymbol{f}_2 d\mu$

3. $-_A(\boldsymbol{f}_1 \wedge \boldsymbol{f}_2)d\mu \leq -_A \boldsymbol{f}_1 d\mu \wedge -_A \boldsymbol{f}_2 d\mu$

4. $-_A \boldsymbol{f} d\mu \geq -_A \boldsymbol{f} d\mu \vee -_A \boldsymbol{f} d\mu$

5. $-_A \boldsymbol{f} d\mu \leq -_A \boldsymbol{f} d\mu \wedge -_A \boldsymbol{f} d\mu$

Proof. 1. Assume that $B \subseteq A$. We have $B \cap F_\alpha \subseteq A \cap F_\alpha$. So,

$$\mu(B \cap F_\alpha) \leq \mu(A \cap F_\alpha).$$

Then, $\alpha \wedge \mu(B \cap F_\alpha) \leq \alpha \wedge \mu(A \cap F_\alpha)$. So,

$$\sup_{\alpha \in [0,\infty]} [\alpha \wedge \mu(B \cap F_\alpha)] \leq \sup_{\alpha \in [0,\infty]} [\alpha \wedge \mu(A \cap F_\alpha)].$$

Hence, $-_B \boldsymbol{f} d\mu \leq -_A \boldsymbol{f} d\mu$.

2. We have

$$\{x/f_1(x) \geq \alpha\} \subseteq \{x/(\boldsymbol{f}_1 \vee \boldsymbol{f}_2)(x) \geq \alpha\}$$

$$\{x/f_2(x) \geq \alpha\} \subseteq \{x/(\boldsymbol{f}_1 \vee \boldsymbol{f}_2)(x) \geq \alpha\}.$$

Then, $A \cap F_\alpha^2 \subseteq A \cap F_\alpha$ and $A \cap F_\alpha^1 \subseteq A \cap F_\alpha$. So,

$$\alpha \wedge \mu(A \cap F_\alpha^2) \leq \alpha \wedge \mu(A \cap F_\alpha)$$

$$\alpha \wedge \mu(A \cap F_\alpha^1) \leq \alpha \wedge \mu(A \cap F_\alpha).$$

Consequently, $\int_A \boldsymbol{f}_2 d\mu \leq \int_A \boldsymbol{f}_1 \vee \boldsymbol{f}_2 d\mu$ and $\int_A \boldsymbol{f}_1 d\mu \leq \int_A \boldsymbol{f}_1 \vee \boldsymbol{f}_2 d\mu$. Therefore, $\int_A \boldsymbol{f}_1 d\mu \vee \int_A \boldsymbol{f}_2 d\mu \leq \int_A (\boldsymbol{f}_1 \vee \boldsymbol{f}_2) d\mu$.

3. We have

$$\{x/(\boldsymbol{f}_1 \wedge \boldsymbol{f}_2)(x) \geq \alpha\} \subseteq \{x/\boldsymbol{f}_1(x) \geq \alpha\}$$

$$\{x/(\boldsymbol{f}_1 \wedge \boldsymbol{f}_2)(x) \geq \alpha\} \subseteq \{x/\boldsymbol{f}_2(x) \geq \alpha\}.$$

Then, $A \cap F_\alpha \subseteq A \cap F_\alpha^1$ and $A \cap F_\alpha \subseteq A \cap F_\alpha^2$. So,

$$\alpha \wedge \mu(A \cap F_\alpha) \leq \alpha \wedge \mu(A \cap F_\alpha^1)$$

$$\alpha \wedge \mu(A \cap F_\alpha) \leq \alpha \wedge \mu(A \cap F_\alpha^2).$$

Consequently, $\int_A \boldsymbol{f}_1 \wedge \boldsymbol{f}_2 d\mu \leq \int_A \boldsymbol{f}_1 d\mu$ and $\int_A \boldsymbol{f}_1 \wedge \boldsymbol{f}_2 d\mu \leq \int_A \boldsymbol{f}_2 d\mu$. Therefore, $\int_A \boldsymbol{f}_1 \wedge \boldsymbol{f}_2 d\mu \leq \int_A \boldsymbol{f}_1 d\mu \wedge \int_A \boldsymbol{f}_2 d\mu$.

4. We have $A \subseteq A \cup B$ and $B \subseteq A \cup B$. So, $A \cap F_\alpha \subseteq (A \cup B) \cap F_\alpha$ and $B \cap F_\alpha \subseteq (A \cup B) \cap F_\alpha$. Then,

$$\alpha \wedge \mu(A \cap F_\alpha) \leq \alpha \wedge \mu((A \cup B) \cap F_\alpha)$$

$$\alpha \wedge \mu(B \cap F_\alpha) \leq \alpha \wedge \mu((A \cup B) \cap F_\alpha).$$

So, $\int_A \boldsymbol{f} d\mu \leq \int_{A \cup B} \boldsymbol{f} d\mu$ and $\int_B \boldsymbol{f} d\mu \leq \int_{A \cup B} \boldsymbol{f} d\mu$. Therefore, $\int_A \boldsymbol{f} d\mu \vee \int_B \boldsymbol{f} d\mu \leq \int_{A \cup B} \boldsymbol{f} d\mu$.

5. $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Then, $(A \cap B) \cap F_\alpha \subseteq A \cap F_\alpha$ and

$(A \cap B) \cap F_\alpha \subseteq B \cap F_\alpha$. So,

$$\alpha \wedge \mu((A \cap B) \cap F_\alpha) \leq \alpha \wedge \mu(A \cap F_\alpha)$$

$$\alpha \wedge \mu((A \cap B) \cap F_\alpha) \leq \alpha \wedge \mu(B \cap F_\alpha).$$

Then, $\int_{A\cap B} f d\mu \leq \int_A f d\mu$ and $\int_{A\cap B} f d\mu \leq \int_B f d\mu$. Hence, $\int_{A\cap B} f d\mu \leq \int_A f d\mu \wedge \int_B f d\mu$.

□

Remark In general, the fuzzy integral lacks some important properties that Lebesgue's integral possesses. For example, Lebesgue's integral has linearity, that is

$$\int_A (f_1 + f_2) d\mu = \int_A f_1 d\mu + \int_A f_2 d\mu$$

and

$$\int_A af\, d\mu = a \int_A f d\mu$$

But the fuzzy integral does not. We can see this in the following example:

Example 3.2.3. *Let $X = [0, 1]$ and $\mathcal{F}$ be the class of all the Borel sets in X (namely $\mathcal{B} \cap [0, 1]$) and μ be the Lebesgue measure. We take $f(x) = x$ for any $x \in X$, and $a = \frac{1}{2}$. Then, we have*

$$F_\alpha = \{x / f(x) \geq \alpha\} = \{x / \frac{x}{2} \geq \alpha\} = [2\alpha, 1].$$

Now,

$$
\begin{aligned}
- af\, d\mu &= - \frac{x}{2} d\mu \\
&= \sup_{\alpha \in [0,\infty]} [\alpha \wedge \mu(A \cap F_\alpha)] \\
&= \sup_{\alpha \in [0,\infty]} [\alpha \wedge \mu([2\alpha, 1])] \\
&= (\alpha \wedge (1 - 2\alpha)) \\
&= \begin{cases} 1 - 2\alpha & \text{if } \alpha \geq \frac{1}{3} \\ \alpha & \text{if } \alpha < \frac{1}{3} \end{cases} \\
&= \frac{1}{3}
\end{aligned}
$$

However,

$$
\begin{aligned}
a - f\, d\mu &= \frac{1}{2} - x\, d\mu \\
&= \frac{1}{2} \sup_{\alpha \in [0,\infty]} [\alpha \wedge \mu(A \cap F_\alpha)] \\
&= \frac{1}{2} \sup_{\alpha \in [0,\infty]} [\alpha \wedge \mu(A \cap [\alpha, 1])] \\
&= \frac{1}{2} \sup_{\alpha \in [0,\infty]} [\alpha \wedge (1 - \alpha)] \\
&= \begin{cases} \alpha & \text{if } \alpha \leq \frac{1}{2} \\ 1 - \alpha & \text{if } \alpha \geq \frac{1}{2} \end{cases} \\
&= \frac{1}{2}
\end{aligned}
$$

Therefore, $- af\, d\mu = a - f\, d\mu$.

Lemma 3.2.4. *Let $A \in \mathcal{F}$, $a \in [0, \infty)$, $f_1 \in F$ and $f_2 \in F$. If*

$$|f_1 - f_2| \le a,$$

on A, then we have $\left| \fint_A f_1 d\mu - \fint_A f_2 d\mu \right| \le a$

Proof. Since $|f_1 - f_2| \le a$, then $-a \le f_1 - f_2 \le a$. So, $f_1 \le f_2 + a$ on A, using properties 3, 5, 6 of the fuzzy integral (Theorem 3.2.1), we have

$$\fint_A f_1 d\mu \le \fint_A (f_2 + a) d\mu \le \fint_A f_2 d\mu + \fint_A a d\mu = \fint_A f_2 d\mu + [a \wedge \mu(A)] \le \fint_A f_2 d\mu + a$$

Similarily, since $f_2 \le f_1 + a$ on A, then

$$\fint_A f_2 d\mu \le \fint_A (f_2 + a) d\mu \le \fint_A f_1 d\mu + \fint_A a d\mu = \fint_A f_1 d\mu + [a \wedge \mu(A)] \le \fint_A f_1 d\mu + a.$$

Therefore, Card $\left(\fint_A f_1 d\mu - \fint_A f_2 d\mu \right) \le a$ □

Lemma 3.2.5. *We have*

$$\fint_A f d\mu \le \alpha \vee \mu(A \cap F_{\alpha^+}) \le \alpha \vee \mu(A \cap F_\alpha) \forall \alpha \in [0, \infty]$$

Proof. We calculate

$$\begin{aligned}
\fint_A f d\mu &= \sup_{\alpha' \in [0, \alpha]} [\alpha \wedge \mu(A \cap F_{\alpha}^{\prime +})] \\
&\le \sup \alpha \vee \sup_{\alpha' \in (\alpha, \infty]} \mu(A \cap F_{\alpha}^{\prime +}) \\
&\le \alpha \vee \mu(A \cap F_\alpha^+) \\
&\le \alpha \vee \mu(A \cap F_\alpha)
\end{aligned}$$

□

Lemma 3.2.6. *We have* $\fint_A f d\mu = \infty \iff \mu(A \cap F_\alpha) = \infty,\ \forall \alpha \in [0, \infty)$

Proof. $\Rightarrow$) If $\fint_A f d\mu = \infty$, then by Lemma 3.2.5, $\alpha \vee \mu(A \cap F_\alpha) = \infty$. So, if $\alpha \in [0, \infty)$, then $\mu(A \cap F_\alpha) = \infty$.

$\Leftarrow$) If $\mu(A \cap F_\alpha) = \infty$, $\forall \alpha \in [0, \infty)$, then,

$$\fint_A f d\mu = \sup_{\alpha \in [0, \infty]} [\alpha \wedge \mu(A \cap F_\alpha)] = \sup_{\alpha \in [0, \infty]} [\alpha \wedge \infty] = \infty$$

□

Lemma 3.2.7. *For any* $\alpha \in [0, \infty)$, *we have*

1. $\fint_A f d\mu \geq \alpha \iff \mu(A \cap F_\beta) \geq \alpha$ *for any* $\beta < \alpha \Leftarrow \mu(A \cap F_\alpha) \geq \alpha$
2. $\fint_A f d\mu < \alpha \iff \exists \beta < \alpha$ *such that* $\mu(A \cap F_\beta) < \alpha \Rightarrow \mu(A \cap F_\alpha) < \alpha \Rightarrow \mu(A \cap F_{\alpha^+}) < \alpha$
3. $\fint_A f d\mu \leq \alpha \iff \mu(A \cap F_{\alpha^+}) \leq \alpha \Leftarrow \mu(A \cap F_\alpha) \leq \alpha$
4. $\fint_A f d\mu > \alpha \iff \mu(A \cap F_{\alpha^+}) > \alpha \Rightarrow \mu(A \cap F_\alpha) > \alpha$
5. $\fint_A f d\mu = \alpha \iff$ *for any* $\beta < \alpha$, $\mu(A \cap F_\beta) \geq \alpha \geq \mu(A \cap F_{\alpha^+}) \Leftarrow \mu(A \cap F_\alpha) = \alpha$

   When $\mu(A) < \infty$, *we have*

6. $\fint_A f d\mu \geq \alpha \iff \mu(A \cap F_\alpha) \geq \alpha$
7. $\fint_A f d\mu = \alpha \iff \mu(A \cap F_\alpha) \geq \alpha \geq \mu(A \cap F_{\alpha^+})$

Proof. 1. We only need to consider the case when $\alpha \in (0, \infty)$. If $\mu(A \cap F_\beta) \geq \alpha$ for any $\beta < \alpha$, then

$$
\begin{aligned}
\fint_A f d\mu &= \sup_{\beta \in [0,\infty)} [\beta \wedge \mu(A \cap F_\beta)] \\
&\geq \sup_{\beta \in [0,\infty)} [\beta \wedge \mu(A \cap F_\beta)] \\
&\geq \sup_{\beta \in [0,\infty)} [\beta \wedge \alpha] \\
&= \sup_{\beta \in [0,\alpha)} \beta \\
&= \alpha
\end{aligned}
$$

Conversely, if $\exists \beta < \alpha$ such that $\mu(A \cap F_\beta) < \alpha$. Then by Lemma 3.2.5, we have $\fint_A f d\mu \leq \beta \vee \mu(A \cap F_\beta) < \alpha$. So

$$\fint_A f d\mu \geq \alpha \iff \mu(A \cap F_\beta) \geq \alpha \forall \beta < \alpha$$

Now let's prove that $\mu(A \cap F_\alpha) \geq \alpha \Rightarrow \mu(A \cap F_\beta) \geq \alpha$, $\forall \beta < \alpha$. Since $\beta < \alpha$, by Lemma 3.1.2, $F_\alpha \subseteq F_{\beta^+} \subseteq F_\beta$. Then, $A \cap F_\alpha \subseteq A \cap F_\beta$ which implies that $\mu(A \cap F_\alpha) \leq \mu(A \cap F_\beta)$ So, $\alpha \leq \mu(A \cap F_\alpha) \leq \mu(A \cap F_\beta)$. Therefore, $\mu(A \cap F_\beta) \geq \alpha$, $\forall \beta < \alpha$

2. From the previous part, by negation we directly get $\fint_A f d\mu < \alpha \iff \exists \beta < \alpha$ such that $\mu(A \cap F_\beta) < \alpha$. Also, from the previous part we have that $\mu(A \cap F_\alpha) \geq \alpha \Rightarrow \mu(A \cap F_\beta) \geq \alpha$, $\forall \beta < \alpha$. So by negation, we get if $\exists \beta < \alpha$ such that $\mu(A \cap F_\beta) < \alpha \Rightarrow \mu(A \cap F_\alpha) < \alpha$. Now, let's prove that if $\mu(A \cap F_\alpha) < \alpha$, then $\mu(A \cap F_{\alpha^+}) < \alpha$. We have $F_{\alpha^+} \subseteq F_\alpha$, so

$A \cap F_{\alpha^+} \subsetneq A \cap F_\alpha$ which implies that

$$\mu(A \cap F_{\alpha^+}) \leq \mu(A \cap F_\alpha) < \alpha.$$

Hence, $\mu(A \cap F_{\alpha^+}) < \alpha$

3. If $\mu(A \cap F_{\alpha^+}) \leq \alpha$, then by Lemma 3.2.5, we have

$$-\!\!\!\!\!\!\int_A f d\mu \leq \alpha \vee \mu(A \cap F_{\alpha^+}) = \alpha.$$

Now, if $\mu(A \cap F_{\alpha^+}) > \alpha \Rightarrow \exists \alpha_0 > \alpha$ such that $\mu(A \cap F_{\alpha_0}) > \alpha$. So, by Definition 3.1.1,

$$-\!\!\!\!\!\!\int_A f d\mu \geq \alpha_0 \wedge \mu(A \cap F_{\alpha_0}) > \alpha.$$

Now, let's prove if $\mu(A \cap F_\alpha) \leq \alpha$, then $\mu(A \cap F_{\alpha^+}) \leq \alpha$. We have $F_{\alpha^+} \subsetneq F_\alpha$, so $A \cap F_{\alpha^+} \subsetneq A \cap F_\alpha$ which implies that

$$\mu(A \cap F_{\alpha^+}) \leq \mu(A \cap F_\alpha) \leq \alpha.$$

Therefore, $\mu(A \cap F_{\alpha^+}) \leq \alpha.$

4. We already proved in the previous part that

$$-\!\!\!\!\!\!\int_A f d\mu > \alpha \iff \mu(A \cap F_{\alpha^+}) > \alpha$$

. Now let's prove if $\mu(A \cap F_{\alpha^+}) > \alpha$, then $\mu(A \cap F_\alpha) > \alpha$. We have $F_{\alpha^+} \subsetneq F_\alpha$, so $A \cap F_{\alpha^+} \subsetneq A \cap F_\alpha$. Then, $\alpha < \mu(A \cap F_{\alpha^+}) \leq \mu(A \cap F_\alpha)$.

Hence, $\mu(A \cap F_\alpha) > \alpha$.

5. We have

$$\fint_A f d\mu = \alpha \iff \fint_A f d\mu \geq \alpha$$

and $\fint_A f d\mu \leq \alpha \Rightarrow \forall\beta < \alpha, \mu(A \cap F_\beta) \geq \alpha$ and $\mu(A \cap F_{\alpha^+}) \leq \alpha$. Hence $\forall\beta < \alpha, \mu(A \cap F_{\alpha^+}) \leq \alpha \leq \mu(A \cap F_\beta)$. Now, $\mu(A \cap F_\alpha) = \alpha \iff \mu(A \cap F_\alpha) \geq \alpha$ and $\mu(A \cap F_\alpha) \leq \alpha$. Thus, by parts 1 and 3, we have

$$\fint_A f d\mu \geq \alpha \quad \text{and} \quad \fint_A f d\mu \leq \alpha.$$

Then, $\mu(A \cap F_\alpha) = \alpha \Rightarrow \fint_A f d\mu = \alpha$.

6. Since $\mu(A) < \infty$, we have from part 1 that

$$\mu(A \cap F_\alpha) \geq \alpha \Rightarrow \fint_A f d\mu \geq \alpha \Rightarrow \mu(A \cap F_\beta) \geq \alpha, \forall\beta < \alpha$$

$$\Rightarrow \fint_A f d\mu \geq \alpha$$

7. $\fint_A f d\mu = \alpha \Rightarrow \fint_A f d\mu \geq \alpha \Rightarrow \mu(A \cap F_\alpha) \geq \alpha$(by part 6). We have $\fint_A f d\mu = \alpha \Rightarrow \mu(A \cap F_{\alpha^+}) \leq \alpha$ (by part 5). Hence,

$$\mu(A \cap F_{\alpha^+}) \leq \alpha \leq \mu(A \cap F_\alpha).$$

By part 3, if $\mu(A \cap F_{\alpha^+}) \leq \alpha \Rightarrow \fint_A f d\mu \leq \alpha$ and by part 6, we have

$$\mu(A \cap F_\alpha) \geq \alpha \Rightarrow \fint_A f d\mu = \sup_{x \in [0,\infty]} [\alpha \wedge \mu(A \cap F_\alpha)] = \sup_{x \in [0,\infty]} \alpha \geq \alpha$$

Therefore, $\int_A f d\mu = \alpha \iff \mu(A \cap F_\alpha) \geq \alpha \geq \mu(A \cap F_{\alpha^+})$

□

In classical measure theory , if two measurable functions f_1 and f_2 are equal *a.e*, then their integrals are equal. What about the fuzzy integral on fuzzy measure space?

Example 3.2.8. *Let* $X = \{0, 1\}$ *and* $\mathcal{F} = \mathcal{P}(X)$. *We define*

$$\mu(E) = \begin{cases} 1 & \text{if } E = X \\ 0 & \text{if } E \not\subseteq X \end{cases}$$

$$f_1(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x = 0 \end{cases}$$

and $f_2(x) = 1$. *Let* $E = \{0\}$, *we have* $\mu(E) = 0$ $(E = X)$ *and* $f_1 = f_2$ *on* $X \setminus E$. *Therefore,* $f_1 = f_2$ *a.e. Then,*

$$\int f_1 d\mu = \sup_{x \in [0, \infty]} [\alpha \wedge \mu(A \cap F_\alpha)]$$

$$F_\alpha = \{x / f_1(x) \geq \alpha\} = \begin{cases} \{x / 1 \geq \alpha\} & \text{if } x=1 \\ \{x / 0 \geq \alpha\} & \text{if } x=0 \end{cases}$$

but $\alpha \in [0, \infty]$*, so,* $F_\alpha = \{x/1 \geq \alpha\} = [0, 1]$*. We have*

$$\begin{aligned}
-\!\!\!\!\int_A f_1 d\mu &= \sup_{x\in[0,\infty]} [\alpha \wedge \mu(A \cap F_\alpha)] \\
&= \sup_{x\in[0,\infty]} [\alpha \wedge \mu(A \cap [0, 1])] \\
&= [1 \wedge \mu(\{1\})] \vee [0 \wedge \mu(\{0, 1\})] \\
&= [1 \wedge 0] \vee [0 \wedge 1] \\
&= 0 \wedge 0 = 0
\end{aligned}$$

Also, $F_\alpha = \{x/f_2(x) \geq \alpha\} = \{x/1 \geq \alpha\} = [0, 1]$*, so*

$$\begin{aligned}
-\!\!\!\!\int f_2 d\mu &= \sup_{x\in[0,\infty]} [\alpha \wedge \mu(A \cap F_\alpha)] \\
&= \sup_{x\in[0,\infty]} [\alpha \wedge \mu(A \cap [0, 1])] \\
&= 1 \wedge \mu(X \cap [0, 1]) \\
&= 1 \wedge \mu(X) = 1 \wedge 1 = 1.
\end{aligned}$$

So, $-\!\!\!\!\int f_1 d\mu = -\!\!\!\!\int f_2 d\mu$.

Definition 3.2.9. $\mu : \mathcal{F} \to [-\infty, \infty]$ *is* ***null additive*** *if and only if*

$$\mu(E \cup F) = \mu(E),$$

whenever $E \in \mathcal{F}, F \in \mathcal{F}, E \cap F = \emptyset$ *and* $\mu(F) = 0$.

Now, we give an important theorem for fuzzy integrals.

Theorem 3.2.10. *Assume that $f_1 = f_2$ a.e. Then,*

$$- f_1 d\mu = - f_2 d\mu \iff \mu \quad \text{is null-additive.}$$

Proof. $\Leftarrow$)If μ is null-additive, then $\mu(\{x/f_1(x) = f_2(x)\}) = 0$, and

$$\begin{aligned} & \mu(\{x/f_2(x) \geq \alpha\}) \\ \leq \ & \mu(\{x/f_1(x) \geq \alpha\}) \cup \{x/f_1(x) = f_2(x)\}) \\ = \ & \mu(\{x/f_1(x) \geq \alpha\}), \end{aligned}$$

for any $\alpha \in [0, \infty]$ Also,

$$\begin{aligned} & \mu(\{x/f_1(x) \geq \alpha\}) \\ \leq \ & \mu(\{x/f_2(x) \geq \alpha\} \cup \{x/f_2(x) = f_1(x)\}) \\ = \ & \mu(\{x/f_2(x) \geq \alpha\}). \end{aligned}$$

So, $\mu(\{x/f_1(x) \geq \alpha\}) = \mu(\{x/f_2(x) \geq \alpha\})$, $\forall \alpha \in [0, \infty]$. Therefore,

$$- f_1 d\mu = - f_2 d\mu$$

$\Rightarrow$) For any $E \in \mathcal{F}, F \in \mathcal{F}$ with $\mu(F) = 0$. If $\mu(E) = \infty$ then by monotonicity of μ, we have $\mu(E \cup F) = \infty = \mu(E)$. Assume $\mu(E) < \infty$, use contradiction to prove $\mu(E \cup F) = \mu(E)$. Assume $\mu(E \cup F) > \mu(E)$. Take $\alpha \in (\mu(E), \mu(E \cup F))$

and

$$f_1(x) = \begin{cases} a & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

$$f_2(x) = \begin{cases} a & \text{if } x \in E \cup F \\ 0 & \text{if } x \notin E \cup F \end{cases}$$

Then, $\mu(\{x / f_1(x) = f_1(x)\}) = \mu(F \mid E) \leq \mu(F) = 0$. That is $f_1 = f_2$ *a.e.* So, we should have $\int f_1 d\mu = \int f_2 d\mu$. But,

$$\begin{aligned} \int f_1 d\mu &= \sup_{x \in [0, \infty]} [\alpha \wedge \mu(A \cap F_\alpha)] \\ &= [0 \wedge \mu(\overline{E})] \vee [a \wedge \mu(E)] \\ &= 0 \vee [a \wedge \mu(E)] \\ &= a \wedge \mu(E) = \mu(E) \end{aligned}$$

$$\begin{aligned} \int f_2 d\mu &= \sup_{x \in [0, \infty]} [\alpha \wedge \mu(A \cap F_\alpha)] \\ &= [0 \wedge \mu(\overline{E \cup F})] \vee [a \wedge \mu(E \cup F)] \\ &= 0 \vee [a \wedge \mu(E \cup F)] \\ &= a = \mu(E) \end{aligned}$$

Then $\int f_1 d\mu = \int f_2 d\mu$, which is a contradiction. □

Corollary 3.2.11. *If μ is null-additive, then $\int_A f_1 d\mu = \int_A f_2 d\mu$ whenever $f_1 = f_2$ a.e on A.*

Proof. If $f_1 = f_2$ ae on A, then $f_1.\chi_A = f_2.\chi_A$ *a.e.* So,

$$\int f_1 \chi_A d\mu = \int f_2 \chi_A d\mu.$$

Hence, $\int_A f_1 d\mu = \int_A f_2 d\mu$ □

Corollary 3.2.12. *If μ is null additive, then for any $f \in F$, we have*

$$\int_{A \cup B} f d\mu = \int_A f d\mu,$$

whenever $A \in \mathcal{F}, B \in \mathcal{F}$ with $\mu(B) = 0$

Proof. We have $f.\chi_{A\cup B} = f.\chi_A$ *a.e.* So, $\int_{A\cup B} f d\mu = \int_A f d\mu$ □

We already discussed in the previous chapter several convergences of measurable function sequences on fuzzy measure spaces. In classical measure theory, there are some concepts of convergence of measurable function sequences that concern the integral. One of them is the mean convergence. Since the fuzzy integral has been defined for measurable functions, we can introduce a concept of fuzzy mean convergence on fuzzy measure measure spaces as follows.

Definition 3.2.13. *Let $\{f_n\} \subseteq F$ and $f \in F$. We say that $\{f_n\}$ fuzzy mean converges (f-mean converges) to f if and only if*

$$\lim_n \int |f_n - f| d\mu = 0$$

However, the following theorem shows that such convergence is not necessary.

Theorem 3.2.14. *The f-mean convergence is equivalent to the convergence in measure on fuzzy measure spaces.*

Proof. If $f_n \xrightarrow{\mu} f$, then for any given $c > 0$, $\exists n_0$ such that

$$\mu(\{x / |f_n(x) - f(x)| \geq \frac{c}{2}\}) < c,$$

whenever $n \geq n_0$. Using Lemma 3.2.7 part 1, we know

$$- |(f_n - f)| d\mu < c.$$

So, $\lim_n - |f_n - f| d\mu = 0$. This shows that $\{f_n\}$ f-mean converges to f. Conversely, if $f_n \xrightarrow{\mu} f$ does not hold, then $\exists c > 0$, $\delta > 0$, and a sequence $\{n_i\}$ such that

$$\mu(\{x / |f_{n_i}(x) - f(x)| \geq c\}) \geq c \wedge \delta > 0,$$

for any n_i, $i = 1, 2, \cdots$. This shows that $\{f_n\}$ does not f mean converge to f. □

3.3 Convergence Theorems of the Fuzzy Integral Sequence

In this section, we will give several convergence theorems of fuzzy integral sequence under some conditions as weak as possible. In these theorems, we assume $\{f_n\} \subseteq F$, and we will use symbol " $\searrow$ " to denote "decreasingly converge to", " " to denote "increasingly converge to" and " $\rightarrow$ " to denote "converge to" for both function sequences and number sequences. We will write $F_\alpha^n = \{x/f_n(x) \geq \alpha\}$ and $F_{\alpha^+}^n = \{x/f_n(x) > \alpha\}$

Lemma 3.3.1. *If $f_n \downarrow f$, then*

$$F_\alpha^n \downarrow \bigcap_{n=1}^{\infty} F_\alpha^n = F_\alpha \quad and \quad F_{\alpha^+} \subseteq F_{\alpha^+}^n \downarrow \bigcap_{n=1}^{\infty} F_{\alpha^+}^n \subseteq F_\alpha.$$

Proof. Let $f_n \downarrow f$. Since $f_n \geq f$, $\forall x \in X$, we have

$$x \in F_{\alpha^+} \Rightarrow f(x) > \alpha \Rightarrow f_n(x) > \alpha \Rightarrow x \in F_{\alpha^+}^n.$$

So, $F_{\alpha^+} \subseteq F_{\alpha^+}^n$. Now since $f_n \geq f$, $\forall x \in X$ and

$$F_{\alpha^+}^1 = \{x/f_1(x) > \alpha\}, F_{\alpha^+}^2 = \{x/f_2(x) > \alpha\},$$

We get $\{F_{\alpha^+}^n\}$ is nonincreasing w.r.t n which implies that $F_{\alpha^+}^n \downarrow \bigcap_{n=1}^{\infty} F_{\alpha^+}^n$. Now let $x \in \bigcap_{n=1}^{\infty} F_{\alpha^+}^n \Rightarrow x \in F_{\alpha^+}^n, \forall n$. Then $f_n(x) > \alpha, \forall n$. So, $f(x) \geq \alpha$ which implies that $f(x) \in F_\alpha$. Hence $\bigcap_{n=1}^{\infty} F_{\alpha^+}^n \subseteq F_\alpha$. Let f_n f. Since $f_n \leq f, \forall x \in X$ □

Theorem 3.3.2. *Let $A \in \mathcal{F}$. If $f_n \downarrow f$ on A and $\exists n_0$ such that*

$$\mu\left(\{x/f_{n_0}(x) > \int_A f d\mu\} \cap A\right) < \infty$$

or if f_n f then

$$\lim_n \int_A f_n d\mu = \int_A f d\mu.$$

Proof. There is no loss of generality if we assume that $A = X$. Write $\int f d\mu = c$

and let $f_n \downarrow f$ with n_0 such that $\mu(\{x/f_{n_0}(x) > c\}) < \infty$. If $c = \infty$, by using monotonicity of fuzzy integral (Theorem 3.2.1 part 3),we have

$$\int f_n d\mu \geq \int f d\mu = \infty.$$

So, the conclusion of this theorem holds. If $c < \infty$, then $\int f_n d\mu \geq c$ for any $n = 1, 2, \cdots$ and therefore,

$$\lim_n \int f_n d\mu \geq c.$$

Assume $\lim_n \int f_n d\mu > c$, then $\exists c > c$ such that $\lim_n \int f_n d\mu > c$, so

$$\int f_n d\mu > c ,$$

for any n. From Lemma 3.2.7 part 2 ,We know that $\mu(F_{c'}^{n} > c)$ for any n. Since $\exists n_0$ such that

$$\mu(F_{c'}^{n_0}) = \mu(\{x/f_{n_0}(x) \geq c\}) \leq \mu(\{x/f_{n_0}(x) > c\}) < \infty,$$

then by applying the continuity from above of μ, by Lemma 3.3.1 we get $\mu(F_c) = \lim_n \mu(F_c{}^n) \geq c$. Using Lemma 3.2.7 part 1, we know $\int f d\mu \geq c > c$. This contradicts $\int f d\mu = c$. Consequently

$$\lim_n \int f_n d\mu = c = \int f d\mu.$$

When $f_n \quad f$, the proof is similar. □

Corollary 3.3.3. *Let $A \in \mathcal{F}$. If $f_n \downarrow f$ on A, then $\exists n_0$ and a constant $c \leq \int_A f d\mu$ such that $\mu(\{x/f_{n_0} > c \cap A\}) < \infty$. Then $\int_A f_n d\mu \downarrow \int_A f d\mu$.*

Corollary 3.3.4. *If $f_n \downarrow f$ and μ is finite, then $\int_A f_n d\mu \downarrow \int_A f d\mu$.*

Corollary 3.3.5. *Let μ be null additive.*

1. *If $f_n \downarrow f$ a.e and $\exists n_0$ and a constant $c \leq \int f d\mu$ such that*

$$\mu(\{x/f_{n_0}(x) > c\}) < \infty,$$

then $\int f_n d\mu \downarrow \int f d\mu$.

2. *If f_n f a.e, then $\int f_n d\mu$ $\int f d\mu$*

Theorem 3.3.6. *Let $A \in \mathcal{F}$. If $f(x) = \liminf_n f_n(x)$, $\forall x \in A$, then*

$$\int_A f d\mu \leq \liminf_n \int_A f_n d\mu.$$

Proof. Let $g_n(x) = \inf_{i \geq n} f_i(x)$, $\forall x \in A$, then g_n f on A. By using Theorem 3.3.2 , we get

$$\lim_n \int_A g_n d\mu = \int_A f d\mu.$$

Since $g_n \leq f_n$ on A, we have $\int_A g_n d\mu \leq \int_A f_n d\mu$ and, therefore $\lim_n \int_A g_n d\mu \leq \liminf_n \int_A f_n d\mu$. Consequently, we have

$$\int_A f d\mu \leq \liminf_n \int_A f_n d\mu.$$

□

In Theorem 3.3.2, when $\{f_n\}$ is a non-increasing sequence, the condition that $\exists n_0 / \mu(\{x/f_{n_0}(x) > \int_A f d\mu\} \cap A) < \infty$ cannot be abandoned casually. Without this condition, the conclusion of this theorem might not hold. We can see this from the following example.

Example 3.3.7. *Let $X = [0, \infty)$ and $\mathcal{F}$ be the class of all Borel sets that are in X ($\mathcal{F} = \mathcal{B} \cap X$), and μ be the Lebesgue measure. Take $f_n(x) = \frac{x}{n}$ for any $x \in X$ and any $n = 1, 2, \cdots$, then $f_n \searrow f \equiv 0$. Such a measurable function sequence $\{f_n\}$ does not satisfy the condition given in Theorem 3.3.2. In fact, we have*

$$\mu\left(\{x/f_n(x) > \int d\mu\}\right) = \mu(\{x/f_n(x) > 0\}) = \mu(X) = \infty$$

for any $n = 1, 2, \cdots$. Consequently, $\int f_n d\mu = \infty$ for any $n = 1, 2, \cdots$ but $\int f d\mu = 0$. That is, $\lim_n \int f_n d\mu = \int f d\mu$

Now, we will make use of the monotone convergence theorem to give a convergence theorem of fuzzy integral sequence for the measurable function sequence which is convergent everywhere.

Theorem 3.3.8. [The everywhere convergence theorem.] *Let $A \in \mathcal{F}$. If $f_n \to f$ on A, and $\exists n_0$ and a constant $c \leq \int_A f d\mu$ such that*

$$\mu\left(\{x/\sup_{n \geq n_0} f_n > c\} \cap A\right) < \infty,$$

then $\int_A f_n d\mu \to \int_A f d\mu$.

Proof. There is no loss of generality if we assume that $A = X$. Let $h_n = \sup_{i \geq n} f_i$ and $g_n = \inf_{i \geq n} f_i$. Then h_n and $g_n (n = 1, 2, \cdots)$ are measurable.

We also have $h_n \downarrow f$ and $g_n \uparrow f$. Since $g_n \le f_n \le h_n$, we have

$$\int g_n d\mu \le \int f_n d\mu \le \int h_n d\mu,$$

and therefore

$$\lim_n \int g_n d\mu \le \liminf_n \int f_n d\mu \le \lim_n \int h_n d\mu.$$

Noting that $\mu(\{x / h_{n_0}(x) > c\}) < \infty$, where $c \le \int f d\mu$, from Theorem 3.3.2 and Corollary 3.3.3, we get

$$\lim_n \int g_n d\mu = \lim_n \int h_n d\mu = \int f d\mu.$$

So $\int_A f_n d\mu \to \int f d\mu$. □

For a measurable function sequence which is convergent *a.e*, we have the following theorem.

Theorem 3.3.9. [“a.e” Convergence Theorem]. *We have $\int_A f_n d\mu \to \int_A f d\mu$ whenever $A \in \mathcal{F}$, $f_n \xrightarrow{a.e} f$ on A and $\exists n_0$ and a constant $c \le \int_A f d\mu$ such that*

$$\mu(\{x / \sup_{n \ge n_0} f_n(x) > c\} \cap A) < \infty$$

if and only if μ is null additive.

Proof. To prove that μ is null additive by Theorem 3.2.10, it is enough to prove that $\int f_1 d\mu = \int f_2 d\mu$ whenever $f_1 = f_2$ *a.e*. But we have $\int f_n d\mu \to$

$\bar{\ }_A\, f d\mu$ whenever $A \in \mathcal{F}$, $f_n \xrightarrow{ae}$ on A and $\exists n_0$, constant $c \leq \bar{\ }_A\, f d\mu$ such that

$\sup_{i \geq n} f_i$ and $g_n = \inf_{i \geq n} f_i$. Then h_n and $g_n (n = 1, 2, \cdots)$ are measurable.

$\mu(\{x / \sup f_n(x) > c\} \cap A) < \infty$. Then $\int f_1 d\mu = \int f_2 d\mu$. Hence, μ is null additive. □

Corollary 3.3.10. *Let μ be finite and subadditive. If $f_n \xrightarrow{a.e} f$, then we have $\int f_n d\mu \to \int f d\mu$.*

Example 3.3.11. *Let $X = [0, \infty)$, $\mathcal{F} = \mathcal{B} \cap X$, and μ be the Lebesgue measure. Consider*

$$f_n(x) = \begin{cases} 1 & \text{if } x > n \\ 0 & \text{if } x \in [0, n] \end{cases}$$

Then, $f_n \downarrow f \equiv 0$. Note that $0 \leq f_n(x) \leq 1$ for any $x \in X$ and any $n = 1, 2, \cdots$ and $\int 1 d\mu = 1 < \infty$. In our case, however, $\int f_n d\mu = 1 n = 1, 2, \cdots$ and $\int f d\mu = 0$. Consequently, we have $\lim_n \int f_n d\mu = \int f d\mu$. So, in this example the function sequence $\{f_n\}$ does not satisfy the finiteness condition on μ given in Theorem 3.3.2.

Definition 3.3.12. *$\mu : \mathcal{F} \to [-\infty, \infty]$ is autocontinuous from above (or from below) if and only if*

$$\lim_n \mu(E \cup F_n) = \mu(E) \quad \left(\text{or} \quad \lim_n \mu(E - F_n) = \mu(E) \right),$$

whenever $E \in \mathcal{F}, F_n \in \mathcal{F}, E \cap F_n = \emptyset$ (or $F_n \subseteq E$, respectively), $n = 1, 2, \cdots$ and $\lim_n \mu(F_n) = 0$. We say that μ is **autocontinuous** $\Longleftrightarrow$ *it is both autocontinuous from above and autocotinuous from below .*

Theorem 3.3.13. *[**Convergence in Measure Theorem.**] We have $\int f_n d\mu \to \int_A f d\mu$ whenever $A \in \mathcal{F}$, $\{f_n\} \subseteq F$, $f \in F$ and $f_n \xrightarrow{\mu} f$ on A if and only if μ is autocontinuous.*

Proof. ⇐) Without any loss of generality, we can assume $A = X$. Let μ be autocontinuous and $f_n \xrightarrow{\mu} f$, and let $c = - f d\mu$. When $c < \infty$, by using Lemma 3.2.7 part 5, for any given $c > 0$, we have

$$\mu(F_{c-}) \geq c \quad \text{and} \quad \mu(F_{c+}) \leq c.$$

On one hand, $F^n_{c+2} \subseteq F_{c+} \cup \{x / |f_n(x) - f(x)| \geq c\}$. Since $f_n \xrightarrow{\mu} f$, we get

$$\mu(\{x / |f_n(x) - f(x)| \geq c\}) \to 0.$$

An application of autocontinuity from above yields that

$$\mu \quad F_{c+} \cup \{x / |f_n(x) - f(x)| \geq c\} \quad \to \mu(F_{c+}).$$

So, $\exists n_0$ such that

$$\begin{aligned} & \mu(F_{c+2}) \\ \leq \;& \mu(F_{c+} \cup \{x / |f_n - f(x)| \geq c\}) \\ \leq \;& \mu(F_{c+}) + c \leq c + c \leq c + 2c, \end{aligned}$$

whenever $n \geq n_0$. It follows, by Lemma 3.2.7 part 3 that

$$- f_n d\mu \leq c + 2c$$

for any $n \geq n_0$. On the other hand, to prove a converse inequality we only need to consider the case when $c > 0$. For any given $c \in (0, \frac{c}{2})$, we have

$$F^{n}_{c-2} \supset F_{c-} - \{x / |\boldsymbol{f}_n(x) - \boldsymbol{f}(x)| \geq c\}.$$

Since, $\boldsymbol{f}_n \xrightarrow{\mu} \boldsymbol{f}$ and μ is autocontinuous from below, $\exists n_0$ such that

$$\mu(F_{c-2}) \geq \mu(F_{c-}) - c \geq c - 2c,$$

whenever $n \geq n_0$. Hence, $\lim_n - \boldsymbol{f}_n d\mu$ exists and $- \boldsymbol{f}_n d\mu \to c$. If $c = \infty$, then from Lemma 3.2.6, we have $\mu(F_\alpha) = \infty$ for any given $N > 0$ and

$$F^{n}_{N} \supset F_{N+1} - \{x / |\boldsymbol{f}_n(x) - \boldsymbol{f}(x)| \geq 1\}.$$

Since $\boldsymbol{f}_n \xrightarrow{u} \boldsymbol{f}$ and μ is autocontinuous from below, $\exists n_0$ such that

$$\mu(F^{n}_{N}) \geq \mu(F_{N+1} - \{x / |\boldsymbol{f}_n(x) - \boldsymbol{f}(x)| \geq 1\}) \geq N,$$

whenever $n \geq n_0$. It follows from Lemma 3.2.7 part 1 that $- \boldsymbol{f}_n d\mu \geq N$ for any $n \geq n_0$. This shows that $- \boldsymbol{f}_n d\mu \to \infty = c$.

$\Rightarrow$) For any $B \in \mathcal{F}$ and $\{B_n\} \subseteq \text{Card}F$ with $\mu(B) \to 0$, we are going to prove that $\mu(B \cap B_n) \to \mu(B)$. Benefiting from the monotonicity of μ, we only need to consider the case when $\mu(B) < \infty$. Take $a > \mu(B)$ and

$$\boldsymbol{f}(x) = \begin{cases} a & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

$$f_n(x) = \begin{cases} a & \text{if } x \in B \cup B \\ 0 & \text{if } x \notin B \cup B_n \end{cases}$$

for any $n = 1, 2, \cdots$. Then, for any given $c > 0$, we have

$$\{x / |f_n(x) - f(x)| \geq c\} \subseteq B_n,$$

for any $n = 1, 2, \cdots$. So, $f_n \xrightarrow{\mu} f$. By the hypothesis of this proposition, **it** should hold that $\int f_n d\mu \to \int f d\mu$. Since

$$\int f_n d\mu = a \wedge \mu(B \cup B_n),$$

and $\int f d\mu = a \wedge \mu(B) = \mu(B)$, we get

$$\mu(B \cup B_n) \to \mu(B).$$

This means that μ is autocontinuous [1]. □

Theorem 3.3.14. [f-mean Convergence Theorem]. *We have*

$$\int f_n d\mu \to \int f d\mu,$$

whenever $\{f_n\} \subseteq F$, $f \in F$ *and* $\{f_n\}$ *f-mean converges to* f *if and only if* μ *is autocontinuous.*

Example 3.3.15. *Let* $X = \{1, 2, \cdots\}$, $C = \mathcal{P}(X)$ *and*

$$\mu(E) = \text{card}(E) \sum_{i \in E} 2^{-i},$$

$\forall E \in C$. First, μ is not continuous from above. Take $\boldsymbol{f}(x) = \chi_{\{1\}}(x)$ and $\boldsymbol{f}_n(x) = \chi_{\{1,n\}}(x)$ for $x \in X$ and $n = 1, 2, \cdots$. Then, for any given $c \in (0, 1)$, we have

$$\mu(\{x/\mathrm{Card}(\boldsymbol{f}_n(x) - \boldsymbol{f}(x)) \geq c\}) = \mu(\{n\}) = 2^{-n} \to 0.$$

Namely $\boldsymbol{f}_n \xrightarrow{\mu} \boldsymbol{f}$. But $\int \boldsymbol{f} d\mu = \frac{1}{2}$ and $\int \boldsymbol{f}_n d\mu = 1$ for any $n = 1, 2, \cdots$. So, $\int \boldsymbol{f}_n d\mu$ does not tend to $\int \boldsymbol{f} d\mu$.

Definition 3.3.16. *Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space, $\boldsymbol{f} \in F$. We call $\boldsymbol{f}$ a fuzzy integrable w.r.t μ if and only if $\int \boldsymbol{f} d\mu < \infty$.*

If we write $L^1(\mu) = \{\boldsymbol{f}/\boldsymbol{f} \in F, \boldsymbol{f}$ is fuzzy integrable w.r.t $\mu\}$, then we have the following theorem.

Theorem 3.3.17. *Let $A \in \mathcal{F}$, μ be uniformly autocontinuous. If $\boldsymbol{f}_n \xrightarrow{\mu} \boldsymbol{f}$ on A, then*

1. $\int_A \boldsymbol{f} d\mu = \infty \iff \exists n_0$ *such that* $\int_A \boldsymbol{f}_n d\mu = \infty$ *for any* $n \geq n_0$
2. $\int_A \boldsymbol{f} d\mu < \infty \iff \exists n_0$ *such that* $\int_A \boldsymbol{f}_n d\mu < \infty$ *for any* $n \geq n_0$

When $A = X$, we can rewrite the above propositions as

1. $\boldsymbol{f} \notin L^1(\mu) \iff \exists n_0$ *such that* $\boldsymbol{f}_n \notin L^1(\mu)$ *for any* $n \geq n_0$
2. $\boldsymbol{f} \in L^1(\mu) \iff \exists n_0$ *such that* $\boldsymbol{f}_n \in L^1(\mu)$ *for any* $n \geq n_0$

Proof. Without any loss of generality, we can assume $A = X$.

1. $\Leftarrow$) Since the uniform autocontinuity implies the autocontinuity from $\boldsymbol{f}_n \xrightarrow{\mu} \boldsymbol{f}$ by using Theorem 3.3.13, we have $\int \boldsymbol{f}_n d\mu \to \int \boldsymbol{f} d\mu$. So, if $\exists n_0$

such that $\int f_n d\mu = \infty$ for any $n \geq n_0$, we get $\int f d\mu = \infty$.

$\Rightarrow$) Conversely, if $\int f d\mu = \infty$, by Lemma 3.2.6,

$$\mu(X \cap F_\alpha) = \mu(F_\alpha) = \infty,$$

$\forall \alpha \in [0, \infty[$, and in particular for $\alpha + 1$, so $\mu(F_{\alpha+1}) = \infty$ for any $\alpha \in [0, \infty)$.. Since $f_n \xrightarrow{\mu} f$ and μ is uniformly autocontinuous, $\exists n_0$ such that

$$\mu(F_{\alpha+1} - \{x / |f_n(x) - f(x)| \geq 1\}) = \infty,$$

for any $\alpha \in [0, \infty)$ whenever $n \geq n_0$. From

$$F_\alpha^n \supset F_{\alpha+1} - \{x / |f_n(x) - f(x)| \geq 1\}$$

for any $\alpha \in [0, \infty)$, we have

$$\mu(F_\alpha^n) \geq \mu(F_{\alpha+1} - \{x / |f_n(x) - f(x)| \geq 1\}) = \infty,$$

for any $\alpha \in [0, \infty)$ whenever $n \geq n_0$. Consequently, we have $\int f_n d\mu = \infty$ for any $n \geq n_0$.

2. An application of reduction to absurdity can show the implication " $\Leftarrow$".

   As to the implication " $\Rightarrow$" ,we can get it from $\int f_n d\mu \to \int f d\mu < \infty$

□

The symbol $f_n \xrightarrow{u} f$ on A will denote that $\{f_n\}$ converges to f on A uniformly.

Theorem 3.3.18. [Uniform Convergence Theorem] *Let $A \in \mathcal{F}$. If $f_n \xrightarrow{\mu} f$ on A, then*

$$\fint_A f_n d\mu \to \fint_A f d\mu$$

Proof. For any given $c > 0$, since $f_n \xrightarrow{\mu} f$ on A, $\exists n_0$ such that

$$\text{Card}(f_n - f) \leq c,$$

on A whenever $n \geq n_0$. Using Lemma 3.2.4, we have

$$\text{Card}\left(\fint_A f_n d\mu - \fint_A f d\mu\right) \leq c,$$

for any $n \geq n_0$. This shows that $\fint_A f_n d\mu \to \fint_A f d\mu$ □

3.4 Transformation Theorem for Fuzzy Integrals

In this section, we will discuss how to transform a fuzzy integral $\fint_A f d\mu$ which is defined on a fuzzy measure space $(X, \mathcal{F}, \mu)$ into another fuzzy integral $\fint g dm$ defined on the Lebesgue measure space $([0, \infty], \overline{B}_+, m)$ where $\overline{B}_+$ is the class of all Borel sets in $[0, \infty]$ and m is the Lebesgue measure.

Theorem 3.4.1. *For any $A \in \mathcal{F}$. Then,*

$$\fint_A f d\mu = \fint \mu(A \cap F_\alpha) dm,$$

where $F_\alpha = \{x / f(x) \geq \alpha\}$ and m is the Lebesgue measure.

Proof. Denote $g(\alpha) = \mu(A \cap F_\alpha)$. From Lemma 3.1.2, we know that $g(\alpha)$ is decreasing w.r.t α. For any $\alpha \in [0, \infty]$, denote

$$B_\alpha = \{E / \sup E = \alpha, E \in \bar{B}_+\}.$$

Then, $\{B_\alpha / \alpha \in [0, \infty]\}$ is a partition of $\bar{B}_+$ and $\sup_{E \in B_\alpha} m(E) = \alpha.$ Thus, from Theorem 3.1.3,

$$\begin{aligned} & - \mu(A \cap F_\alpha) dm \\ = \ & - g(\alpha) dm \\ = \ & \sup_{E \in B_+} [\inf_{\beta \in E} g(\beta) \wedge m(E)] \\ = \ & \sup_{\alpha \in [0, \infty]} \ \sup_{E \in B_\alpha} [\inf_{\beta \in E} g(\beta) \wedge m(E)]. \end{aligned}$$

Since $g(\beta)$ is decreasing, we have $g(\alpha^-) \geq \inf_{\beta \in E} g(\beta) \geq g(\alpha)$ for any $E \in B_\alpha$ where $g(\alpha^-) = \lim_{\beta \to \alpha^-} g(\beta)$. So, on one hand, we have

$$\begin{aligned} - \mu(A \cap F_\alpha) dm \ & \geq \sup_{\alpha \in [0, \infty]} [g(\alpha) \wedge \sup_{E \in B_\alpha} m(E)] \\ & = \sup_{\alpha \in [0, \infty]} [g(\alpha) \wedge \alpha] \\ & = \sup_{\alpha \in [0, \infty]} [\alpha \wedge \mu(A \cap F_\alpha)] \\ & = \underset{A}{-} f d\mu \end{aligned}$$

On the other hand, for any given $c > 0$,

$$
\begin{aligned}
-\mu(A \cap F_\alpha)dm &\leq \sup_{\alpha \in [0,\infty]} [g(\alpha^-) \wedge \sup_{E \in B_\alpha} m(E)] \\
&= \sup_{\alpha \in [0,\infty]} [g(\alpha^-) \wedge \alpha] \\
&\leq \sup_{\alpha \in [\ ,\infty]} [\alpha \wedge g(\alpha^-)] \vee c \\
&\leq \sup_{\alpha in [\ ,\infty]} [\alpha \wedge g(\alpha - c)] \vee c \\
&\leq \sup_{(\alpha - \ ) \in [0,\infty]} [(\alpha - c) \wedge g(\alpha - c)] + c \\
&= \sup_{(\alpha - \ ) \in [0,\infty]} [(\alpha - c) \wedge \mu(A \cap F_{\alpha -}\ )] + c \\
&= -_A f d\mu + c
\end{aligned}
$$

As $c \to 0$, we get $- \mu(A \cap F_\alpha)dm = -_A f d\mu$ □

3.5 Fuzzy Measures Defined by Fuzzy Integrals

In this section, we discuss how to define a fuzzy measure by using the fuzzy integral of a given measurable function w.r.t another given fuzzy measure.

Theorem 3.5.1. *Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space, $f \in F$. Then, the set function v defined by*

$$v(A) = -_A f d\mu,$$

for any $A \in \mathcal{F}$ is a lower semi-continuous fuzzy measure on $(X, \mathcal{F})$. Furthermore, if μ is finite, then v is a finite fuzzy measure on $(X, \mathcal{F})$

Proof. From Theorem 3.2.1, we know that $v(\emptyset) = 0$, v is monotone. We only need to prove that v is continuous from below. Let $\{E_n\}$ be an increasing set sequence in $\mathcal{F}$, $E_n \uparrow E \in \mathcal{F}$. Then,we have

$$f.\chi_{E_n} \uparrow f.\chi_E.$$

From Theorem 3.3.2 , we have

$$\lim_n v(E_n) = \lim_n \int_{E_n} f d\mu = \lim_n \int f.\chi_{E_n} d\mu = \int f.\chi_E d\mu = \int_E f d\mu = v(E)$$

Furthermore, suppose that μ is finite. For any given decreasing set sequence $\{E_n\}$ in $\mathcal{F}$ with $E_n \downarrow E \in \mathcal{F}$. From $f.\chi_{E_n} \downarrow f.\chi_E$ and Theorem 3.3.2, we have also $\lim_n v(E_n) = v(E)$. That is, v is continuous from above. Consequently, v is a fuzzy measure. The finiteness of v follows from

$$v(X) = \int f d\mu \leq \mu(X) < \infty.$$

□

The following example shows that the set function v may not be continuous from above.

Example 3.5.2. *Let $X = [0, \infty)$ and $\mathcal{F}$ be the class of all Borel sets in $[0, \infty)$ and μ be the Lebesgue measure. Define the function f by $f(x) = 1$ for all $x \in X$. Taking $E_n = [n, \infty)$, $n = 1, 2, \cdots$, we have $E_n \downarrow \emptyset$ and*

$$v(E_n) = \int_{E_n} f(x)dx = \int_{[0,\infty)} 1 d\mu = 1,$$

for $n = 1, 2, \cdots$. But $v(\emptyset) = \int_{\emptyset} f(x)d\mu = 0$. So, v is not continuous from *above.*

Chapter 4

Application of Fuzzy Measure and Fuzzy Integral in Students Failure Decision Making

Fuzzy Integrals, in general, and Sugeno Integrals, in particular, are well-known aggregation operators. They can be used in a great variety of decision making applications. In real life problems, most of the criteria have independent or interactive characteristics, which cannot be evaluated using additive measures (see [2]). The student failure is one of the issues that all academic institutes face, and there are many reasons for this failure. Some major reasons are given in Table 1 and Table 2.

Table 1: Failure Reasons and Criteria

Sr. No.	Failure Reasons	Criteria
1)	Lack of concentration	Intelligence Quotient (IQ) or Learning Ability (C_1)
2)	Poor Classroom Attendance or irregular due to Travelling or Financial problem.etc.	Attendance or Regularity (C_2)
3)	Lack of Motivation, Wrong teaching habits etc	Subject Liking (C_3)
4)	Careless behavior of students, Lack of Time-Management and willingness. Lack of maturity, Peer Relationships etc.	Responsibility (C_4)
5)	Examination Phobia, Overconfidence, Wrong reading and Writing habits, Mental Stress, Accident, illness, Sudden death of family member	Unavoidable conditions (C_5)

Table 2: Linguistic Scales for the Importance Weight

If the criteria is less	Extremely	0.0
	Highly	0.1
	Very	0.2
	Strongly	0.3
	Quite	0.4
Medium	Medium	0.5
If the criteria is more	Quite	0.6
	Strongly	0.7
	Very	0.8
	Highly	0.9
	Extremely	1.0

Let $\{A, B, C, ...J\}$ be the set of 10 students observed for the five criteria C_1, C_2, C_3, C_4, C_5. Here grades are given to each student for different criteria by taking their IQ test (for C_1), attendance report (for C_2), by giving questionnaire (for C_3), and by discussing with the students, their friends, parents, and teachers (for C_4 and C_5).

Table 3: Criteria Wise Students Grades

Criteria → / Students ↓	C1	C2	C3	C4	C5
A	0.9	0.2	0.7	0.2	0.8
B	0.1	0.9	0.8	0.2	0.8
C	0.1	0.2	0.9	0.8	0.2
D	0.2	0.1	0.6	0.8	0.2
E	0.4	0.8	0.1	0.5	0.3
F	0.6	0.7	0.8	0.5	0.3
G	0.6	0.2	0.3	0.4	0.5
H	0.7	0.5	0.8	0.8	0.1
I	0.4	0.2	0.1	0.5	0.3
J	0.4	0.2	0.1	0.3	0.9

Example: For student D the value of the criteria C_1 is 0.2 means the student D has very less IQ or learning ability Where,

C_1: Intelligence Quotient (IQ) or Learning Ability
C_2: Attendance or Regularity
C_3: Subject Liking
C_4: Responsibility
C_5: Unavoidable condition

First, we construct λ-fuzzy measure as a set of criteria. Let $X = \{C_1, C_2, C_3, C_4, C_5\}$. Now, experts were asked to rate the degree of importance of these five criteria (passing grades) in a short survey. Table 4 gives the judgement of relative importance of passing grades by experts.

Table 4: Judgment of Relative Importance of Passing Grades by Experts

C_1	C_2	C_3	C_4	C_5
0.5	0.8	0.6	0.6	0.5

We assume that

$$g_\lambda(\{C_1\}) = 0.5, g_\lambda(\{C_2\}) = 0.8, g_\lambda(\{C_3\}) = 0.6, g_\lambda(\{C_4\}) = 0.6, g_\lambda(\{C_5\}) = 0.5,$$

where λ is to be determined. Here Sugeno's λ-fuzzy measure is used to compute the interdependency between the selected criteria. We calculate the parameter λ using Equation in in Theorem 1.2.13. There are 5 roots but since $\lambda \in (-1, \infty)$, the accepted roots are 0 and -0.991368. If $\lambda = 0$, then g_λ becomes additive measure which means there is no relation between the criteria C_1, C_2, C_3, C_4, C_5 which is not the reality. Therefore, $\lambda = -0.991368 \in (-1, \infty)$. Now let's calculate the interdependencies between two or more criteria. For example,

$$\begin{aligned} g_\lambda(\{C_1, C_2\}) &= g_\lambda(\{C_1\}) + g_\lambda(\{C_2\}) + \lambda g_\lambda(\{C_1\})g_\lambda(\{C_2\}) \\ &= 0.5 + 0.8 + (-0.991368)(0.5)(0.8) = 0.9034. \end{aligned}$$

Similarly, one can calculate the other values. Table 5 and 6 shows the calculated values, which indicates the interdependencies between two or more criteria.

Table 5: The Interdependencies measures among C_i's

Between two criteria	Interdependencies measure or λ-Measure	Among three criteria	Interdependencies measure or λ-Measure
C_1, C_2	0.9034	C_1, C_2, C_3	0.9660
C_1, C_3	0.8026	C_1, C_2, C_4	0.9660
C_1, C_4	0.8026	C_1, C_2, C_5	0.9565
C_1, C_5	0.7521	C_1, C_3, C_4	0.8852
C_2, C_3	0.9241	C_1, C_3, C_5	0.9047
C_2, C_4	0.9241	C_1, C_4, C_5	0.9047
C_2, C_5	0.9034	C_2, C_3, C_4	0.9745
C_3, C_4	0.8431	C_2, C_3, C_5	0.9660
C_3, C_5	0.8026	C_2, C_4, C_5	0.9660
C_4, C_5	0.8026	C_3, C_4, C_5	0.8852

Table 6: The Interdependencies Measures Among Four Criteria's

Among four criteria	Interdependencies measure or λ-Measure
C_1, C_2, C_3, C_4	0.9915
C_1, C_2, C_3, C_5	0.9872
C_1, C_2, C_4, C_5	0.9872
C_1, C_3, C_4, C_5	0.9660
C_2, C_3, C_4, C_5	0.9915

Based on the above tables, the pairs C_2, C_3, and C_2, C_4 received the highest interdependency measure whereas the pair C_1, C_5 has the least degree of relations. Also, from Table 5, we notice that C_2, C_3, C_4 received the highest interdependency measure. From Table 6, we can see that C_1, C_2, C_3, C_4 and C_2, C_3, C_4, C_5 have the highest interdependence measure. Now, we combine the 5 criteria using Sugeno Integration with respect to λ-fuzzy measure. Let's calculate the aggregate values of criteria by using Sugeno integral for student

A. We have

$$-\int f dg_\lambda = \sup_{1 \le i \le n} (f(c_i^*) \wedge g_\lambda(A_i)),$$

where

$$f(c_1^*) = C_2 = 0.2, f(c_2^*) = C_4 = 0.2, f(c_3^*) = C_3 = 0.7, f(c_4^*) = C_5 = 0.8, f(c_5^*) = C_1 = 0.9.$$

Hence

$$\begin{aligned}
-\int f dg_\lambda &= \sup \big((f(c_1^*) \wedge g_\lambda(\{c_1^*, c_2^*, c_3^*, c_4^*, c_5^*\})), (f(c_2^*) \wedge g_\lambda(\{c_2^*, c_3^*, c_4^*, c_5^*\}), \\
&\quad (f(c_3^*) \wedge g_\lambda(\{c_3^*, c_4^*, c_5^*\})), (f(c_4^*) \wedge g_\lambda(\{c_4^*, c_5^*\})), (f(c_5^*) \wedge g_\lambda(\{c_5^*\})) \\
&= \sup[(0.2 \wedge 1), (0.2 \wedge 0.9915), (0.7 \wedge 0.8852), (0.8 \wedge 0.8026), (0.9 \wedge 0.5)] \\
&= \sup[0.2, 0.2, 0.7, 0.8, 0.5] \\
&= 0.8
\end{aligned}$$

So, this application presents the measures of relative importance and interdependencies among the five main criteria for student's failure prediction and it shows that the regularity is one of the most important criteria in the examination.

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