MEAN VALUE THEOREMS: GENERALIZATIONS AND ASSOCIATED FUNCTIONAL EQUATIONS

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Abstract of the Thesis

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The **Lagrange's Mean Value Theorem** is a very important result in Analysis. It originated from Rolle's theorem, which was proved by the French mathematician Michel Rolle (1652-1719) for polynomials in 1691. This theorem appeared for the first time in the book *"Méthode pour résoudre les égalités"* without a proof and without any special emphasis. Rolle's Theorem got its recognition when Joseph Lagrange (1736-1813) presented his mean value theorem in his book *"Théorie des functions analytiques"* in 1797.

It received further recognition when Augustin Louis Cauchy (1789-1857) proved his mean value theorem in his book entitled *"Équations différentielles ordinaires"*. Most of the results in Cauchy's book were established using the Mean Value Theorem or indirectly Rolle's Theorem. Since their discovery, many papers have appeared dealing (directly or indirectly) with of the Rolle's Theorem and Lagrange's Mean Value Theorem. Recently, many functional equations, motivated by various Mean Value Theorems, were studied.

The main goal of this thesis is to prove several Mean Value Theorems, present some of their applications and generalizations and study their Associated Functional Equations.

In Chapter 1, we prove the Lagrange's Mean Value Theorem and present some of its applications. Many examples are given to illustrate its importance in Analysis. Then, we discuss and solve functional equations associated to it. After proving the Topological and Weak Topological Mean Value Theorems and deduce some regularity results, we finish Chapter 1 by establishing the Flett's Mean Value Theorem.

In Chapter 2, we prove the Cauchy's Mean Value Theorem which generalizes Lagrange's Mean Value Theorem. We then study a functional equation associated to the Cauchy's Mean Value Theorem. Mainly, we characterize all pairs of smooth functions for which the mean value is taken at a point having a well-determined position in the interval. As an application, a partial answer to a question, posed by Sahoo and Riedel, is obtained. As we did in Chapter 1, we finish Chapter 2 by establishing the Cauchy's Mean Value Theorem for divided differences.

Chapter 3 introduces the Ostrowski's inequality via the Cauchy's Mean Value Theorem, with some applications and generalizations. Chapter 4 deals with a variation of the Lagrange's mean value theorem due to Dimitri Pompeiu. It is called the Pompeiu's Mean Value Theorem. This theorem has been the source of motivations for many Stamate type functional equations. We finish Chapter 4 by studying some functional equations motivated by the Simpson's rule for numerical integration.

To my family

Contents

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Chapter 1

Lagrange’s Mean Value Theorem: Applications and Associated Functional Equations

In this chapter, we recall the Lagrange’s Mean Value Theorem and some of its interesting applications. We then discuss and solve different functional equations that can be motivated using this mean value theorem. Moreover, we prove the Lagrange’s Mean Value Theorem for divided differences and present some of its applications in the study of means. After proving the topological and weak topological mean value theorems, we present one of the generalizations of the Lagrange’s Mean Value Theorem, namely the Flett’s Mean Value Theorem.

1.1 The Lagrange's Mean Value Theorem

The Lagrange's Mean Value Theorem is one of the most important theorems in calculus. It was discovered by Joseph Louis Lagrange. The first statement of the theorem appears in [1] by the physicist André-Marie Ampère. The idea of proving the theorem using Rolle's Theorem by considering a suitable function was given by Ossian Bonnet [2]. The Indian mathematician Bhaskara II (1114-1185) is credited with knowledge of Rolle's Theorem [3, page 156]. Although the theorem is named after Michel Rolle, Rolle's proof covered only the case of polynomial functions. His proof did not use the methods of differential calculus but was considered to be fallacious at that point in his life.

Theorem 1.1.1. [Rolle's Theorem]. *Consider a real-valued continuous function* f *on* $[x_1, x_2]$*. Assume that* f *is differentiable on* (x_1, x_2) *and* $f(x_1) = f(x_2)$*, then there exists a point* $c \in (x_1, x_2)$ *such that* $f'(c) = 0$*.*

Proof. Since the function f is continuous on $[x_1, x_2]$, by the Intermediate Value Theorem [4, page 134], f attains its maximum and minimum values on $[x_1, x_2]$. If both values occur at the end points x_1 and x_2, the maximum and minimum values are equal and the function is constant, which means that $f'(c) = 0$ for all $c \in (x_1, x_2)$. If the maximum or the minimum occurs at a point $c \in (x_1, x_2)$, we have $f'(c) = 0$. □

Geometrically, Rolle's Theorem means that if there is a horizontal secant line to the graph of the function f, then there is a horizontal tangent to the graph of f at a point between the two points of intersection of the secant line with the graph of f. Rolle's Theorem can be generalized by rotating the graph of f, which yields the Lagrange's Mean Value Theorem.

Theorem 1.1.2. [Lagrange's Mean Value Theorem]. *Let f be a differentiable real-valued function on an interval I. Then, for any two different points $x_1, x_2 \in I$, there exists a point c depending on x_1 and x_2 such that*

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} = f'(c(x_1, x_2)). \tag{1.1.1}$$

Proof. We consider the function h defined on I by

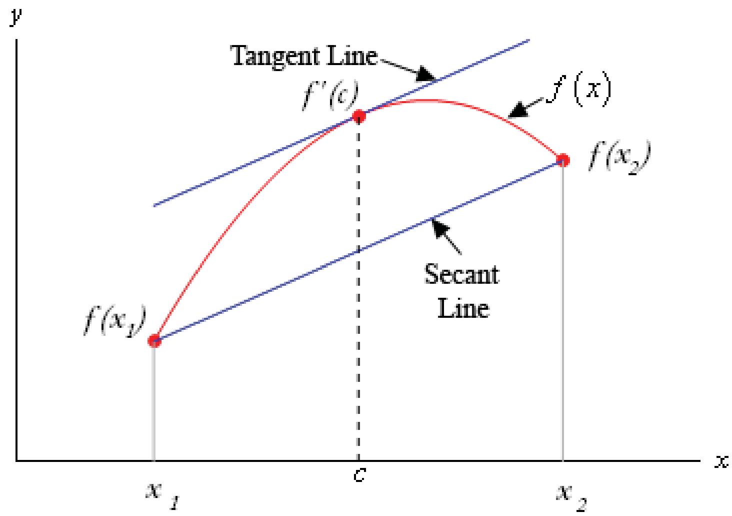
$$h(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) + f(x_1).$$

This is the equation of the line intersecting the graph of f at $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Consider now the function g on I defined by $g = f - h$ (the function g is the result of rotating f and shifting it down to the x-axis). Since both f and h are continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2) so is g. It is also clear that $g(x_1) = g(x_2) = 0$, hence by Rolle's Theorem 1.1.1, there exists a point $c \in (x_1, x_2)$ such that $g'(c) = 0$. This means that

$$f'(c) - \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0,$$

which is $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$. □

Geometrically, the Lagrange's MVT means that the tangent line to the graph of the function f at $c(x_1, x_2)$ is parallel to the secant line joining the points (x_1, $f(x_1)$) and (x_2, $f(x_2)$). This is illustrated in the following figure:



As mentioned, the above proof of the Lagrange's Mean Value Theorem is based on a rotated version of the Rolle's Theorem. However, there are several other proofs of the Lagrange's Mean Value Theorem that do not use Rolle's Theorem (see [5–7]).

1.2 Applications and Examples

Applications of the Lagrange's MVT are numerous. In this section we discuss some of these applications and give examples [4, 8]. First, using the Lagrange's MVT, one can prove basic results in calculus. We recall some of these results in the following proposition:

Proposition 1.2.1. *Let f be a real-valued continuous function on $[a, b]$ that is differentiable on (a, b).*

1. *If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.*

2. *If g is another differentiable real-valued function on (a, b) such that $f'(x) = g'(x)$ for all $x \in (a, b)$, then f and g differ by a constant on $[a, b]$.*

3. *If $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing function on $[a, b]$. If $f'(x) < 0$ for all $x \in (a, b)$, then f is strictly decreasing function on $[a, b]$.*

4. *If $f''(x) > 0$ for all $x \in (a, b)$, then f is concave upward on the interval $[a, b]$.*

Proof. 1. Let x_1, x_2 be any two points in (a, b) and suppose $f(x_1) = f(x_2)$, then by the Lagrange's MVT, Theorem 1.1.2, there is a c in (a, b) such that $f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1} = 0$ contradicting the hypothesis that $f'(x) = 0$ for all $x \in (a, b)$.

2. Let $h(x) = f(x) - g(x)$, then $h'(x) = 0$ on (a, b), so from the first part of this proposition, we have $h(x) = c$, for some constant c. Thus, f and g differ by a constant.

3. Let x_1, x_2 be in $[a, b]$ such that $x_1 < x_2$, then by the Lagrange's MVT there is a point $c \in (x_1, x_2)$, such that $\frac{f(x_1)-f(x_2)}{x_2-x_1} = f'(c) < 0$ and since $x_2 - x_1 > 0$, we have $f(x_2) - f(x_1) > 0$ and hence f is strictly increasing.

4. Using Taylor Series expansion, we have

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2} f''(c), \tag{1.2.1}$$

for $a < c < b$. Since $f^{tt}(c) > 0$, Equality (1.2.1) becomes

$$f(x) > f(x_0) + (x - x_0)f^{t}(x_0). \quad (1.2.2)$$

Replacing x by x_1 and x_0 by $x_0 =: \lambda x_1 + (1 - \lambda)x_2$ for some $0 < \lambda < 1$, Inequality (1.2.2) can be written as

$$f(x_1) > f(x_0) + (1 - \lambda)(x_1 - x_2)f^{t}(x_0). \quad (1.2.3)$$

Now, replacing x by x_2 and x_0 by $x_0 =: \lambda x_1 + (1 - \lambda)x_2$ in Inequality (1.2.2), we get

$$f(x_2) > f(x_0) + \lambda(x_2 - x_1)f^{t}(x_0). \quad (1.2.4)$$

Multiplying Inequality (1.2.3) by λ and Inequality (1.2.4) by $1 - \lambda$ and adding them, we obtain

$$\begin{aligned} \lambda f(x_1) + (1 - \lambda)f(x_2) &> \lambda f(x_0) + \lambda f^{t}(x_0)(1 - \lambda)(x_1 - x_2) \\ &\quad + (1 - \lambda)f(x_0) + (1 - \lambda)\lambda f^{t}(x_0)(x_2 - x_1). \end{aligned}$$

Thus we have $\lambda f(x_1) + (1 - \lambda)f(x_2) > \lambda f(x_0) + (1 - \lambda)f(x_0) = f(x_0)$, which means that f is concave.

□

The Fundamental Theorem of Calculus can also be established by invoking the Lagrange's MVT.

Theorem 1.2.1. [Fundamental Theorem of Calculus]. *Let f be a continuous function on $[a, b]$. Then,*

$$\int_a^b f(x)dx = F(b) - F(a),$$

where F is an antiderivative of f, i.e., $F' = f$.

Proof. Let f be a continuous function on the interval $[a, b]$ and F be an antiderivative of f. Let n be a positive integer and divide $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. Let $x_0 = a, x_1, x_2, \ldots, x_n = b$ be the endpoints of these subintervals. Then,

$$\begin{aligned}
F(b) - F(a) &= F(x_n) - F(x_0) \\
&= F(x_n) + \left(-F(x_{n-1}) + F(x_{n-1}) - \ldots - F(x_1) + F(x_1)\right) - F(x_0) \\
&= \left(F(x_n) - F(x_{n-1})\right) + \ldots + \left(F(x_1) - F(x_0)\right) \\
&= \sum_{j=1}^{n} \left(F(x_j) - F(x_{j-1})\right).
\end{aligned}$$

By the Lagrange's MVT, Theorem 1.1.2, there exists a number c_j in each subinterval $[x_{j-1}, x_j]$ such that

$$F'(c_j) = \frac{F(x_j) - F(x_{j-1})}{x_j - x_{j-1}} = \frac{F(x_j) - F(x_{j-1})}{\Delta x}.$$

Then, $F(x_j) - F(x_{j-1}) = F'(c_j)\,\Delta x$. Since F is an antiderivative of f, we have $F'(c_j) = f(c_j)$, therefore

$$F(b) - F(a) = \sum_{j=1}^{n} f(c_j)\,\Delta x.$$

By taking the limit as $n \to +\infty$, we obtain $F(b) - F(a) = \int_a^b f(x)dx.$ □

Besides these theoretical applications, the Lagrange's MVT has other applications. The next examples illustrate some of these applications.

Example 1.2.1. *The Lagrange's MVT can be used tp prove* **Bernouilli's Inequality***: If* $x > -1$*, then*

$$(1 + x)^n \geq 1 + nx, \quad \text{for all} \quad n \in \mathbb{R}.$$

Proof. Assume that $x \geq 0$ and let $f(t) = (1 + t)^n$, for $t \in [0, x]$. Thus f satisfies the hypothesis of the Lagrange's MVT and we have a point $c \in (0, x)$ with

$$f(x) - f(0) = (x - 0)f'(c).$$

It means that

$$(1 + x)^n - 1 = xn(1 + c)^{n-1} \geq nx,$$

and hence $(1 + x)^n \geq 1 + nx$. The case when $-1 < x < 0$ can be handled similarly by considering $f(t) = (1 + t)^n$ for $t \in [x, 0]$. □

Example 1.2.2. *The Lagrange's MVT can be used to prove the inequality*

$$x \geq 1 + \ln(x) \quad \text{for} \quad x > 0. \tag{1.2.5}$$

Equality holds if and only if $x = 1$. This inequality is widely used in information theory to establish the non negativity of the direct divergence. This inequality is further used for proving the arithmetic mean is greater than or equal to the geometric mean.

Proof. Consider the function f on $[1, b]$ defined by $f(t) = \ln(t)$ where $b > 1$. The function f satisfies the hypothesis of the Lagrange's MVT. Hence, there exists a point $c \in (1, b)$ such that $f(b) - f(1) = (b-1)f'(c)$ which is $\ln(b) = \frac{b-1}{c}$. Since $c \in (1, b)$, we get $\frac{b-1}{b} < \frac{b-1}{c} < \frac{b-1}{1}$ and thus

$$1 - \frac{1}{b} < \ln(b) < b - 1. \tag{1.2.6}$$

Using the left-hand side of Inequality (1.2.6), we obtain $\frac{1}{b} > 1 + \ln(\frac{1}{b})$. Since $b > 1$, it follows that $0 < \frac{1}{b} < 1$. Letting $x = \frac{1}{b}$, we have $x > 1 + \ln(x)$ if $0 < x < 1$. Next, the right-hand side of Inequality (1.2.6) yields $\ln(b) < b - 1$ for $b > 1$. This can be written as $x > \ln(x)$ for $x > 1$. If $x = 1$, then clearly the left hand side of Inequality (1.2.5) is equal to right hand side of Inequality (1.2.5). Thus, we have shown that for any $x > 0$, Inequality (1.2.5) holds with equality if and only if $x = 1$. □

Example 1.2.3. *The Lagrange's MVT can be used in establishing the following inequality*

$$a^{\alpha} < a\alpha + b(1-\alpha) \; b^{\alpha-1}, \tag{1.2.7}$$

where $0 < \alpha < 1$ and a, b are positive real numbers. This inequality is used while proving the Hölder inequality in Analysis.

Proof. We define a function f by $f(t) = t^{\alpha}$ for $t > 0$ and $0 < \alpha < 1$. The function f is continuous on $[a, b]$. Applying the Lagrange's MVT to f, we obtain

$$\frac{b^\alpha - a^\alpha}{b - a} = \alpha c^{\alpha-1} \tag{1.2.8}$$

for some $c \in (a, b)$. This gives $c^{\alpha-1} > b^{\alpha-1}$. Hence, since $\alpha > 0$, we have $\alpha c^{\alpha-1} > \alpha b^{\alpha-1}$. This with Inequality (1.2.8) gives

$$b^\alpha - a^\alpha > (b - a)\alpha b^{\alpha-1},$$

which after some simplifications yields Inequality (1.2.7). □

Example 1.2.4. *The Lagrange's MVT can be used to show that, for $x > 0$, the function $(1+\frac{1}{x})^x$ is an increasing function of x while the function $(1+\frac{1}{x})^{x+1}$ is a decreasing function of x.*

Proof. Consider the function f defined by $f(t) = \ln t$ for $t > 0$. Applying the Lagrange's MVT to f, we get, for $x > 0$,

$$\ln(x + 1) - \ln x = \frac{1}{c},$$

for some $c \in (x, x + 1)$. Now, we have

$$\begin{aligned}
\frac{d}{dx} \ln \left(1 + \frac{1}{x}\right)^x &= \frac{d}{dx} x(\ln(x + 1) - \ln x) \\
&= \ln(x + 1) - \ln x + x\left(\frac{1}{1 + x} - \frac{1}{x}\right) \\
&= \ln(x + 1) - \ln x - \frac{1}{x + 1} \\
&= \frac{1}{c} - \frac{1}{x + 1} > 0.
\end{aligned}$$

Since $\ln x$ is an increasing function, we conclude that $(1+\frac{1}{x})^x$ is an increasing function of x. To show that $(1+\frac{1}{x})^{x+1}$ is a decreasing function of x, we proceed in a similar manner and show that

$$\begin{aligned}
\frac{d}{dx}\ln\left(1+\frac{1}{x}\right)^{x+1} &= \frac{d}{dx}[(x+1)(\ln(x+1)-\ln x)] \\
&= \ln(x+1)-\ln x+(x+1)\left(\frac{1}{x+1}-\frac{1}{x}\right) \\
&= \ln(x+1)-\ln x-\frac{1}{x} \\
&= \frac{1}{c}-\frac{1}{x}<0.
\end{aligned}$$

Hence $(1+\frac{1}{x})^{x+1}$ is a decreasing function of the variable x. □

Example 1.2.5. *The Lagrange's MVT can be used in establishing the formula*

$$\int_0^b x^\alpha dx = \frac{b^{\alpha+1}}{\alpha+1}, \tag{1.2.9}$$

for $\alpha \geq 0$ and $b>0$. This approach has some advantages over the traditional approach found in many calculus textbooks.

Proof. Consider the function f defined by $f(t)=\frac{t^{\alpha+1}}{\alpha+1}$. By the Lagrange's MVT, there exists $c \in (k-1,k)$ such that for every positive integer k, we have

$$\frac{k^{\alpha+1}}{\alpha+1}-\frac{(k-1)^{\alpha+1}}{\alpha+1}=c^\alpha. \tag{1.2.10}$$

Since $c \in (k-1,k)$, we see that $(k-1)^{\alpha+1} < c^\alpha < k^\alpha$. Using Equation (1.2.10), we get

$$(k-1)^{\alpha+1} < \frac{k^{\alpha+1}}{\alpha+1}-\frac{(k-1)^{\alpha+1}}{\alpha+1} < k^\alpha.$$

Summing over k from 1 to n, we obtain

$$\sum_{k=1}^{n}(k-1)^{\alpha} < \frac{k^{\alpha+1}}{\alpha+1} < \sum_{k=1}^{n} k^{\alpha}.$$

Thus, one can deduce that

$$\frac{1}{\alpha+1} < \frac{1^{\alpha}+2^{\alpha}+\ldots+n^{\alpha}}{n^{\alpha+1}} < \frac{1}{\alpha+1}+\frac{1}{n}.$$

Letting $n \to \infty$, we obtain $\lim_{n\to\infty} \frac{1^{\alpha}+2^{\alpha}+\ldots+n^{\alpha}}{n^{\alpha+1}} = \frac{1}{\alpha+1}$. The definition of definite integral implies that

$$\int_0^b x^{\alpha}dx = \lim_{n\to\infty} \frac{1^{\alpha}+2^{\alpha}+\ldots+n^{\alpha}}{n^{\alpha+1}} b^{\alpha+1}.$$

Thus we have $\int_0^b x^{\alpha}dx = \frac{b^{\alpha+1}}{\alpha+1}$. □

Example 1.2.6. *Let f be a function defined on (a, b), and suppose $f'(c)$ exists for some $c \in (a, b)$. Let g be differentiable on an interval containing $f(c+h)$ for h sufficiently small, and suppose g' in continuous at $f(c)$. Then $g \circ f$ differentiable at c and we have*

$$(g \circ f)'(c) = g'(f(c))f'(c).$$

Proof. In fact, since g differentiable, by the Lagrange's Mean Value Theorem, we get

$$g(f(h+c)) - g(f(c)) = g'(\theta)[f(c+h) - f(c)],$$

for some θ strictly between $f(c+h)$ and $f(c)$. Now since f is differentiable at c, we have $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = f'(c)$. As f is continuous, $\lim_{h\to 0} f(c+h) = f(c)$ and thus $\lim_{h\to 0} \theta = f(c)$ since $f(c+h) < \theta < f(c)$. Using the continuity of g' at $f(c)$, we have $\lim_{h\to 0} g'(\theta) = g'(\lim_{h\to 0} \theta) = g'(f(c))$. Hence,

$$\begin{aligned} g'(f(c))f'(c) &= \left(\lim_{h\to 0} g'(\theta)\right) \left(\lim_{h\to 0} \frac{f(c+h) - f(c)}{h}\right) \\ &= \lim_{h\to 0} \left(g'(\theta) \frac{f(c+h) - f(c)}{h} \right) \\ &= \lim_{h\to 0} \left(\frac{g(f(c+h)) - g(f(c))}{h} \right) \\ &= (g \circ f)'(c). \end{aligned}$$

□

Example 1.2.7. *The Lagrange's MVT can also be used to introduce an infinite family of means, known as* **Stolarsky's mean** *[9]. Define* $f(x) = x^{\alpha}$, *where* α *is a real parameter. We apply the Lagrange's Mean Value Theorem to* f *on the interval* $[x, y]$. *There exists a point* c *with* $x < c < y$

$$f'(c_{\alpha}(x, y)) = \frac{f(x) - f(y)}{x - y}$$

which is

$$c_{\alpha}(x, y) = \left(\frac{x^{\alpha} - y^{\alpha}}{\alpha(x - y)}\right)^{\frac{1}{\alpha - 1}}.$$

Note that we have used $c_{\alpha}(x, y)$ *instead of* c *to emphasize the dependence of* c *on* x, y *and* α. *From this, one obtains an infinite family of means by varying the parameter* α. *These means are known as Stolarsky's means.*

For instance, if $\alpha = -1$, then one gets the geometric mean

$$c_{-1}(x, y) = \sqrt{xy}.$$

If $\alpha = 2$, one gets the arithmetic mean

$$c_2(x, y) = \frac{x + y}{2}.$$

If $\alpha \to 0$, one gets the logarithmic mean

$$\lim_{\alpha \to 0} c_\alpha(x, y) = \frac{x - y}{\ln x - \ln y}.$$

If $\alpha \to 1$, one gets the identric mean

$$\lim_{\alpha \to 0} c_\alpha(x, y) = \frac{1}{e}\left(\frac{y^y}{x^x}\right)^{\frac{1}{y-x}}.$$

In Example 1.2.7, we have seen that one can construct an infinite class of means given two positive real numbers x and y using the Lagrange's Mean Value Theorem. It is easy to extend the definitions of the arithmetic and geometric means to n positive real numbers. It is obvious that the arithmetic mean of n positive numbers is

$$A(x_1, x_2, \cdots, x_n) = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

whereas the geometric mean is

$$G(x_1, x_2, \cdots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n}.$$

However, it is not so obvious to find an appropriate formula for the logarithmic mean of n positive numbers. In the remaining portion of this section we discuss how one can extend the definition of the logarithmic mean in the case of more than two positive real numbers. Recall that the logarithmic mean of positive real numbers x and y is given by

$$L(x, y) = \begin{cases} \frac{x-y}{\ln x - \ln y} & \text{if } x = y, \\ x & \text{if } x = y. \end{cases}$$

Given the positive real numbers x, y and z one can construct an infinite class of means by using a different quotient which approximates a second derivative. Using the mean value for divided difference, Theorem 1.4.2 (see Section 1.4), we have

$$\frac{f(x)}{(x-y)(x-z)} + \frac{f(y)}{(y-z)(y-x)} + \frac{f(z)}{(z-x)(z-y)} = \frac{1}{2}f''(c),$$

where $\min\{x, y, z\} < c < \max\{x, y, z\}$. As in the previous example, we let $f(t) = t^{\alpha}$ to obtain

$$c_{\alpha}(x, y, z) = \left(\frac{2}{\alpha(\alpha-1)} \frac{z^{\alpha}(y-x) + y^{\alpha}(x-z) + x^{\alpha}(x-y)}{(x-y)(z-x)(y-x)}\right)^{\frac{1}{\alpha-2}}.$$

If we put $\alpha = 3$, then we get

$$c_3(x, y, z) = \frac{x+y+z}{3}.$$

With $\alpha = -1$, we get

$$c_{-1}(x, y, z) = \sqrt[3]{xyz}.$$

Two generalizations of the logarithmic mean can be constructed by considering the limiting cases of $\alpha = 0$ and $\alpha = 1$. For instance

$$\lim_{\alpha\to 0} c_\alpha(x, y, z) = \sqrt{\frac{(z-y)(z-x)(y-x)}{2\left(x\ln(\frac{z}{y}) + y\ln(\frac{x}{z}) + z\ln(\frac{y}{x})\right)}},$$

$$\lim_{\alpha\to -1} c_\alpha(x, y, z) = \sqrt{\frac{(z-y)(z-x)(y-x)}{2\left(xy\ln(\frac{z}{y}) + xz\ln(\frac{x}{z}) + xy\ln(\frac{y}{x})\right)}}.$$

A generalization of the identric mean can be obtained by considering the limiting case when $\alpha = 2$. For example,

$$\lim_{\alpha\to 2} c_\alpha(x, y, z) = \exp\left(-\frac{3}{2} + \frac{z^2\ln z}{(z-x)(z-y)} + \frac{y^2\ln y}{(y-x)(y-z)} + \frac{x^2\ln x}{(x-y)(x-z)}\right).$$

Besides these generalizations, one can also generalize the logarithmic mean by examining the appropriate integral representation of the function $L(x, y)$. It can be checked that

$$L(x, y) = \int_0^1 \left(\frac{x}{y}\right)^t y\,dt.$$

In view of this integral, one can define the logarithmic mean between three positive real numbers as

$$L(x, y, z) = \int_0^1\int_0^1 \left(\frac{x}{z}\right)^t \left(\frac{y}{z}\right)^s z\,dt\,ds.$$

Evaluating the above integral we obtain the following explicit form of L as

$$L(x, y, z) = \frac{(z - x)(z - y)}{z(\ln z - \ln x)(\ln z - \ln y)}.$$

The above can be generalized in the case of n positive real numbers $x_1, x_2, \cdots, x_n$ as

$$L(x_1, x_2, \cdots, x_n) = \int_0^1 \int_0^1 \cdots \int_0^1 \left(\frac{x_1}{x_n}\right)^{t_1} \left(\frac{x_2}{x_n}\right)^{t_2} \cdots \left(\frac{x_{n-1}}{x_n}\right)^{t_{n-1}} x_n \, dt_1 \, dt_2 \cdots dt_{n-1}.$$

It should also be noted that the function $L(x, y)$ can also be represented by the following integral:

$$L(x, y) = \left[\int_0^1 \frac{dt}{tx + (1 - t)y}\right]^{-1}.$$

In view of the above integral representation of the logarithmic mean, we have the following extension

$$L(x_1, x_2, \cdots, x_n) = \left[(n - 1)! \int_{\Gamma_n} \left(\sum_{i=1}^{n} t_i x_i\right)^{-1} dt\right]^{-1},$$

where $\Gamma_n = \left\{(t_1, t_2, \cdots, t_n) \text{ such that } t_i \geq 0, \sum_{i=1}^{n} t_i \leq 1\right\}$, $t_n = 1 - \sum_{i=1}^{n-1} t_i$, and $dt = dt_1 \, dt_2 \cdots dt_n$.

1.3 Functional Equations Associated to the Lagrange's MVT

In this section, we study a functional equation that arises from the Lagrange's MVT and then we investigate a methodical study of this functional equation and its various generalizations. These functional equations characterize polynomials of different degrees.

Definition 1.3.1. *For distinct real numbers $x_1, x_2, \ldots, x_n$, the divided difference of the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined as $f[x_1] = f(x_1)$ and*

$$f[x_1, x_2, ..., x_n] = \frac{f[x_1, x_2, ..., x_{n-1}] - f[x_2, x_3, ..., x_n]}{x_1 - x_n}$$

for all $n \geq 2$.

It is easy to see that $f[x_1, x_2] = \frac{f(x_1)-f(x_2)}{x_1-x_2}$ and

$$\begin{aligned}
f[x_1, x_2, x_3] &= \frac{f[x_1, x_2] - f[x_2, x_3]}{x_1 - x_3} \\
&= \frac{\frac{f(x_1)-f(x_2)}{x_1-x_2} - \frac{f(x_2)-f(x_3)}{x_2-x_3}}{x_1 - x_3} \\
&= \frac{(x_2 - x_3)f(x_1) - (x_2 - x_3)f(x_2) - (x_1 - x_2)f(x_2) + (x_1 - x_2)f(x_3)}{(x_1 - x_2)(x_2 - x_3)(x_1 - x_3)} \\
&= \frac{(x_2 - x_3)f(x_1) + (x_3 - x_1)f(x_2) + (x_1 - x_2)f(x_3)}{(x_1 - x_2)(x_2 - x_3)(x_1 - x_3)}
\end{aligned}$$

Using the definition of divided difference, Equality (1.1.1) in the Lagrange's

MVT takes the form:

$$f[x_1, x_2] = f'(c(x_1, x_2)). \tag{1.3.1}$$

Of course, c depends on x_1 and x_2 and one may ask for what f the mean value c depends on x_1 and x_2 in a given manner. From this point of view, Equality (1.3.1) appears as a **functional equation with unknown function f and given** c. The following theorem was established by J. Aczél [10] and also independently by Sh. Haruki [11].

Theorem 1.3.1 ([10]). *The functions $f, h : \mathbb{R} \longrightarrow \mathbb{R}$ satisfy the functional equation*

$$f[x, y] = h(x + y), x = y \tag{1.3.2}$$

if and only if $f(x) = ax^2 + bx + c$ and $h(x) = ax + b$ where a, b, c are arbitrary real constants.

Proof. Using the definition of the divided difference of f, Equation (1.3.2) can be written as

$$f(x) - f(y) = (x - y)h(x + y) \quad \text{for} \quad x = y, \tag{1.3.3}$$

which is also true for $x = y$ since $f(x) - f(x) = (x - x)h(x + x) = 0$. If f satisfies Equation (1.3.3), so does $f + c$, where c is an arbitrary constant. Therefore we may assume without loss of generality $f(0) = 0$, Putting $y = 0$

in Equation (1.3.3) we see that

$$f(x) = xh(x). \tag{1.3.4}$$

Hence using Equation (1.3.4), Equation (1.3.3) becomes

$$xh(x) - yh(y) = (x - y)h(x + y). \tag{1.3.5}$$

Again if h satisfies Equation (1.3.5) so also $h + c$, where c is an arbitrary constant. So suppose $h(0) = 0$, therefore letting $x = -y$ in Equation (1.3.5), we obtain

$$-yh(y) = yh(y).$$

That is h is an odd function. Taking this into consideration and replacing y by $-y$ in Equation (1.3.5), we get

$$xh(x) + yh(-y) = (x + y)h(x - y).$$

Since h is odd, we get

$$xh(x) - yh(y) = (x + y)h(x - y). \tag{1.3.6}$$

Comparing Equation (1.3.6) with Equation (1.3.5), we obtain

$$(x - y)h(x + y) = (x + y)h(x - y). \tag{1.3.7}$$

Substituting $u = x + y$ and $v = x - y$ in Equation (1.3.7) we obtain $vh(u) = uh(v)$ for all $u, v \in \mathrm{R}$. Thus $h(u) = au$. If we do not assume $h(0) = 0$, then we have in general $h(u) = au + b$. By Equation (1.3.4), this gives $f(x) = x(ax + b)$ and if we do not assume $f(0) = 0$, then $f(x) = ax^2 + bx + c$. So we have proved that all solutions of Equation (1.3.2) are of the form $f(x) = ax^2 + bx + c$ and $h(x) = ax + b$. The converse of this theorem is straightforward. □

Corollary 1.3.1. *The function $f : \mathrm{R} \longrightarrow \mathrm{R}$ satisfies the functional equation*

$$f(x) - f(y) = (x - y)f^t\left(\frac{x + y}{2}\right), \quad \text{for} \quad x = y,$$

if and only if $f(x) = ax^2 + bx + c$ where a, b, c are real constants.

Theorem 1.3.2 ([12, 13]). *If the quadratic polynomial $f(x) = ax^2 + bx + c$ with $a = 0$, is a solution of the functional equation*

$$f(x + h) - f(x) = hf^t(x + \theta h) \qquad (0 < \theta < 1), \tag{1.3.8}$$

assumed for all $x \in \mathrm{R}$ and $h \in \mathrm{R}$, then $\theta = \frac{1}{2}$. Conversely, if a function f satisfies the above functional differentiable equation with $\theta = \frac{1}{2}$, then the only solution is a polynomial of degree at most two.

Proof. Suppose the polynomial $f(x) = ax^2 + bx + c$ is a solution of Equation (1.3.8). Then Equation (1.3.8) can be written as

$$a(x + h)^2 + b(x + h) + c - ax^2 - bx - c = h[2a(x + \theta h) + b],$$

which is $ah^2(1 - 2\theta) = 0$. Since $a = 0$ and $h = 0$, we have $1 - 2\theta = 0$ therefore

$\theta = \frac{1}{2}$. This proves the if part of the theorem. Next we prove the converse of the theorem. Letting $\theta = \frac{1}{2}$ and $h = y - x$ in Equation (1.3.8), we have that $f(x) - f(y) = (x - y)f^{t}(\frac{x+y}{2})$, for $x = y$. Thus by Corollary 1.3.1, f is a polynomial of degree at most two. □

Theorem 1.3.3. *Let s and t be given real numbers. Then all differentiable functions f on the real line which satisfy*

$$f[x, y] = f^{t}(sx + ty)$$

for all real numbers x, y with $x = y$ are of the form

$$f(x) = \begin{cases} ax^2 + bx + c, & \text{if } s = t = \frac{1}{2} \\ bx + c, & \text{otherwise} \end{cases}$$

where a, b, c are arbitrary constants.

P. L. Kannapan, P. K. Sahoo and M. S. Jacobson [14] established a generalization of Theorem 1.3.3. For this reason, we omit the proof of Theorem 1.3.3 and we prove its generalization.

Theorem 1.3.4 ([14]). *Let s and t be the real parameters. The functions $f, g, h : \mathrm{R} \longrightarrow \mathrm{R}$ satisfy*

$$\frac{f(x) - g(y)}{x - y} = h(sx + ty) \tag{1.3.9}$$

for all $x, y \in \mathrm{R}$, $x = y$ if and only if

$$
f(x) = \begin{cases} ax + b, & \text{if } s = 0 = t \\ ax + b, & \text{if } s = 0, t = 0 \\ ax + b, & \text{if } s = 0, t = 0 \\ atx^2 + ax + b, & \text{if } s = t = 0 \\ \frac{A(tx)}{t} + b, & \text{if } s = -t = 0 \\ \beta x + b, & \text{if } s^2 = t^2 \end{cases}
$$

$$
g(y) = \begin{cases} ay + b, & \text{if } s = 0 = t \\ ay + b, & \text{if } s = 0, t = 0 \\ ay + b, & \text{if } s = 0, t = 0 \\ aty^2 + ay + b, & \text{if } s = t = 0 \\ \frac{A(ty)}{t} + b, & \text{if } s = -t = 0 \\ \beta y + b, & \text{if } s^2 = t^2 \end{cases}
$$

$$
h(y) = \begin{cases} \text{arbitrary with } h(0) = a, & \text{if } s = 0 = t \\ a, & \text{if } s = 0, t = 0 \\ a, & \text{if } s = 0, t = 0 \\ ay + b, & \text{if } s = t = 0 \\ \frac{A(ty)}{y} + \frac{(c-b)t}{y}, & \text{if } s = -t = 0, y = 0 \\ \beta, & \text{if } s^2 = t^2, \end{cases}
$$

where $A : \mathbb{R} \longrightarrow \mathbb{R}$ *is an additive function and* a, b, c, α, β *are arbitrary real* *constants.*

Proof. We consider several cases depending on parameters s and t.

<u>Case 1:</u> Suppose $s = 0 = t$, then Equation (1.3.9) yields

$$\frac{f(x) - g(y)}{x - y} = h(0),$$

which is $f(x) - ax = g(y) - ay$, where $a = h(0)$. From the above, we obtain

$$f(x) = ax + b \quad \text{and} \quad g(y) = ay + b, \tag{1.3.10}$$

where b is an arbitrary constant. Letting Equation (1.3.10) into Equation (1.3.9), we see that h is an arbitrary function with $a = h(0)$. Thus we obtain the solution as asserted in Theorem 1.3.4 for the case $s = 0 = t$.

<u>Case 2:</u> Suppose $s = 0$ and $t = 0$ (the case $s = 0$ and $t = 0$ can be handled in a similar manner). Then from Equation (1.3.9), we get

$$\frac{f(x) - g(y)}{x - y} = h(ty). \tag{1.3.11}$$

Putting $y = 0$ in Equation (1.3.11), we get $\frac{f(x)-g(0)}{x-0} = h(0)$ or $\frac{f(x)-b}{x} = a$, which is

$$f(x) = ax + b, \quad x = 0, \tag{1.3.12}$$

where $h(0) = a$ and $g(0) = b$. Letting Equation (1.3.12) into Equation (1.3.11), we obtain

$$ax + b - g(y) = (x - y)h(ty), \tag{1.3.13}$$

for all $x = y$ and $x = 0$. Equating the coefficients of x and the constant terms in Equation (1.3.13), we get

$$h(ty) = a \quad \text{and} \quad g(y) = h(ty)y + b = ay + b, \tag{1.3.14}$$

for all $y \in \mathrm{R}$. Letting $x = 0$ in Equation (1.3.11) and using Equation (1.3.14), we see that $\boldsymbol{f}(0) = b$. Thus Equation (1.3.12) holds for all all $x \in \mathrm{R}$. Hence from Equation (1.3.12) and Equation (1.3.14), we get the solution of Equation (1.3.9) for this case as asserted in the theorem.

<u>Case 3:</u> Suppose $s = 0$ and $\boldsymbol{t} = 0$. Letting $x = 0$ in Equation (1.3.9) we get

$$g(y) = yh(yt) + b, \tag{1.3.15}$$

for all $y = 0$. Similarly, letting $y = 0$ in Equation (1.3.9) we get

$$\boldsymbol{f}(x) = xh(sx) + c, \tag{1.3.16}$$

for all $x = 0$, where $g(0) = c$. Inserting Equation (1.3.15) and Equation (1.3.16) in Equation (1.3.9) and simplifying we obtain

$$xh(sx) + c - yh(ty) - b = (x - y)h(sx + ty), \tag{1.3.17}$$

for all non zero x and y with $x = y$. Replacing x by $\frac{x}{s}$ and y by $\frac{y}{t}$ in Equation (1.3.17), we get

$$\frac{x}{s}h(x) + c - \frac{y}{t}h(y) - b = (\frac{x}{s} - \frac{y}{t})h(x + y), \tag{1.3.18}$$

for all nonzero x and y with $tx = sy$.

<u>Subcase 3.1:</u> Suppose $s = t$, hence Equation (1.3.18) yields

$$xh(x) - yh(y) = t(b - c) + (x - y)h(x + y). \tag{1.3.19}$$

Interchanging x with y in Equation (1.3.19) and adding the resulting equation to Equation (1.3.19) we get $b = c$. Thus Equation (1.3.19) reduces to

$$xh(x) - yh(y) = (x - y)h(x + y), \tag{1.3.20}$$

for all real nonzero x and y with $x = y$. Replacing y with $-y$ in Equation (1.3.20), we obtain

$$xh(x) + yh(-y) = (x + y)h(x - y), \tag{1.3.21}$$

for all nonzero x and y with $x + y = 0$. Letting $y = -x$ in Equation (1.3.20), we see that

$$xh(x) + xh(-x) = 2xh(0). \tag{1.3.22}$$

Subtracting Equation (1.3.20) from Equation (1.3.21) and using Equation

(1.3.22) we get

$$2yh(0) = (x + y)h(x - y) - (x - y)h(x + y), \qquad (1.3.23)$$

for all nonzero x, y with $x + y$ and $x - y = 0$. Writing $u = x + y$ and $v = x - y$ in Equation (1.3.23), we get

$$(u - v)h(0) = uh(v) - vh(u),$$

which is

$$v[h(u) - h(0)] = u[h(v) - h(0)],$$

for all nonzero $u, v, u - v$ and $u + v$. Thus

$$h(u) = \alpha u + a, \qquad (1.3.24)$$

for all nonzero $u \in R$ (where $a = h(0)$). Notice that Equation (1.3.24) also holds for $u = 0$. Using Equation (1.3.24) in Equation (1.3.9), we get

$$\boldsymbol{f}(x) - g(y) = (x - y)(\alpha tx + \alpha ty + a),$$

for all $x = y$. Thus we obtain the asserted solution $\boldsymbol{f}(x) = g(x) = \alpha tx^2 + ax + b$ and $h(y) = \alpha y + a$ where α, a and b are arbitrary constants.

<u>Subcase 3.2:</u> Suppose $s = -\boldsymbol{t}$. Then Equation (1.3.18) yields

$$xh(x) + yh(y) + (b - c)\boldsymbol{t} = (x + y)h(x + y), \qquad (1.3.25)$$

for all real nonzero x and y with $x = y$. Define

$$A(x) = \begin{cases} xh(x) + (b - c)t, & \text{if } x = 0, \\ 0, & \text{if } x = 0. \end{cases} \tag{1.3.26}$$

Then by Equation (1.3.26), Equation (1.3.25) reduces to

$$A(x) + A(y) = A(x + y), \tag{1.3.27}$$

for all real nonzero x and y and $x+y$. Next we show that A in Equation (1.3.27) is additive on the set of reals. In order for A to be additive it must satisfy $A(x) + A(-x) = A(0) = 0$ or $xh(x) - xh(-x) + 2(b - c)t = 0$. Interchanging y with $-y$ in Equation (1.3.25), we get

$$xh(x) - yh(-y) + (b - c)t = (x - y)h(x - y). \tag{1.3.28}$$

Subtracting Equation (1.3.28) from Equation (1.3.25), we get

$$yh(y) + yh(-y) = (x + y)h(x + y) - (x - y)h(x - y).$$

Thus using Equation (1.3.26), we get

$$A(y) - A(-y) = A(x + y) - A(x - y), \tag{1.3.29}$$

for all nonzeo $x, y, x + y$ and $x - y$. Replacing x by $-x$ in Equation (1.3.29) we obtain

$$A(y) - A(-y) = A(-x + y) - A(-x - y). \tag{1.3.30}$$

From Equation (1.3.29) and Equation (1.3.30) we get

$$A(x + y) - A(-(x + y)) = A(x - y) - A(-(x - y)). \tag{1.3.31}$$

Letting $u = x + y$ and $v = x - y$ in Equation (1.3.31), we see that $A(u) + A(-u) = A(v) + A(-v)$ for all real nonzero $u, v, u - v$ and $u + v$. Thus

$$A(u) + A(-u) = \gamma, \tag{1.3.32}$$

for all real nonzero u (where γ is a constant). Using Equation (1.3.26), we see from Equation (1.3.32) that

$$xh(x) - xh(-x) + 2(b - c)\boldsymbol{t} = \gamma, \tag{1.3.33}$$

for all real nonzero x. From Equation (1.3.9) with $s = -\boldsymbol{t}$, we get

$$f(x) - g(y) = (x - y)h(-(x - y)t). \tag{1.3.34}$$

Interchanging x with y, we get

$$f(y) - g(x) = -(x - y)h((x - y)t). \tag{1.3.35}$$

Adding Equation (1.3.34) to Equation (1.3.35) and using Equation (1.3.33), we get

$$\begin{aligned} f(x) - g(y) + f(y) - g(x) &= (x-y)h(-(x-y)t) - (x-y)h((x-y)t) \\ &= \frac{-\gamma}{t} + 2(b-c). \end{aligned} \tag{1.3.36}$$

Using Equation (1.3.15) and Equation (1.3.26), we obtain

$$A(tx) = t[g(x) - c] \quad \text{for} \quad x = 0. \tag{1.3.37}$$

Similarly using Equation (1.3.16) and Equation (1.3.26), we get

$$A(-tx) = -t[f(x) - b], \qquad \text{for} \quad x = 0. \tag{1.3.38}$$

So from Equation (1.3.37) and Equation (1.3.38), we see that

$$f(x) - g(x) = -\frac{(-tx) + A(tx)}{t} + b - c = -\frac{\gamma}{t} + b - c.$$

Hence from above, we get

$$f(x) - g(x) + f(y) - g(y) = -2\frac{\gamma}{t} + 2(b-c). \tag{1.3.39}$$

Comparing Equation (1.3.36) and Equation (1.3.39), we get $\gamma = 0$ and thus Equation (1.3.32) yields: $A(x) + A(-x) = 0$ for real nonzero x. Evidently the above also holds for $x = 0$. Hence A is an additive function on the set of reals.

From Equation (1.3.26), Equation (1.3.15) and Equation (1.3.16), we obtain

$$\begin{cases} f(x) = \frac{A(tx)}{t} + b, \\ g(y) = \frac{A(ty)}{t} + c, \\ h(y) = \frac{A(y)}{y} + \frac{(c-b)t}{y}, \end{cases}$$

where b and c are arbitrary constants.

<u>Subcase 3.3:</u> Suppose $s^2 = t^2$, that is $s = \pm t$. Interchanging x with y in Equation (1.3.18), we get

$$\frac{y}{s}h(y) - \frac{x}{t}h(x) + c - b = (\frac{y}{s} - \frac{x}{t})h(x+y), \tag{1.3.40}$$

for all nonzero x and y with $ty = sx$. Subtracting Equation (1.3.40) from Equation (1.3.18) and using $s + t = 0$ we get $xh(x) - yh(y) = (x - y)h(x + y)$ which is the same as Equation (1.3.20). Thus

$$h(x) = \alpha x + b, \tag{1.3.41}$$

where α and b are arbitrary constants. Letting Equation (1.3.41) into Equation (1.3.40) and simplifying the resulting expression, we get

$$\alpha xy(\frac{1}{s} - \frac{1}{t}) = b - c,$$

for all nonzero x and y with $tx = sy$ and $sx = ty$. Since $s = t$, we see that

$\alpha = 0$ and $b = c$. Hence Equation (1.3.41) becomes

$$h(x) = b. \tag{1.3.42}$$

From Equation (1.3.42), Equation (1.3.15) and Equation (1.3.16), we obtained the asserted form of f, g and h. □

Remark 1.3.1. *If $g = f$, the subcase 3.2 can be simplified. In fact for $g = f$, the left side of Equation (1.3.9) for $s = -t$ is symmetric in x and y. Using this symmetry one can conclude that h is an even function. The evenness of h implies that A in Equation (1.3.20) is additive.*

Remark 1.3.2. *In the subcase 3.3, $h(y)$ is undefined at $y = 0$.*

Remark 1.3.3. *The functional equation $A(x + y) = A(x) + A(y)$ has discontinuous solutions in addition to the continuous solutions of the form $A(x) = ax$ [8], where a is an arbitrary real constant. Since an additive function appears in the solution of Equation (1.3.9) for the subcase $s = -t$, it follows that Equation (1.3.9) has discontinuous solutions. However, all continuous solutions of Equation (1.3.9) are polynomials of low degree.*

The following corollary is obvious from the above theorem.

Corollary 1.3.2. *The functions $\varphi, f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfy the functional equation*

$\boldsymbol{f}[x_1, x_2] = \varphi^t(sx_1 + tx_2))$ *for all* $x, y \in \mathbb{R}$ *with* $x = y$ *if and only if:*

$$\boldsymbol{f}(x) = \begin{cases} ax + b, & \text{if } \ s = t = 0 \\ ax + c, & \text{if } \ s = 0, t = 0 \\ ax + c, & \text{if } \ s = 0, t = 0 \\ atx^2 + ax + b, & \text{if } \ s = t = 0 \\ \frac{A(tx)}{t} + b, & \text{if } \ s = -t = 0 \\ \beta x + b, & \text{if } \ s^2 = t^2 \end{cases}$$

$$\varphi(y) = \begin{cases} \text{arbitrary}, & \text{if } \ s = t = 0 \\ a, & \text{if } \ s = 0, t = 0 \\ a, & \text{if } \ s = 0, t = 0 \\ \alpha y + a, & \text{if } \ s = t = 0 \\ \frac{A(y)}{y} + b, & \text{if } \ s = -t = 0, y = 0 \\ \beta, & \text{if } \ s^2 = t^2, \end{cases}$$

where $A : \mathbb{R} \longrightarrow \mathbb{R}$ *is an additive function and* a, b, c, α, β *are arbitrary real constants.*

The following corollary addresses a recreational problem posed by W. Rudin [13].

Corollary 1.3.3. *The function* $\boldsymbol{f} : \mathbb{R} \longrightarrow \mathbb{R}$ *satisfies the equation*

$$\boldsymbol{f}^t(sx + ty) = \frac{\boldsymbol{f}(y) - \boldsymbol{f}(x)}{y - x}$$

for all $x, y \in \mathbb{R}$ with $x = y$ if and only if

$$\boldsymbol{f}(x) = \begin{cases} ax^2 + bx + c, & \text{if } s = \frac{1}{2} = t \\ bx + c, & \text{if otherwise} \end{cases}$$

where a, b, c are arbitrary real constants.

Polynomials are basic objects in mathematics. In many applications, functions are used for modeling of real world problems. If these functions are sufficiently smooth, they can be approximated by polynomials, in some range and within some accuracy.

Theorem 1.3.1 and Theorem 1.3.4 characterize polynomials of low degree. Originally Equation (1.3.2) appeared in the form

$$\boldsymbol{f}(x) - \boldsymbol{f}(y) = (x - y)h(x + y),$$

and thus it was not clear what the generalization of it would be for higher order polynomials. Ideas came from the notions of divided difference. Bailey [15] generalized Theorem 1.3.1 and Theorem 1.3.4 and established the following:

Theorem 1.3.5 ([15]). *If $\boldsymbol{f}$ is a differentiable function satisfying, the functional equation*

$$\boldsymbol{f}[x, y, z] = h(x + y + z), \tag{1.3.43}$$

then $\boldsymbol{f}$ is a polynomial of degree at most three.

Proof. Using the definition of divided difference and Equation (1.3.43) one obtains:

$$\begin{aligned} & f(x)(y-z)+f(y)(z-x)+f(z)(x-y) \\ = \; & (x-y)(y-z)(x-z)h(x+y+z). \qquad (1.3.44) \end{aligned}$$

If f satisfies Equation (1.3.44) so also $f+d$, where d is an arbitrary constant. Therefore, we may assume without loss of generality $f(0) = 0$. Under this assumption we set $z = 0$ in Equation (1.3.44) and obtain

$$yf(x) - xf(y) = xy(x-y)h(x+y). \qquad (1.3.45)$$

Rewriting Equation (1.3.45) we get

$$\frac{f(x)}{x} - \frac{f(y)}{y} = (x-y)h(x+y). \qquad (1.3.46)$$

Now under the assumption that f is differentiable, h is continuous and thus, if we allow y to approach 0 on each side of Equation (1.3.46), we obtain

$$f^{t}(0) - \frac{f(x)}{x} = -xh(x).$$

Therefore, if we define:

$$q(x) = \begin{cases} \frac{f(x)}{x}, & \text{if } x = 0 \\ f^{t}(0), & \text{if } x = 0 \end{cases},$$

we have $f(x) = xq(x)$ for all x and $q(y) - q(x) = (y - x)h(x + y)$. By Theorem 1.3.1, we obtain $q(x) = ax^2 + bx + c$ so that $f(x) = ax^3 + bx^2 + cx$. Removing the assumption that $f(0) = 0$ we have $f(x) = ax^3 + bx^2 + cx + d$, as asserted in the theorem. □

In 1992 and without being aware of the result of Gestici and Neogu [16], Bailey [15] posed the question whether every continuous (or differentiable) f satisfying the functional equation

$$f[x_1, x_2, ..., x_n] = g(x_1 + x_2 + ... + x_n) \tag{1.3.47}$$

is a polynomial of degree at most n. Using some elementary techniques, Kannappan and Sahoo [17] have solved Bailey's problem. First, we solve Bailey's problem for $n = 3$ and then in the next theorem we present the solution of Equation (1.3.47).

Theorem 1.3.6. *Let f satisfy the functional equation*

$$f[x_1, x_2, x_3] = g(x_1 + x_2 + x_3) \tag{1.3.48}$$

for all $x_1, x_2, x_3 \in$ R with $x_1 = x_2, x_2 = x_3$ and $x_3 = x_1$. Then f is a polynomial of degree at most three and g is linear.

Proof. If $f(x)$ is a solution of Equation (1.3.48) so also $f(x) + a_0 + a_1 x$. Hence we may assume without loss of generality that

$$f(0) = 0 \tag{1.3.49}$$

and

$$\boldsymbol{f}(\alpha) = 0, \tag{1.3.50}$$

for some $\alpha = 0$ ($\alpha \in \mathrm{R}$). Note that there are many choices for such an α. First substitute $(x, 0, \alpha)$ for (x_1, x_2, x_3) in Equation (1.3.48) to get $\boldsymbol{f}[x, 0, \alpha] = g(x, 0, \alpha)$. Using the definition of divided difference, Equation (1.3.49) and Equation (1.3.50) for $x = 0$ and $x = \alpha$, we get

$$\boldsymbol{f}(x)(0 - \alpha) + \boldsymbol{f}(0)(\alpha - x) + \boldsymbol{f}(\alpha)(x - 0) = (x - 0)(0 - \alpha)(x - \alpha)g(x + 0 + \alpha),$$

which can be written as

$$\boldsymbol{f}(x) = -x(\alpha - x)g(x + \alpha). \tag{1.3.51}$$

Next, we substitute $(x, 0, y)$ for (x_1, x_2, x_3) in Equation (1.3.48) for all $x, y = 0$ and $x = y$, to get

$$\frac{\boldsymbol{f}(x)}{x(x - y)} - \frac{\boldsymbol{f}(y)}{y(x - y)} = g(x + y). \tag{1.3.52}$$

Define $q(x) = \frac{f(x)}{x}$ for $x \in \mathrm{R}^*$. Equation (1.3.52) reduces to

$$q(x) - q(y) = (x - y)g(x + y), \tag{1.3.53}$$

for all $x, y \in \mathrm{R}^*$ with $x = y$. Note that Equation (1.3.53) is valid even for

$x = y$. Putting $y = -x$ in Equation (1.3.53) to get

$$q(x) - q(-x) = 2xg(0), \tag{1.3.54}$$

for all $x = 0$. Next we replace y by $-y$ in Equation (1.3.53) to get

$$q(x) - q(-y) = (x + y)g(x - y), \tag{1.3.55}$$

with $x, y \in R^*$ and $x + y = 0$. Thus, we conclude that Equation (1.3.55) holds for $x, y \in R^*$. Subtract Equation (1.3.53) from Equation (1.3.55) to get

$$q(y) - q(-y) = (x + y)g(x - y) - (x - y)g(x + y),$$

for all $x, y \in R^*$. Using Equation (1.3.54) we obtain, for all $x, y \in R^*$,

$$\begin{aligned} & 2yg(0) = (x + y)g(x - y) - (x - y)g(x + y) \\ \Longrightarrow\ & (x - y)g(x + y) + yg(0) = (x + y)g(x - y) - yg(0) \\ \Longrightarrow\ & (x - y)g(x + y) + yg(0) - xg(0) = (x + y)g(x - y) - yg(0) - xg(0), \end{aligned}$$

that is

$$(x + y)[g(x - y) - g(0)] = (x - y)[g(x + y) - g(0)]. \tag{1.3.56}$$

Fix a nonzero $u \in R$. Choose a $v \in R$ such that $\frac{u+v}{2} = 0$ and $\frac{u-v}{2} = 0$. There are plenty of choices for such v. Let $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$ so that

$$u = x + y \quad \text{and} \quad v = x - y. \tag{1.3.57}$$

Letting Equation (1.3.57) into Equation (1.3.56), we get

$$u[g(v) - g(0)] = v[g(u) - g(0)],$$

for all $v = u, -u$ (here we note that v can be zero since $x = y$ is allowed). Hence for fixed $u = u_1$, we get

$$g(v) = a_1 v + b_1,$$

for $v \in \mathbb{R} \setminus \{u_1, -u_1\}$. Again for $u = u_2$, we get

$$g(v) = a_2 v + b_2,$$

for $v \in \mathbb{R} \setminus \{u_2, -u_2\}$. Since the sets $\{u_1, -u_1\}$ and $\{u_2, -u_2\}$ are disjoint, we get

$$g(v) = av + b, \tag{1.3.58}$$

for all $v \in \mathbb{R}$. Now using Equation (1.3.58) in Equation (1.3.51), we get

$$f(x) = (x^2 - x\alpha)g(x + \alpha) = (x^2 - x\alpha)[a(x + \alpha) + b] = ax^3 + bx^2 + cx,$$

where $c = -a\alpha^2 - b\alpha$. Removing the assumption that $f(0) = 0$, we get

$$f(x) = ax^3 + bx^2 + cx + d, \tag{1.3.59}$$

for all $x = 0, \alpha$. By Equation (1.3.49), Equation (1.3.50) and Equation (1.3.59), we conclude that f is a polynomial of degree at most three for all $x \in \mathbb{R}$.

□

Next we find the solution of Equation (1.3.47) without any assumptions on the unknown functions f and g. The following lemma is needed to solve Bailey's problem.

Lemma 1.3.1. *Let S be a finite subset of $\mathbb{R}$ symmetric about zero (that is, $-S = S$) and let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions satisfying the functional equation*

$$f(x) - f(y) = (x - y)g(x + y), \tag{1.3.60}$$

for all $x, y \in \mathbb{R} \setminus S$, then

$$f(x) = ax^2 + bx + c \quad \text{and} \quad g(y) = ay + b, \tag{1.3.61}$$

for all $x \in \mathbb{R} \setminus S$ and $y \in \mathbb{R}$, where a, b, c are some constants.

Proof. Replacing y by $-x$ in Equation (1.3.60), we obtain

$$f(x) - f(-x) = 2xg(0), \tag{1.3.62}$$

for $x \in \mathbb{R} \setminus S$. Again replacing y by $-y$ in Equation (1.3.60) we get

$$f(x) - f(-y) = (x + y)g(x - y),$$

for $x, y \in \mathbb{R} \setminus S$, which after subtracting from Equation (1.3.60) and Equation (1.3.62) gives

$$(x + y)(g(x - y) - g(0)) = (x - y)(g(x + y) - g(0)), \qquad (1.3.63)$$

for all $x, y \in \mathbb{R} \setminus S$. Fix a nonzero $u \in \mathbb{R}$. Let $v \in \mathbb{R}$ such that $\frac{(u \pm v)}{2} \notin S$ and put $x = \frac{(u+v)}{2}$ and $y = \frac{(u-v)}{2}$. Then $x + y = u$ and $x - y = v$ and use Equation (1.3.63) to get

$$u \quad g(v) - g(0) \quad = v \quad g(u) - g(0) \quad , \qquad (1.3.64)$$

for all $v \in \mathbb{R} \setminus (2S \pm u)$, where $2S \pm u$ denotes the set

$$\{2s + u, \ s \in S\} \cup \{2s - u, \ s \in S\}.$$

For each fixed u, Equation (1.3.64) shows that g is linear in v, that is of the form $av + b$, except on the finite set $2S \pm U$. To conclude that g is linear on the reals, one has to note that, if one takes two suitable different values of u, which is now treated as a parameter, the exceptional sets involved are disjoint and so $g(v) = av + b$ for all real v with the same constants everywhere.

Substituting this for g in Equation (1.3.60), we obtain

$$f(x) - ax^2 - bx = f(y) - ay^2 - by \tag{1.3.65}$$

for all $x, y \in \mathbb{R} \setminus S$. Choosing any $y \in \mathbb{R} \setminus S$ in Equation (1.3.65) yields that $f(x) = ax^2 + bx + c$ for $x \in \mathbb{R} \setminus S$, for some constant c, which is the required form of f in Equation (1.3.61). □

The following theorem addresses the problem posed by Bailey [15] in 1992. We point out that Schwaiger [18] has also established this theorem independently.

Theorem 1.3.7. *[17] Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying the functional equation*

$$f[x_1, x_2, ..., x_n] = g(x_1 + x_2 + ... + x_n) \tag{1.3.66}$$

for distinct $x_1, x_2, ..., x_n$, that is, for $x_i \neq x_j$ ($i \neq j$, $i, j = 1, 2, ..., n$). Then f is a polynomial of degree at most n and g is linear, that is, a polynomial of first degree.

Proof. It is easy to see that if f is a solution of Equation (1.3.66) so also $f(x) + \sum_{k=0}^{n-2} a_k x^k$. So, we can assume that $f(0) = f(y_1) = ... = f(y_{n-2}) = 0$ for $y_1, y_2, ..., y_{n-2}$ distinct and different from zero. Obviously there are plenty of choices for $0, y_1, ..., y_{n-2}$. Replacing in Equation (1.3.66) $(x_1, x_2, ..., x_n)$ by $(x, 0, y_1, .., y_{n-2})$ and then by $(x, 0, y_1, .., y_{n-3})$, we get

$$f(x) = -x(y_1 - x)...(y_{n-2} - x)g\left(x + \sum_{k=1}^{n-2} y_k\right) \tag{1.3.67}$$

and

$$\frac{f(x)}{x(x-y)(y_1-x)\dots(y_{n-3}-x)} - \frac{f(yx)}{y(x-y)(y_1-y)\dots(y_{n-3}-y)}$$
$$= g\left(x+y+\sum_{k=1}^{n-3} y_k\right)$$

respectively for $x = y$ and $x = 0, y, y_1, \dots, y_{n-2}$. Now the above equation can be rewritten as

$$l(x) - l(y) = (x-y)g\left(x+y+\sum_{k=1}^{n-3} y_k\right),$$

where $l(x) = \frac{f(x)}{x(y_1-x)\dots(y_{n-3}-x)}$ for $x, y = 0, y_1, \dots y_{n-3}$. Then by Lemma 1.3.1 and the arbitrary choice of $x, y = 0, y_1, \dots y_{n-3}$, we get that g is linear (and $l(x)$ is quadratic). Hence by Equation (1.3.67), f is a polynomial of degree at most n. This proves the theorem. □

Remark 1.3.4. *The functional equation*

$$f[x, y] = h(c(x, y)) \tag{1.3.68}$$

has been studied by taking c to be geometric mean and harmonic mean of x and y (see [19]). Further the functional equation (1.3.68) was treated in [20] assuming $c(x, y)$ to be a quasiarithmatic mean.

1.4 The Lagrange's MVT for Divided Differences

In this section we prove the Lagrange's MVT for divided differences and then present some applications toward the study of means. We begin this section with an integral representation of divided differences. Results of this section can be found in [8, 21, 22].

Theorem 1.4.1. *Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ has a continuous n^{th} derivative in the interval $\min\{x_0, x_1, ..., x_n\} \leq x \leq \max\{x_0, x_1, ..., x_n\}$. If the points $x_0, x_1, ..., x_n$ are all distinct, then*

$$\begin{aligned} \int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} & f^{(n)}\left(x_0 + \sum_{k=1}^{n} t_k(x_k - x_{k-1})\right) dt_n \\ &= f[x_0, x_1, ..., x_n], \end{aligned} \tag{1.4.1}$$

where $n \geq 1$.

Proof. We prove this theorem by induction. If $n = 1$, the representation given in Equation (1.4.1) reduces to

$$f[x_0, x_1] = \int_0^1 f'(t_1(x_1 - x_0) + x_0)dt_1.$$

Since $x_1 \neq x_0$, introducing a new variable z for $t_1(x_1 - x_0) + x_0$, we get

$dz = (x_1 - x_0)dt_1$. Hence we have

$$\begin{aligned}
\int_0^1 f'\left(t_1(x_1 - x_0) + x_0\right) dt_1 &= \int_{x_0}^{x_1} f'(z)\frac{dz}{x_1 - x_0} \\
&= \frac{\int_{x_0}^{x_1} f'(z)dz}{x_1 - x_0} \\
&= \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f[x_0, x_1].
\end{aligned}$$

Next we assume that the integral representation in Equation (1.4.1) holds for $n - 1$, that is

$$\int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-2}} f^{(n-1)}\left(x_0 + \sum_{k=1}^{n-1} t_k(x_k - x_{k-1})\right) dt_{n-1} = f[x_0, x_1, \cdots, x_{n-1}].$$

We will show that Equation (1.4.1) holds for the integer n. Let

$$w = t_n(x_n - x_{n-1}) + t_{n-1}(x_{n-1} - x_{n-2}) + \cdots + t_1(x_1 - x_0) + x_0$$

to be the new variable. Hence $dt_n = \frac{dw}{x_n - x_{n-1}}$ for $x_n = x_{n-1}$. If $t_n = 0$, the $w = w_0$, where $w_0 = t_{n-1}(x_{n-1} - x_{n-2}) + \cdots + t_1(x_1 - x_0) + x_0$. Similarly, if $t_n = t_{n-1}$, then $w = w_1$, where $w_1 = t_{n-1}(x_{n-1} - x_{n-2}) + \cdots + t_1(x_1 - x_0) + x_0$. Now applying the induction hypothesis, we have

$$\int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} f^{(n)}\left(x_0 + \sum_{k=1}^{n} t_k(x_k - x_{k-1})\right) dt_n$$

$$= \int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-2}} \frac{f^{(n-1)}(w_1) - f^{(n-1)}(w_0)}{x_n - x_{n-1}} dt_{n-1}$$

$$= \frac{f[x_0, x_1, \cdots, x_{n-2}, x_n] - f[x_0, x_1, \cdots, x_{n-2}, x_{n-1}]}{x_n - x_{n-1}}$$

$$= f[x_0, x_1, \cdots, x_n].$$

This completes the proof of the theorem. □

From the above integral representation, we see that the integrand is a continuous function of the variables $x_0, x_1, \cdots, x_n$, and therefore the left side, $f[x_0, x_1, \cdots, x_n]$. For example, if $n = 1$, then the continuous extension of $f[x_0, x_1]$ is

$$f[x_0, x_1] = \begin{cases} \frac{f(x_0) - f(x_1)}{x_0 - x_1} & \text{if } x_1 = x_0, \\ f'(x_0) & \text{if } x_1 = x_0, \end{cases}$$

provided $f(x)$ has the first derivative. Because of this unique extension now we can allow some of the nodes, that is $x_0, x_1, \cdots, x_n$ to coalesce if f is suitably differentiable. Now we present the mean value theorem for divided differences.

Theorem 1.4.2. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a real valued function with continuous n^{th} derivative and $x_0, x_1, ..., x_n$ in $[a, b]$. Then there exists a point c in the following interval $[\min\{x_0, x_1, ..., x_n\}, \max\{x_0, x_1, ..., x_n\}]$ such that $f[x_0, x_1, ..., x_n] = \frac{f^{(n)}(c)}{n!}$.*

Proof. Since $f^{(n)}(x)$ is continuous on $[a, b]$, the function $f^{(n)}(x)$ has a maximum and a minimum on $[a, b]$. Let $m = \min f^{(n)}(x)$ and $M = \max f^{(n)}(x)$. Then from the integral representation of $f[x_0, x_1, ..., x_n]$, we have

$$m \prod_{k=1}^{n} \int_0^{t_{k-1}} dt_k \leq f[x_0, x_1, ..., x_n] \leq M \prod_{k=1}^{n} \int_0^{t_{k-1}} dt_k,$$

where $t_0 = 1$. Using the fact that

$$\prod_{k=1}^{n} \int_0^{t_{k-1}} dt_k = \int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n = \frac{1}{n!},$$

we obtain from the above inequalities

$$m \leq f[x_0, x_1, ..., x_n](n!) \leq M.$$

Since $f^{(n)}(x)$ in continuous, by applying the intermediate value theorem to it, we have

$$f[x_0, x_1, ..., x_n](n!) = f^{(n)}(c),$$

for some $c \in (\min\{x_0, x_1, \cdots, x_n\}, \max\{x_0, x_1, \cdots, x_n\})$. This yields the asserted result $f[x_0, x_1, ..., x_n] = \frac{f^{(n)}(c)}{n!}$ and the proof of this theorem is now complete. □

The mean value theorem for divided differences can be used for defining the functional means. We have seen from our discussion that some of the nodes in the divided differences can coalesce if f is suitably differentiable. For example, if f is differentiable, then

$$f[b, b, a, a] = \frac{f'(b) - 2f[b, a] + f'(a)}{(b - a)^2}$$

To see this consider,

$$
\begin{aligned}
f[b,b,a,a] &= \frac{f[b,b,a]-f[b,a,a]}{b-a} \\
&= \frac{1}{b-a}\; f[b,b,a]-f[b,a,a] \\
&= \frac{1}{b-a}\; \frac{f[b,b]-f[b,a]}{b-a} - \frac{f[b,a]-f[a,a]}{b-a} \\
&= \frac{1}{(b-a)^2}\; f[b,b]-2f[b,a]+f[a,a] \\
&= \frac{f'(b)-2f[b,a]+f'(a)}{(b-a)^2}.
\end{aligned}
$$

1.5 Topological and Weak Topological Mean Value Theorems

In her paper [23], I. Rosenholz has proved, as a consequence of the Jordan curve theorem, the following interesting mean value theorem for the plane.

Theorem 1.5.1. *[23] Let us assume that $\alpha : [a,b] \longrightarrow \mathbb{R}^2$ is a differentiable non-stop arc in the plane (i.e., α is assumed to be injective and to have non zero derivative $\alpha'(t)$ for all $t \in (a,b)$). Then there exists a* **positive number** *M and $t_0 \in (a,b)$ such that $\alpha'(t_0) = M(\alpha(b) - \alpha(a))$.*

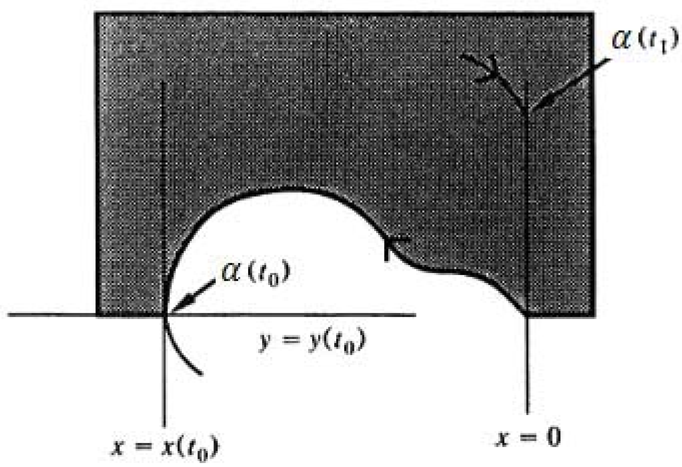
Proof. First note that we can always consider that $\alpha(a) = (0,0)$ and $\alpha(b) = (0,1)$. Otherwise, consider $\gamma(t) = (0,1)\frac{\alpha(t)-\alpha(a)}{\alpha(b)-\alpha(a)}$. Then γ is also differentiable nonstop arc in the plane and we have $\gamma(a) = (0,0)$ and $\gamma(b) = (0,1)$. Let t_1 in $[a,b]$ be chosen so that $\alpha(t_1)$ lies on the y-axis and is the most northerly such point. (It is possible that t_1 equals b, which is a situation we shall have to address-but certainly t_1 does not equal a.) There are two possibilities: Case

1: There is an increasing sequence $(s_n)_n$ such that $\lim s_n = t_1$ and $\alpha(s_n)$ lies on the y-axis for all n; and Case 2: There is no such a sequence.

Case 1: We may assume in this case that t_1 is less than b. (If not, choose a c less than b so that $\alpha(c)$ is due north of $\alpha(a)$, and choose a new t_1 for the path α restricted to the interval $[a, c]$. Note that with this new choice of t_1, we might now be in Case 2.) So α is differentiable at t_1. Then $\alpha^t(t_1) = \lim \frac{\alpha(s_n)-\alpha(t_1)}{s_n-t_1}$. But since $\alpha(s_n)$ lies on the y-axis below $\alpha(t_1)$, and $s_n < t_1$, we may conclude that $\alpha^t(t_1)$ is of the form $(0, y^t(t_1))$ with $y^t(t_1) \geq 0$. Finally, since $\alpha^t(t_1) = (0, 0)$ by hypothesis, $y^t(t_1)$ must be greater than 0, and thus, at time t_1, we have $\alpha^t(t_1) = y^t(t_1)(\alpha(b) - \alpha(a))$.

Case 2: In this case, there is a $t_2 < t_1$ such that $\alpha(t_2)$ lies on the y-axis (below $\alpha(t_1)$, of course) and $\alpha(t)$ does not lie on the y-axis for all t between t_2 and t_1. Then for $t_2 < t < t_1$, α lies entirely to the left of the y-axis or entirely to the right. Without loss of generality assume the former, and concentrate on the restriction of α to the interval $[t_2, t_1]$. Since α is continuous on this interval, there is a t_0 such that $t_2 < t_0 < t_1$ and $\alpha(t_0)$ has minimal x-coordinate. Then $x^t(t_0) = 0$. Therefore, since $\alpha^t(t_0) = (0, 0)$, $\alpha^t(t_0)$ points due north or due south. If $\alpha^t(t_0)$ points north, we are done. So assume, to arrive at a contradiction, that $\alpha^t(t_0)$ points south.

Now $\alpha([t_2, t_1])$ lies in the vertical strip $\{(x, y), x(t_0) \leq x \leq 0\}$. Furthermore, since $\alpha^t(t_0)$ points due south, then for all $t < t_0$ (resp., $t > t_0$) sufficiently close to t_0, $\alpha(t)$ lies above (resp., below) the line $y = y(t_0)$, and as close to vertical as you please. But $\alpha([t_2, t_0])$ separates the vertical strip, and in fact separates points nearly due south of $\alpha(t_0)$ from $\alpha(t_1)$. One easy way to see this is to extend $\alpha([t_2, t_0])$ to a simple closed curve by going due west from $\alpha(t_0)$, then going due north far enough to avoid the image of α, next going due east to get to the right of the y-axis, then proceeding south to the height of $\alpha(t_2)$, and finally due west to $\alpha(t_2)$ (see the below figure).



Points of $\alpha((t_0, t_1])$ almost due south of $\alpha(t_0)$ lie on the outside of this simple closed curve and $\alpha(t_1)$ lies on the inside of this simple closed curve. So by the Jordan Curve Theorem, $\alpha((t_0, t_1])$, which lies inside the strip, must cross this simple closed curve, necessarily at a point of $\alpha([t_2, t_0))$. This contradicts the hypothesis that α is an arc and, therefore, $\alpha^t(t_0)$ points due north, as claimed, completing the proof. □

This theorem contains as a particular case the classical Lagrange's MVT, which is usually stated for curves of the form $\alpha(t) = (t, f(t))$ and claims that if f is continuous on $[a, b]$ and differentiable on (a, b), there exists a point

$t_0 \in (a, b)$ such that the tangent line to the curve $\alpha(t) = (t, f(t))$ at $\alpha(t_0)$ is parallel to the vector joining the end points of the curve $(a, f(a))$ and $(b, f(b))$.

In her paper Rosenholtz named Theorem 1.5.1 **"Topological Mean Value Theorem"** because its proof is strongly based on the Jordan Curve Theorem for the plane, and also because the result has some interesting topological consequences. For example, she proved that if $\alpha : [a, b] \longrightarrow \mathbb{R}^2$ is a differentiable non-stop arc in the plane then the derivative directions

$$\frac{\alpha'(t)}{\|\alpha'(t)\|}, \; t \in [a, b]$$

form a connected subset of the unit circle [23, Corollary 3]. Note that we are not assuming that α is of class C^1.

On the other hand, it is very easy to prove a mean value type theorem which also holds for non Jordan curves. To be more precise, the following result holds:

Theorem 1.5.2. (Weak topological mean value theorem) *Let us assume that $\alpha : [a, b] \longrightarrow \mathbb{R}^2$ is a differentiable non-stop curve. Then there exists a number $M = 0$ and $t_0 \in (a, b)$ such that $\alpha'(t_0) = M(\alpha(b) - \alpha(a))$.*

Proof. There is no loss of generality if we assume that $[a, b] = [0, T]$, $\alpha(0) = (0, 0)$ and $\alpha(T) = (0, A)$. Then for each $t \in [0, T]$ the square of the distance from $\alpha(t)$ to the y-axis $\Lambda = \{(0, y) : y \in \mathbb{R}\}$ is given by $d(\alpha(t), \Lambda)^2 = \alpha_1^2(t)$ and, since $[0, T]$ is compact, $d(\alpha(t), \Lambda)^2$ attains its maximum at some point $t_0 \in [0, T]$. It follows that $\alpha_1'(t_0) = 0$ and hence, since α' vanishes nowhere, the theorem follows. $\square$

Remark 1.5.1. *If $\alpha(t)$ is of class at least C^1, then there is another nice proof of Theorem 1.5.2. Indeed, we may proceed as follows: we assume that $[a, b] = [0, T]$, $\alpha(0) = (0, 0)$, $\alpha(T) = (0, A)$ and that $\alpha = \alpha(s)$ is parametrized by the arc-length parameter. It follows that $\alpha'(s) = (\cos\theta(s), \sin\theta(s))$ for a certain continuous function $\theta : [0, T] \longrightarrow \mathbb{R}$, and*

$$\alpha(s) = \left(\int_0^s \cos\theta(r)dr, \int_0^s \sin\theta(r)dr \right), \quad \text{for all } s.$$

Here we have used that $\alpha(0) = (0, 0)$. Now $\alpha(T) = (0, A)$ implies

$$\int_0^T \cos\theta(r)dr = 0.$$

Hence there exists some $s_0 \in (0, T)$ such that $\cos\theta(s_0) = 0$. This ends the proof.

We say that Theorem 1.5.2 is weaker than the topological mean value theorem since if we restrict ourselves to consider Jordan arcs, it says nothing about the sign of M.

As we have already said, in her proof of Theorem 1.5.1, Rosenholtz used the Jordan Curve Theorem. However it is interesting to notice that on the sphere, where the Jordan Curve Theorem holds, the analogous of Theorem 1.5.1 is false.

Next, we will prove that Theorem 1.5.1 is only possible for plane curves. In fact, we will prove that Theorem 1.5.2 characterizes the planes of $\mathbb{R}^3$ in a certain sense. To do this we first introduce the following concept:

Definition 1.5.1. *Let $S \subset \mathbb{R}^3$ be a smooth surface of class at least C^1. We say that S satisfies the weak topological mean value theorem if for each differentiable non-stop curve $\alpha : [a, b] \longrightarrow S$, there exists a number $M = 0$ and a $t_0 \in (a, b)$ such that $\alpha^t(t) = M(\alpha(b) - \alpha(a))$.*

The main result in this section is the following:

Theorem 1.5.3. *Let us assume that $S \subset \mathbb{R}^3$ be a smooth surface of class at least C^1 which satisfies the weak topological mean value theorem. Then S is an open subset of a plane.*

Proof. Suppose that S is not an open subset of a plane. There is no loss of generality in assuming that S has a chart

$$x : D_\delta := \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 < \delta\} \longrightarrow S$$

of the form $x(u, v) = (u, v, \boldsymbol{f}(u, v))$, for a certain function $\boldsymbol{f} \in C^1(D_\delta)$, such that $\boldsymbol{f}(0, 0) = \boldsymbol{f}_u(0, 0) = \boldsymbol{f}_v(0, 0) = 0$, and $\boldsymbol{f}(a, b) = \boldsymbol{f}_0 = 0$ for a certain $(a, b) \in D_\delta$. If the curve $\alpha(t) = x(u(t), v(t))$ satisfies $\alpha(0) = (0, 0, 0)$, $\alpha(T) = (a, b, \boldsymbol{f}_0)$ and $\alpha^t(t_0)$ is parallel to $(a, b, \boldsymbol{f}_0)$ then the curve $\beta(t) = (u(t), v(t))$ satisfies $\beta(0) = (0, 0)$, $\beta(T) = (a, b)$ and $\beta^t(t_0)$ is parallel to (a, b). Furthermore if $\beta^t(t_0) = \lambda(a, b)$ for a certain constant $\lambda = 0$, then the identity $\alpha^t(t_0) = \lambda(a, b, \boldsymbol{f}_0)$ also holds. This obviously implies that

$$\boldsymbol{f}_u(\beta(t_0))a + \boldsymbol{f}_v(\beta(t_0))b = \boldsymbol{f}_0.$$

Now the function $h(u, v) = \boldsymbol{f}_u(u, v)a + \boldsymbol{f}_v(u, v)b$ is continuous on D_δ and vanishes at $(u, v) = (0, 0)$, so that there exists a point $p_0 = (u_0, v_0) \in D_\delta \setminus \{0, 0\}$ such that $h(p_0) = \boldsymbol{f}_0$. In fact, we can assume that

$$p_0 = c_1(a, b) + c_2(b, -a),$$

for some constants c_1 and c_2 in $(0, 1)$. We set $\beta(t) = t(a, b) + \rho(t)(b, -a)$ where $\rho : [0, 1] \longrightarrow [0, 1]$ is a smooth function such that $\rho(0) = p(1) = 0$, $\rho(c_1) = c_2$ and $t = c_1$ is the unique critical point of p. Then

$$\beta^t(t) = (a, b) + \rho^t(t)(b, -a)$$

is parallel to (a, b) only for $t = c_1$. We define $\alpha(t) = x(\beta(t))$. It is clear that α is differentiable of class at least C^1 and $\alpha^t(t) = (0, 0, 0)$ for all t. If $\alpha^t(t_0)$ is parallel to $(a, b, \boldsymbol{f}_0)$, then $\beta^t(t_0)$ is parallel to (a, b) and $\boldsymbol{f}_u(\beta(t_0))a+\boldsymbol{f}_v(\beta(t_0))b = \boldsymbol{f}_0$. But $\beta^t(t_0) = \lambda(a, b)$ implies $t_0 = c_1$. Hence $\beta(t_0) = p_0$ and

$$\boldsymbol{f}_u(\beta(t_0))a + \boldsymbol{f}_v(\beta(t_0))b = h(p_0) = \boldsymbol{f}_0,$$

a contradiction. □

Remark 1.5.2. *The previous results do not need, in any way, more regularity than C^1. In particular, no notion of curvature is necessary for the surface under consideration. In so far, Theorem 1.5.3 can also be regarded as a regularity result, since the conclusion gives, in particular, that the validity of the weak topological mean value theorem implies smoothness of the surface.*

1.6 Flett's Mean Value Theorem

In Section 1.1, we examined Lagrange's MVT which says that for every real-valued function $f : [a, b] \longrightarrow \mathbb{R}$, continuous on $[a, b]$ and differentiable on (a, b), there exists a $c \in (a, b)$ such that

$$f(b) - f(a) = (b - a)f'(c) \qquad (1.6.1)$$

The Lagrange's MVT was derived using Rolle's theorem which states that for every real-valued function $g : [a, b] \longrightarrow \mathbb{R}$ with $g(a) = g(b)$, continuous on $[a, b]$ and differentiable on the open interval (a, b), there exists a $c \in (a, b)$ such that

$$g'(c) = 0$$

If we replace f in the Lagrange's MVT by another function g defined as

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}x$$

then $g(a) = g(b)$ and the two results are now equivalent. Hence, from here after we shall not distinguish between Rolle's theorem and Lagrange's MVT, and consider them both as the MVT. Let us examine how the MVT is deduced from Rolle's Theorem. The usual proof of the MVT is to apply the Rolle's theorem to functions suitably designed to yield the desired result. Frequently, no mention is made of how these functions are discovered. Notice Equation

(1.6.1) can be represented as a determinant, namely

$$f(b) - f(a) - (b-a)f'(c) = \begin{vmatrix} 1 & f'(c) & 0 \\ b & f(b) & 1 \\ a & f(a) & 1 \end{vmatrix}.$$

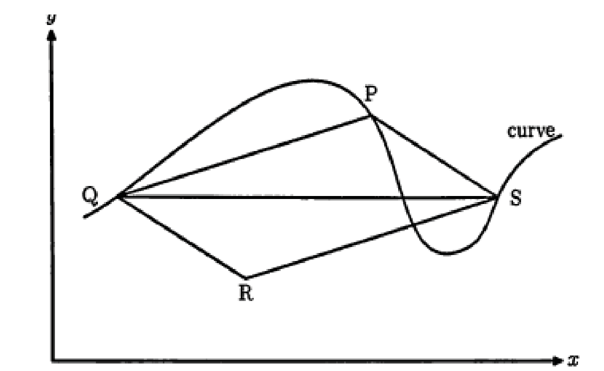
Similarly, the auxiliary function,

$$g(x) = f(x) - \frac{f(b) - f(a)}{b-a}(x-a)$$

used for deducing the MVT can also be represented as a determinant, namely

$$g(x) = -\frac{1}{b-a}\begin{vmatrix} x & f(x) & 1 \\ b & f(b) & 1 \\ a & f(a) & 1 \end{vmatrix}$$

It is easy to see that one may use the above determinant instead of g and can deduce the same result. From analytic geometry we know that the above determinant represents the area of the parallelogram PQRS (see the figure below).



A Geometrical Illustration of the Auxiliary Function

This interpretation is very insightful and allows one to generalize the MVT even further. For one such a generalization of the MVT see [24].

We present two consequences of the mean value theorem not included in the most textbooks. These two results appeared in [25].

Lemma 1.6.1. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) except possibly at finitely many points. Then there exists a point $c \in (a, b)$ such that*

$$|f(b) - f(a)| \leq (b - a)|f'(c)|$$

Proof. Assume that there is only one point $d \in (a, b)$ where the function f is not differentiable. By applying the MVT to f on $[a, d]$ and $[d, b]$ respectively, we obtain

$$f(d) - f(a) = (d - a)f'(c_1)$$

and

$$f(b) - f(d) = (b - d)f'(c_2)$$

for some $c_1 \in (a, d)$ and $c_2 \in (d, b)$. Adding the above two, we get

$$f(b) - f(a) = (d - a)f'(c_1) + (b - d)f'(c_2)$$

and from which we get:

$$
\begin{aligned}
|f(b) - f(a)| &\le (d-a)|f'(c_1)| + (b-a)|f'(c_2)| \\
&\le (d-a)|f'(c)| + (b-d)|f'(c)| \\
&= (b-a)|f'(c)|
\end{aligned}
$$

where $|f'(c)| = \max\{|f'(c_1)|, |f'(c_2)|\}$. The proof can obviously be extended to the case when f is not differentiable at more than one point. □

Lemma 1.6.2. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on the open interval (a, b) except possibly at a finite number, n, of points. Then there exist $n+1$ points $c_1, c_2, \cdots, c_{n+1} \in (a, b)$ and $n+1$ positive numbers $\alpha_1, \alpha_2, \cdots, \alpha_{n+1}$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_{n+1} = 1$ and*

$$f(b) - f(a) = (b-a)\sum_{i=1}^{n+1} \alpha_i f'(c_i)$$

Proof. Assume that there is only one point $d \in (a, b)$ where the function f is not differentiable. By applying the MVT to f on $[a, d]$ and $[d, b]$ respectively, we obtain

$$f(d) - f(a) = (d-a)f'(c_1)$$

and

$$f(b) - f(d) = (b-d)f'(c_2)$$

for some $c_1 \in (a, d)$ and $c_2 \in (d, b)$. Adding the above two, we get

$$f(b) - f(a) = (d-a)f'(c_1) + (b-d)f'(c_2).$$

Rewriting this we obtain

$$f(b) - f(a) = \left(\frac{d-a}{b-a} f'(c_1) + \frac{b-d}{b-a} f'(c_2)\right)(b-a),$$

which is

$$f(b) - f(a) = [\alpha_1 f'(c_1) + \alpha_2 f'(c_2)](b-a),$$

where $\alpha_1 = \frac{d-a}{b-a}$ and $\alpha_2 = \frac{b-d}{b-a}$. Clearly $\alpha_1 + \alpha_2 = 1$, $\alpha_1 > 0$ and $\alpha_2 > 0$. This proof can obviously be extended to the case when f is not differentiable at $n > 1$ points. □

In 1958, T. M. Flett [26] proved the following result which is a variant of Lagrange's MVT.

Theorem 1.6.1. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be differentiable on $[a, b]$ and $f'(a) = f'(b)$. Then there exists a point $c \in (a, b)$ such that:*

$$f(c) - f(a) = (c-a) f'(c).$$

Proof. Without loss of generality, we shall assume that $f'(a) = f'(b) = 0$. If this is not the case we work with $f(x) - xf'(a)$. Consider the function $g : [a, b] \longrightarrow \mathbb{R}$ defined by:

$$g(x) = \begin{cases} \frac{f(x)-f(a)}{x-a} & \text{if } x \in (a, b], \\ f'(a) & \text{if } x = a, \end{cases} \tag{1.6.2}$$

Evidently g is continuous on $[a, b]$ and differentiable on (a, b). Further, from

(1.6.2) we have

$$g^t(x) = -\frac{f(x) - f(a)}{(x - a)^2} + \frac{f^t(x)}{x - a},$$

which is

$$g^t(x) = -\frac{g(x)}{x - a} + \frac{f^t(x)}{x - a}, \qquad (1.6.3)$$

for all $x \in (a, b]$. In view of (1.6.2), to establish the theorem we have to show that there exists a point $c \in (a, b)$ such that $g^t(c) = 0$. From (1.6.2), we see that $g(a) = 0$. If $g(b) = 0$, then by Rolle's Theorem there exists a point $c \in (a, b)$ such that $g^t(c) = 0$ and the result is established. If $g(b) = 0$, then either $g(b) > 0$ or $g(b) < 0$. Suppose $g(b) > 0$, then from (1.6.3), we see that

$$g^t(b) = \frac{g(b)}{b - a} < 0.$$

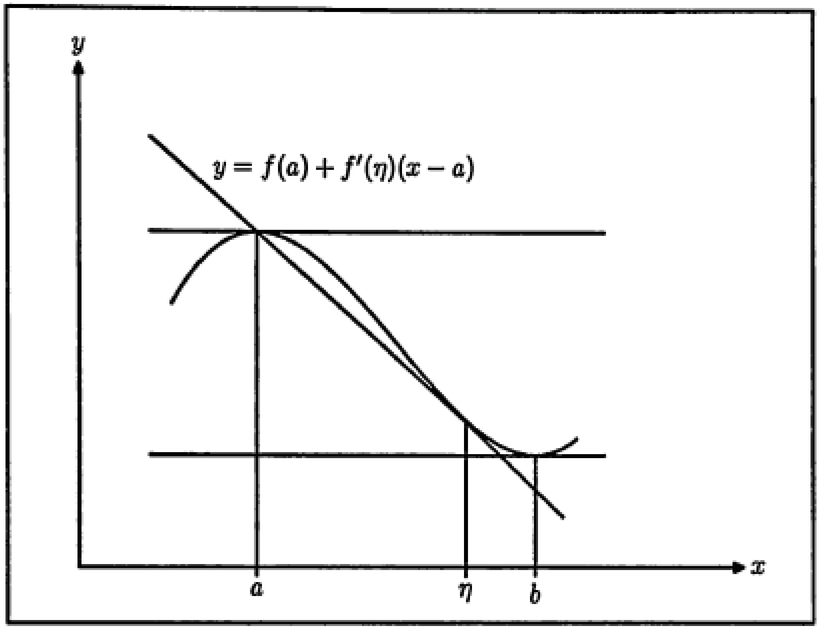
Since g is continuous and $g^t(b) < 0$, there exists a point $x_1 \in (a, b)$ such that $g(x_1) > g(b)$. Hence we have

$$g(a) < g(b) < g(x_1)$$

and by the Intermediate Value Theorem there exist a point $x_0 \in (a, x_1)$ such that $g(x_0) = g(b)$. Now applying the Rolle's Theorem to the function g on the interval $[x_0, b]$, we have $g^t(c) = 0$ for some $c \in (a, b)$. A similar argument applies if $g(b) < 0$. □

The geometrical interpretation of this theorem is the following. If the curve $y = f(x)$ has a continuity turning tangent in $a < x < b$, and if the tangents at

$x = a$ and $x = b$ are parallel, then there is an intermediate point c such that the tangent there passes through the point a. The figure below geometrically illustrates the Flett's MVT.



A Geometrical Illustration of Flett's Theorem

The following theorem removes the boundary assumption on the derivative of f, that is $f'(a) = f'(b)$.

Theorem 1.6.2. *If* $f : [a, b] \longrightarrow \mathbb{R}$ *is a differentiable function, then there exists a point* $c \in (a, b)$ *such that*

$$f(c) - f(a) = (c - a)f'(c) - \frac{1}{2}\frac{f'(b) - f'(a)}{b - a}(c - a)^2.$$

Proof. Defining an auxiliary function $\psi : [a, b] \longrightarrow \mathbb{R}$ as

$$\psi(x) = f(x) - \frac{1}{2}\frac{f^t(b) - f^t(a)}{b - a}(x - a)^2$$

we see that ψ is differentiable on $[a, b]$ and $\psi^t(x) = f^t(x) - \frac{f^t(b)-f^t(a)}{b-a}(x - a)$. From this, it is easy to check that $\psi^t(a) = f^t(a)$. Applying Flett's MVT to ψ, we get $\psi(c) - \psi(a) = (c - a)\psi^t(c)$ for some $c \in (a, b)$. Using the definition of the auxiliary function, we get the asserted result

$$f(c) - f(a) = (c - a)f^t(c) - \frac{1}{2}\frac{f^t(b) - f^t(a)}{b - a}(c - a)^2.$$

□

The auxiliary function ψ used in the above theorem is obtained by considering the difference between $f(x)$ and a quadratic approximation, $A + B(x - a) + C(x - a)^2$, of $f(x)$ and then imposing the boundary condition on the derivative of ψ, namely $\psi^t(a) = \psi^t(b)$. The boundary condition on the derivative of ψ yields $c = \frac{1}{2}\frac{f^t(b)-f^t(a)}{b-a}$. The constants A and B are arbitrary and we have chosen them to be zero for the sake of convenience. In the next theorem, we present the generalization of Flett's theorem due to D.H. Trahan [27]. To prove his result, we need two basic lemmas.

Lemma 1.6.3. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) and $[f(b) - f(a)]f^t(b) \leq 0$, then there exists a point $c \in (a, b]$ such that $f^t(c) = 0$.*

Proof. If $f(b) = f(a)$, then by Rolle's Theorem there exist a point $c \in (a, b)$ such that $f^t(c) = 0$. If $f^t(b) = 0$, then letting $c = b$ we have $f^t(c) = 0$.

Next suppose $[f(b) - f(a)]f'(b) < 0$. This implies that either $f'(b) < 0$ and $f(b) > f(a)$ or $f'(b) > 0$ and $f(b) < f(a)$. In the first case, since f is continuous on $[a, b]$ and $f(b) > f(a)$ with decreasing at b, the function f has a maximum at $c \in (a, b)$. Hence $f'(c) = 0$. Similarly, the second case f has a minimum at some point $c \in (a, b)$ and hence $f'(c) = 0$. □

The following lemma is obvious from the one above.

Lemma 1.6.4. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) and $[f(b) - f(a)]f'(b) < 0$, then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.*

Now we present a generalization of the Flett's MVT.

Theorem 1.6.3. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be differentiable on $[a, b]$ and*

$$\left(f'(b) - \frac{f'(b) - f'(a)}{b - a}\right)\left(f'(a) - \frac{f'(b) - f'(a)}{b - a}\right) \geq 0, \qquad (1.6.4)$$

then there exists a point $c \in (a, b]$ such that $f(c) - f(a) = (c - a)f'(c)$.

Proof. Let us define a function $h : [a, b] \longrightarrow \mathbb{R}$ by

$$h(x) = \begin{cases} \frac{f(x) - f(a)}{x - a}, & \text{if } x \in]a; b], \\ f'(a), & \text{if } x = 0, \end{cases}$$

then h is continuous on the interval $[a, b]$ and it is differentiable on $(a, b]$. Differentiating h, we get $h'(x) = -\frac{f(x) - f(a)}{(x - a)^2} + \frac{f'(x)}{x - a}$ for all $x \in (a, b]$. We have

$$[h(b) - h(a)]h'(b) = \frac{-1}{b - a}\left(f'(b) - \frac{f(b) - f(a)}{b - a}\right)\left(f'(a) - \frac{f(b) - f(a)}{b - a}\right).$$

By (1.6.4), we see that $[h(b) - h(a)]h'(b) \leq 0$. Hence by applying Lemma 1.6.4, we obtain $h'(c) = 0$ for some $c \in (a, b]$. Using the definition of h, we get $\boldsymbol{f}(c) - \boldsymbol{f}(a) = (c - a)\boldsymbol{f}'(c)$. □

Theorem 1.6.3 is a generalization of Theorem 1.6.1 since the latter can be deduced from the former. To see this, define h as in Theorem 1.6.3, that is:

$$h(x) = \begin{cases} \frac{\boldsymbol{f}(x)-\boldsymbol{f}(a)}{x-a}, & \text{if } x \in (a, b], \\ \boldsymbol{f}'(a), & \text{if } x = a \end{cases} \tag{1.6.5}$$

then h is continuous on the interval $[a, b]$ and it is differentiable on $(a, b]$. Differentiating h, we get $h'(x) = -\frac{\boldsymbol{f}(x)-\boldsymbol{f}(a)}{(x-a)^2} + \frac{\boldsymbol{f}'(x)}{x-a}$ for all $x \in a < x \leq b$. First consider the case when

$$\boldsymbol{f}(b) - \boldsymbol{f}(a) = (b - a)\boldsymbol{f}'(b). \tag{1.6.6}$$

Using (1.6.5) and (1.6.6), we have

$$\begin{aligned} h(b) - h(a) &= \frac{\boldsymbol{f}(b) - \boldsymbol{f}(a)}{b - a} - \boldsymbol{f}'(a) \\ &= \boldsymbol{f}'(b) - \boldsymbol{f}'(a) \\ &= 0 \end{aligned}$$

Hence we have $h(b) = h(a)$. Applying Rolle's Theorem to h, we get $h'(c) = 0$ for some $c \in (a, b]$. This yields $\boldsymbol{f}(c) - \boldsymbol{f}(a) - (c - a)\boldsymbol{f}'(c)$. Next, we consider the case

$$f(b) - f(a) = (b - a)f^t(b).$$

Hence, we have $f^t(b) - \frac{f(b)-f(a)}{b-a} > 0$ or $f^t(b) - \frac{f(b)-f(a)}{b-a} < 0$. Therefore using the fact that $f^t(b) = f^t(a)$, we get

$$\left(f^t(b) - \frac{f(b) - f(a)}{b - a}\right)\left(f^t(a) - \frac{f(b) - f(a)}{b - a}\right) > 0.$$

Thus using Theorem 1.6.3, we have Theorem 1.6.1. It can be shown $c = b$.

Chapter 2

Cauchy's Mean Value Theorem and its Associated Functional Equations

We start this chapter by proving the Cauchy's MVT and its generalization. Then, we focus on solving the associated functional equation, studied by Z. Balogh, O. Ibrogimov and B. Mityagin in [28]. In fact, they characterized all pairs of sufficiently smooth functions for which the mean value in the Cauchy's MVT is taken at a point which has a well-determined position in the interval. As an application of this result, a partial answer to a question posed by Sahoo and Riedel is obtained. We finish the chapter by establishing the Cauchy's MVT for divided differences.

2.1 The Cauchy's MVT and its Generalization

A. L. Cauchy [29] gave the following generalization of the Lagrange's MVT which now bears his name.

Theorem 2.1.1. *For all real-valued functions* f *and* g *differentiable on a real interval* I *and for all pairs* $x_1 = x_2$ *in* I, *there exists a point* c *depending on* x_1 *and* x_2 *such that*

$$[f(x_1) - f(x_2)]g^t(c) = [g(x_1) - g(x_2)]f^t(c). \tag{2.1.1}$$

Proof. Define the function h by

$$h(x) = [f(x_1) - f(x_2)]g(x) - [g(x_1) - g(x_2)]f(x),$$

for all $x \in I$. Then h is differentiable on I and furthermore, we have $h(x_1) = f(x_2)g(x_1) - g(x_2)f(x_1) = h(x_2)$. By Rolle's Theorem 1.1.1, there is a $c \in (x_1, x_2)$, such that

$$0 = h^t(c) = [f(x_1) - f(x_2)]g^t(c) - [g(x_1) - g(x_2)]f^t(c),$$

which is the desired result. □

Since the Cauchy's MVT involves two functions, it is natural to wonder if it can be extended to three or more functions. What formulas similar to Equation (2.1.1) can we have ? Next, we show and then extend the following result:

Theorem 2.1.2. *[30] Let α and β be two real numbers such that $\alpha + \beta = 1$. If $f(x)$, $g(x)$, $h(x)$ are three functions continuous on $[a, b]$ and differentiable on (a, b) such that $g(b) = g(a)$ and $h(b) = h(a)$, then there exists a point c in (a, b) such that*

$$f'(c) = \alpha g'(c)\frac{f(b) - f(a)}{g(b) - g(a)} + \beta h'(c)\frac{f(b) - f(a)}{h(b) - h(a)}. \qquad (2.1.2)$$

Observe that Equation (2.1.2) follows by letting $\gamma = -1$ and setting $f_1 = f$, $f_2 = g$ and $f_3 = h$ in Theorem 2.1.3.

Theorem 2.1.3. *Let α, β and γ be three real numbers such that $\alpha+\beta+\gamma = 0$. If f_1, f_2 and f_3 are three continuous on $[a, b]$ and differentiable on (a, b) such that $f_i(a) = f_i(b)$ for $i = 1, 2, 3$, then there exists a point c in (a, b) such that*

$$\frac{\gamma}{f_1(b) - f_1(a)}f_1'(c) + \frac{\alpha}{f_2(b) - f_2(a)}f_2'(c) + \frac{\beta}{f_3(b) - f_3(a)}f_3'(c) = 0 \qquad (2.1.3)$$

Proof. Consider the function $k(x)$ given by

$$\begin{aligned} k(x) \;=\; & \gamma\; f_2(b) - f_2(a) \quad f_3(b) - f_3(a) \quad f_1(x) - f_1(a) \\ & +\alpha\; f_1(b) - f_1(a) \quad f_3(b) - f_3(a) \quad f_2(x) - f_2(a) \\ & +\beta\; f_1(b) - f_1(a) \quad f_2(b) - f_2(a) \quad f_3(x) - f_3(a)\; . \end{aligned}$$

It is easily checked that $k(a) = 0$ and

$$k(b) = (f_1(b) - f_1(a))(f_2(b) - f_2(a))(f_3(b) - f_3(a))(\alpha + \beta + \gamma) = 0.$$

By Rolle's Theorem 1.1.1, there exists a point $c \in (a, b)$ such that $k'(c) = 0$.

Thus Equation (2.1.3) follows. □

For n functions we have the following generalization.

Theorem 2.1.4. *Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be n real numbers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 0$. If $f_1(x), f_2(x), \cdots, f_n(x)$ are n functions continuous on $[a, b]$ and differentiable on (a, b) such that $f_i(a) = f_i(b)$ for $i = 1, 2, \cdots, n$, then there exists a point c in (a, b) such that*

$$\sum_{j=1}^{n} \frac{\alpha_j}{f_j(b) - f_j(a)} f'_j(c) = 0. \tag{2.1.4}$$

Proof. Consider the function $k(x)$ given by

$$\begin{aligned} k(x) &= \alpha_1 \big(f_2(b) - f_2(a)\big) \cdots \big(f_n(b) - f_n(a)\big)\big(f_1(x) - f_1(a)\big) \\ &\quad + \alpha_2 \big(f_1(b) - f_1(a)\big)\big(f_2(x) - f_2(a)\big) \cdots \big(f_n(b) - f_n(a)\big) \\ &\quad + \cdots \\ &\quad + \alpha_n \big(f_1(b) - f_1(a)\big)\big(f_2(b) - f_2(a)\big) \cdots \big(f_n(x) - f_n(a)\big). \end{aligned}$$

It is easily checked that $k(a) = 0$ and

$$k(b) = (f_1(b) - f_1(a))(f_2(b) - f_2(a)) \cdots (f_n(b) - f_n(a))(\alpha_1 + \alpha_2 + \cdots + \alpha_n) = 0.$$

By Rolle's Theorem 1.1.1, there exists a point $c \in (a, b)$ such that $k'(c) = 0$. Thus Equation (2.1.4) follows. □

Theorem 2.1.4 can be useful in proving the existence of solutions of certain equations. The corollary below is a direct consequence of Theorem 2.1.4 for

$\alpha_1 = \frac{1-n}{n}$ and $\alpha_2 = \cdots = \alpha_n = \frac{1}{n}$.

Corollary 2.1.1. *If $f_1(x), f_2(x), \cdots, f_n(x)$ are n functions continuous on $[a, b]$ and differentiable on (a, b) and $f_i(a) = f_i(b)$ for $i = 1, 2, \cdots, n$, then the following equation has at least one solution in (a, b).*

$$\frac{1-n}{f_1(b) - f_1(a)} f_1^t(x) + \frac{1}{f_2(b) - f_2(a)} f_2^t(x) + \cdots + \frac{1}{f_n(b) - f_n(a)} f_n^t(x) = 0$$

Example 2.1.1. *Let*

$$F(x) = -3x + \frac{\pi}{2}\cos\frac{\pi x}{2} + \frac{e^x}{e-1} + \frac{1}{(x+1)\ln 2}.$$

The equation $F(x) = 0$ has at least one solution in $(0, 1)$ because

$$F(x) = -\frac{3}{2}\frac{(x^2)^t}{1^2 - 0^2} + \frac{\left(\sin(\frac{\pi}{2}x)\right)^t}{\sin\frac{\pi}{2}.1 - \sin\frac{\pi}{2}.0} + \frac{(e^x)^t}{e^1 - e^0} + \frac{[\ln(x+1)]^t}{\ln(1+1) - \ln(0+1)}$$

Remark 2.1.1. *In the above example, since $F(0) > 0$ and $F(1) > 0$, this conclusion is not an obvious consequence of the Intermediate Value Theorem for continuous functions.*

2.2 Functional Equations Associated to the Cauchy's MVT

Like the Lagrange's MVT, if one asks for what f and g the mean value c depends on x_1 and x_2 in a given manner, then Equation (2.1.1) appears as a functional equation. It can be formulated as follows:

Problem: for a given $c(x_1, x_2)$, find all pairs (f, g) of differentiable functions $f, g : \mathrm{R} \longrightarrow \mathrm{R}$ satisfying the following equation

$$[f(x_2) - f(x_1)]g'(c(x_1, x_2)) = [g(x_2) - g(x_1)]f'(c(x_1, x_2)) \qquad (2.2.1)$$

for all $x_1, x_2 \in \mathrm{R}$.

This functional equation was investigated by Aumann [31] by assuming

$$c(x_1, x_2) = \psi^{-1}\left(\frac{\psi(x_1) + \psi(x_2)}{2}\right) \qquad (2.2.2)$$

where ψ is a continuous and strictly monotonic function. The function c in Equation (2.2.2) is called a " quasiarithmetic mean " of x_1 and x_2. The recent contribution of Pàles [32] provides the solution of the Functional Equation (2.2.1) under additional assumptions. In this section, we provide a different approach to the Cauchy's MVT. In fact, we investigate the following problem [28]:

Problem 1: Find all pairs (F, G) of differentiable functions $F, G : \mathrm{R} \longrightarrow \mathrm{R}$ satisfying the following equation

$$[F(b) - F(a)]g(\alpha a + \beta b) = [G(b) - G(a)]f(\alpha a + \beta b) \qquad (2.2.3)$$

for all $a, b \in \mathrm{R}$, where $f = F'$, $g = G'$, $\alpha, \beta \in (0, 1)$ are fixed and $\alpha + \beta = 1$. Here, and in the rest of the chapter we will use "lower case" notations for the derivatives $f = F'$ and $g = G'$. As it will turn out, the most challenging situation corresponds to $c = \frac{a+b}{2}$ in which case the main result is the following:

Theorem 2.2.1. *Assume that $F, G : \mathbb{R} \longrightarrow \mathbb{R}$ are three times differentiable functions with derivative $F' = f$, $G' = g$ such that*

$$[F(b) - F(a)]g\left(\frac{a+b}{2}\right) = [G(b) - G(a)]f\left(\frac{a+b}{2}\right), \tag{2.2.4}$$

for all $a, b \in \mathbb{R}$. Then one of the following possibilities holds:

a. *$\{1, F, G\}$ are linearly dependent on $\mathbb{R}$.*

b. *$F, G \in \operatorname{span}\{1, x, x^2\}$, $x \in \mathbb{R}$.*

c. *There exists a non-zero real number μ such that*

$$F, G \in \operatorname{span}\{1, e^{\mu x}, e^{-\mu x}\}, x \in \mathbb{R}.$$

d. *There exists a non-zero real number μ such that*

$$F, G \in \operatorname{span}\{1, \sin(\mu x), \cos(\mu x)\}, x \in \mathbb{R}.$$

Proof of Theorem 2.2.1 will be done in several steps.

2.2.1 Case 1: The Cauchy's MVT with Fixed Mean Value

Let us introduce the sets

$$U_f := \{x \in \mathbb{R} : f(x) = 0\}, \quad U_g := \{x \in \mathbb{R} : g(x) = 0\}, \tag{2.2.5}$$

and also their complements $Z_f := \mathbb{R} \setminus U_g$ and $Z_g := \mathbb{R} \setminus U_f$. Observe that if U_g is empty, i.e. G is constant on $\mathbb{R}$, then Equation (2.2.3) holds for trivial reasons (both sides are identically zero) for an arbitrary differentiable function F. Of course, we can change the roles of G and F and claim: if F is constant then Equation (2.2.3) holds for any differentiable function G. Assume therefore that $U_g = 0$. Then there is a sequence of mutually disjoint open intervals $\{I_\sigma\}_{\sigma\in\Sigma}$, $\Sigma \subset \mathbb{N}$, such that

$$U_g = \bigcup_{\sigma\in\Sigma} I_\sigma \tag{2.2.6}$$

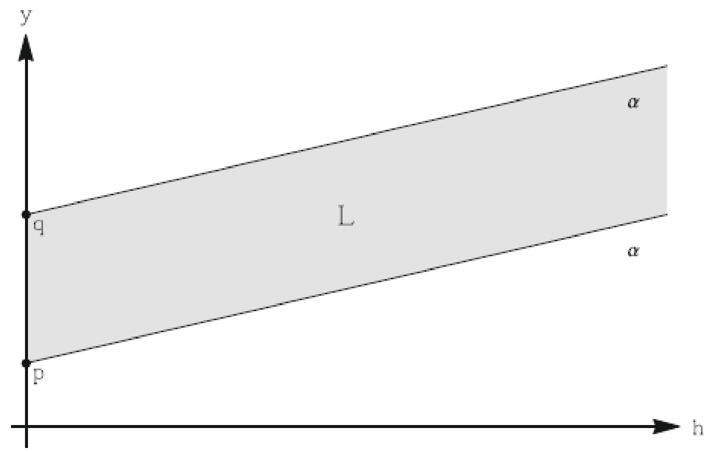
Proposition 2.2.1. *If $U_g = \emptyset$ and $U_f \cap U_g = \emptyset$, then $U_f = \emptyset$ i.e. $f \equiv 0$ on $\mathbb{R}$ and thus F is constant.*

Proof. By assumption, there is a non-empty interval $(p, q) \subset U_g$ such that $g(x) = 0$ on (p, q), but $f(x) = 0$ for all $x \in [p, q]$. Then with the changing of variables $h = b - a$, $x = \alpha a + \beta b$, Equation (2.2.3) yields

$$F(x + \alpha h) - F(x - \beta h) = 0, \tag{2.2.7}$$

for all $x \in (p, q)$, $h > 0$. Denoting $x + \alpha h$ by y for $x \in [p, q]$ and $h > 0$ we get $F(y) - F(y - h) = 0$ if (h, y) lies within the semi-strip

$$L := \{(h, y) : h > 0, p + \alpha h < y < q + \alpha h\}.$$



The semi-strip L

Then for $y > p$ choosing $h > 0$ such that $(h, y) \in L$, we have

$$\frac{\partial}{\partial y}F(y) = \frac{\partial}{\partial y}F(y-h) = -\frac{\partial}{\partial y}F(y-h) = -\frac{\partial}{\partial y}F(y) = 0,$$

so $F(y)$ is a constant, say $F(y) = F(\frac{p+q}{2})$ for $y > p$. However by Equation (2.2.7), we have $F(q + \alpha h) = F(q - \beta h)$ and thus $F(y)$ is the same constant for all $y < q$. Therefore, $f(y) = F'(y) = 0$ for all $y \in \mathbb{R}$. □

Proposition 2.2.1 shows that the condition $U_f \cap U_g = \emptyset$ holds only if at least one of the sets U_f and U_g is empty. Then we have the simple cases described at the beginning of this subsection.

Proposition 2.2.2. *Let* (F, G) *be a solution of the Problem 1 satisfying*

$$U_f \cap U_g = \emptyset, \tag{2.2.8}$$

and consider the representation (2.2.6). If $\{F, G, 1\}$ *are linearly dependent as* *functions on* $\boldsymbol{I}_\sigma$ *for every* $\sigma \in \Sigma$, *then* $\{F, G, 1\}$ *are linearly dependent on* R.

Proof. For $\sigma_1, \sigma_2 \in \Sigma$ with $\sigma_1 = \sigma_2$, consider the intervals

$$\boldsymbol{I}_{\sigma_1} := (p_1, q_1), \ \boldsymbol{I}_{\sigma_2} := (p_2, q_2) \tag{2.2.9}$$

with $p_1 < q_1 \leq p_2 < q_2$ and assume that $\{F, G, 1\}$ are linearly dependent on $\boldsymbol{I}_{\sigma_1}$ and $\boldsymbol{I}_{\sigma_2}$. Then it follows that there are constants $A_1, A_2, B_1, B_2 \in$ R such that

$$F(x) = A_1 G(x) + B_1, x \in \boldsymbol{I}_{\sigma_1} \tag{2.2.10}$$

$$= A_2 G(x) + B_2, x \in \boldsymbol{I}_{\sigma_2}. \tag{2.2.11}$$

With the changing of variables $h = b - a$, $x = \alpha a + \beta b$, Equation (2.2.3) yields

$$[F(x + \alpha h) - F(x - \beta h)]g(x) = [G(x + \alpha h) - G(x - \beta h)]f(x),$$

for all $x \in$ R and $h > 0$. Since $f(x) = A_2 g(x)$ if $x \in \boldsymbol{I}_{\sigma_2}$ by Equation (2.2.11) and $g(x) = 0$ for $x \in \boldsymbol{I}_{\sigma_2}$, we have for all $x \in \boldsymbol{I}_{\sigma_2}$, $h > 0$

$$F(x + \alpha h) - F(x - \beta h) = A_2[G(x + \alpha h) - G(x - \beta h)]. \tag{2.2.12}$$

If at the same time $x - \beta h \in \boldsymbol{I}_{\sigma_1}$, then $F(x - \beta h) = A_1 G(x - \beta h) + B_1$ by Equation (2.2.10). Inserting this value into Equation (2.2.12), we obtain

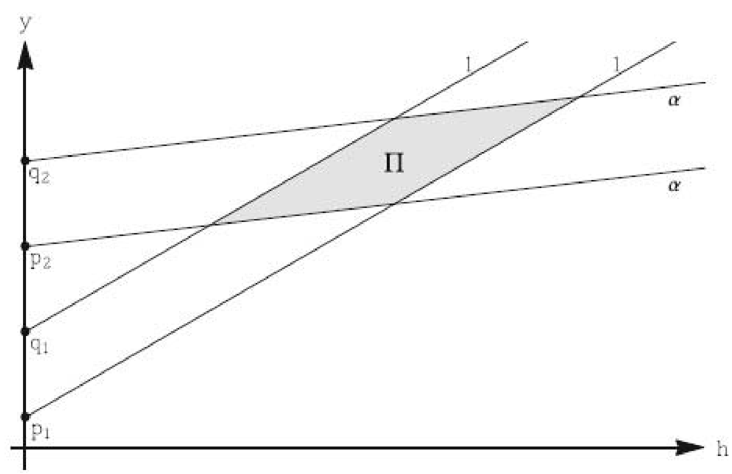
$$F(x + \alpha h) = A_2 G(x + \alpha h) + (A_1 - A_2)G(x - \beta h) + B_1, \tag{2.2.13}$$

for

$$x \in I_{\sigma_2}, \;\; x - \beta h \in I_{\sigma_1}, \;\; h > 0. \qquad (2.2.14)$$

Put $y = x + \alpha h$, then $x - \beta h = y - h$ and Equation (2.2.14) means that (h, y) lies within the parallelogram Π.

$$\Pi := \{(h, y), p_2 + \alpha h < y < q_2 + \alpha h, p_1 + h < y < q_1 + h\}.$$



The parallelogram Π

Since $\beta \in (0, 1)$, Equation (2.2.9) guarantees that $\Pi = \emptyset$, and Equation (2.2.13) implies

$$F(y) = A_2 G(y) + (A_1 - A_2) G(y - h) + B_1 \quad \text{for all} \quad (h, y) \in \Pi.$$

Therefore, at any point of Π, we have

$$0 = \frac{\partial}{\partial y} F(y) = -(A_1 - A_2)G'(y-h) = (A_2 - A_1)g(y-h).$$

But $y - h \in I_{\sigma_1}$ by Equation (2.2.14) , so $g(y - h) = 0$ and thus

$$A_2 - A_1 = 0. \tag{2.2.15}$$

So far our analysis says nothing about B_1, B_2 in Equation (2.2.10) and Equation (2.2.11) but since $\sigma_1, \sigma_2 \in \Sigma$ were arbitrary, Equation (2.2.15) together with Equation (2.2.10) and Equation (2.2.11) imply

$$f(x) = Ag(x) \quad \text{for some constant} \quad A \in \mathrm{R} \quad \text{and all} \quad x \in U_g. \tag{2.2.16}$$

On the other hand, by changing the roles of F and G in the above analysis, we come to the conclusion that

$$g(x) = K f(x) \quad \text{for some constant } K \in \mathrm{R} \quad \text{and all} \quad x \in U_f. \tag{2.2.17}$$

By Equation (2.2.8), there is a point $x_0 \in U_g \cap U_f$ so $AK = 1$ and these coefficients are not zero. But then Equation (2.2.16) implies $U_g \subset U_f$ and Equation (2.2.17) implies $U_f \subset U_g$; therefore, $U_g = U_f$ and $Z_g = Z_f$. The latter means that

$$f(x) = Ag(x), \; g(x) = K f(x) \quad \text{if} \quad x \in Z_g = Z_f$$

by trivial reasons (all these values are zeros) so with Equation (2.2.16) and Equation (2.2.17) these identities are valid on the entire $\mathbb{R} = U_f \cup Z_f = U_g \cup Z_g$. In particular, it follows that $\{F, G, 1\}$ are linearly dependent on $\mathbb{R}$. □

2.2.2 Case 2: The Cauchy's MVT with Fixed Asymmetric Mean Value

In this subsection, we consider the asymmetric case i.e. in Equation (2.2.3) we take

$$\alpha, \beta \in (0, 1) \quad \text{with} \quad \alpha = \frac{1}{2} \text{ and } \beta = 1 - \alpha. \qquad (2.2.18)$$

The following proposition describes all pairs (F, G) of two times continuously differentiable functions satisfying Equation (2.2.3) under the assumption (2.2.18) on α, β in the intervals where $g = G'$ does not vanish.

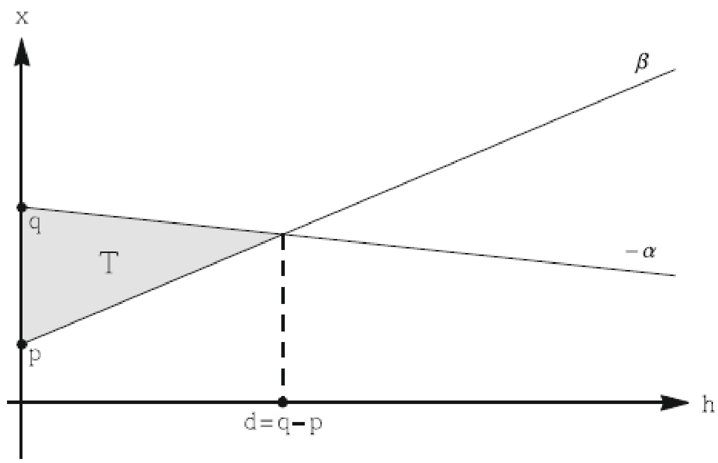
Proposition 2.2.3. *Let (F, G) be a solution of the Problem 1 with α, β satisfying (2.2.18) and $I = (p, q)$, $-\infty \leq p < q \leq \infty$, be an interval where the derivative $g(x)$ does not vanish. If F, G are twice continuously differentiable on I, then $\{F, G, 1\}$ are linearly dependent on I.*

Proof. With the changing of variables $h = b - a$, $x = \alpha a + \beta b$, Equation (2.2.3) yields

$$[F(x + \alpha h) - F(x - \beta h)]g(x) = [G(x + \alpha h) - G(x - \beta h)]f(x), \qquad (2.2.19)$$

if $x \in I$ and $h > 0$ are such that $x + \alpha h$, $x - \beta h \in I$. The latter condition yields that Equation (2.2.19) holds if (h, x) lies within the open triangle

$$T := \{(h, x) : 0 < h < q - p,\ p + \beta h < x < q - \alpha h\}. \qquad (2.2.20)$$



The triangle T

By differentiating both sides of Equation (2.2.19) with respect to h twice, we obtain the following the following relation in T.

$$[\alpha^2 \boldsymbol{f}^{t}(x + \alpha h) - \beta^2 \boldsymbol{f}^{t}(x - \beta h)]g(x) = [\alpha^2 g(x + \alpha h) - \beta^2 g^{t}(x - \beta h)]\boldsymbol{f}(x).$$

All the functions are continuous so the latter holds on the closure $\overline{T}$ as well, in particular on the interval $\{h = 0,\ p < x < q\}$. Therefore, with $\beta^2 - \alpha^2 = 1 - 2\alpha = 0$ by Equation (2.2.18), we get

$$\boldsymbol{f}^{t}(x)g(x) = g^{t}(x)\boldsymbol{f}(x),$$

for all $x \in \boldsymbol{I} = (p, q)$. We can divide both sides by $g^2(x)$ and conclude that

$(f/g)^t = 0$ on I. This implies that $f/g = A$ for some constant $A \in$ R, and

$$F^t(x) = f(x) = Ag(x) = AG^t(x)$$

, for $x \in I$. After integrating we get $F(x) = AG(x) + B(x)$, $x \in I$. □

Now, we state and prove the main result of this subsection.

Theorem 2.2.2. *Let* (F, G) *be a solution of the Problem 1 with* α, β *satisfying (2.2.18). IF* F, G *are twice continuously differentiable on* R *then* $\{F, G, 1\}$ *are linearly dependent on* R *i.e. there exist constants* $A, B, C \in$ R *such that not all of them are zeros and*

$$AF(x) + BG(x) + C = 0, \quad \textit{for all } x \in \text{R}. \qquad (2.2.21)$$

Proof. Consider the following cases:

<u>Case 1:</u> $U_g = \emptyset$. In this case G is a constant on R and Equation (2.2.3) holds for any differentiable function F. Hence Equation (2.2.21) holds, for example, with $A = 0, B = 1, C = -G$ and thus $\{F, G, 1\}$ are linearly dependent on R.

<u>Case 2:</u> $U_g = \emptyset$ but $U_g \cap U_f = \emptyset$. In this case Proposition 2.2.1 yields that F is a constant on R and Equation (2.2.3) holds for any differentiable function G. Hence Equation (2.2.21) holds, for example, with $A = 1, B = 0, C = -F$ and thus $\{F, G, 1\}$ are again linearly dependent on R.

<u>Case 3:</u> $U_g \cap U_f = \emptyset$. In this case Proposition 2.2.2 and Proposition 2.2.3 immediately imply that $\{F, G, 1\}$ are linearly dependent on R. □

2.2.3 Case 3: The Cauchy's MVT with Symmetric Mean Value

In this subsection we consider the problem of describing all pairs (F, G) of smooth functions for which the mean value in Equation (2.2.3) is taken at the midpoint of the interval. The first result gives a necessary (and also sufficient in case $\{F, G, 1\}$ are not linearly dependent) condition on such pairs in the intervals where $g = G^t$ does not vanish.

Proposition 2.2.4. *Assume that $F, G : \mathbb{R} \longrightarrow \mathbb{R}$ are three times differentiable functions with derivative $F^t = f$, $G^t = g$. Let $I \subset \mathbb{R}$ be such an interval that $g = 0$ for all $x \in I$ and Equation (2.2.4) holds for all $a, b \in I$. Then there exist constants $A, K \in \mathbb{R}$ and $x_0 \in I$ such that*

$$f(x) = \left(A + K \int_{x_0}^{x} \frac{dt}{g^2(t)} \right) g(x), \quad \textit{for all } x \in I. \tag{2.2.22}$$

Moreover if Equation (2.2.22) holds with $K = 0$, then Equation (2.2.4) holds if and only if

$$\int_{x-h}^{x+h} g(t) \left(\int_{x_0}^{x} \frac{du}{g^2(u)} \right) dt = \left(\int_{x-h}^{x+h} g(t) dt \right) \left(\int_{x_0}^{x} \frac{du}{g^2(u)} \right), \tag{2.2.23}$$

for all $x, h \in \mathbb{R}$ such that $x, x + h, x - h \in I$.

Proof. With the changing of variables $x = \frac{a+b}{2}$, $h = \frac{b-a}{2}$ we can rewrite Equation (2.2.4) as

$$[F(x + h) - F(x - h)]g(x) = [G(x + h) - G(x - h)]f(x), \tag{2.2.24}$$

for all $x, h \in \mathbb{R}$ with the property that $x, x+h, x-h \in I$. By differentiating this equality three times with respect to h we get:

$$[f''(x+h) + f''(x-h)]g(x) = [g''(x+h) + g''(x-h)]f(x).$$

Setting $h = 0$, we obtain

$$0 = f''(x)g(x) - f(x)g''(x) = \left(f'(x)g(x) - f(x)g'(x)\right)'$$

for all $x \in I$, and thus $f'(x)g(x) - f(x)g'(x) = K$ for some K constant. Then $\left(\frac{f}{g}(x)\right)' = \frac{K}{g^2(x)}, x \in I$ and integration over (x_0, x) with any $x_0 \in I$ yields Equation (2.2.22). Now assume Equation (2.2.22) holds with a nonzero constant K. Then we have

$$\begin{aligned} F(x+h) - F(x-h) &= \int_{x-h}^{x+h} f(t)dt \\ &= \int_{x-h}^{x+h} \left(A + K\int_{x_0}^{t} \frac{du}{g^2(u)}\right) g(t)dt \\ &= A\int_{x-h}^{x+h} g(t)dt + K\int_{x-h}^{x+h} g(t)\left(\int_{x_0}^{t} \frac{du}{g^2(u)}\right) dt \end{aligned}$$

and

$$\begin{aligned} G(x+h) - G(x-h)\frac{f(x)}{g(x)} &= \int_{x-h}^{x+h} g(t)dt \left(A + K\int_{x_0}^{x} \frac{du}{g^2(u)}\right) \\ &= A\int_{x-h}^{x+h} g(t)dt + K\int_{x-h}^{x+h} g(t)dt \left(\int_{x_0}^{x} \frac{du}{g^2(u)}\right) \end{aligned}$$

By comparing the last two relations, it is easy to see that Equation (2.2.23) is equivalent to Equation (2.2.4). □

The following example illustrates that there are non-trivial functions satisfying Equation (2.2.23) (and hence Equation (2.2.4)) on R.

Example 2.2.1. *Consider* $g(t) = e^t$ *on* $I = \mathbb{R}$ *and let* $A = 0$, $K = 1$, $x_0 = 0$. *The integral condition (2.2.23) reads as the following identity*

$$\int_{x-h}^{x+h} e^t \left(\int_0^t e^{-2u}du\right) dt = \left(\int_{x-h}^{x+h} e^t dt\right)\left(\int_0^x e^{-2u}du\right).$$

A direct computation gives $f(x) = \sinh(x) = \frac{e^x - e^{-x}}{2}$ *and consequently*

$$F(x) = \cosh(x) = \frac{e^x + e^{-x}}{2}, \qquad G(x) = e^x,\ x \in \mathbb{R}.$$

One can check directly that this pair (F, G) *satisfies Equation (2.2.4), giving a non-trivial example of such pairs.*

Now we assume that $K = 0$ and analyze Equation (2.2.23) for all $x, h, \in \mathbb{R}$ such that $x, x+h, x-h \in I$. Differentiating it with respect to h, we obtain

$$g(x+h)\int_{x_0}^{x+h}\frac{du}{g^2(u)} + g(x-h)\int_{x_0}^{x-h}\frac{du}{g^2(u)} = [g(x+h)+g(x-h)]\int_{x_0}^{x}\frac{du}{g^2(u)}.$$

Differentiate two more times with respect to h gives

$$g''(x+h)\int_{x_0}^{x+h}\frac{du}{g^2(u)} + g''(x-h)\int_{x_0}^{x-h}\frac{du}{g^2(u)} = [g''(x+h)+g''(x-h)]\int_{x_0}^{x}\frac{du}{g^2(u)},$$

for all $x \in I$ and $h \in \mathbb{R}$ such that , $x, x+h, x-h \in I$. Setting $h = x - x_0$ in these two equations, we obtain

$$g(2x-x_0)\int_{x_0}^{2x-x_0}\frac{du}{g^2(u)} = [g(2x-x_0)+g(x_0)]\int_{x_0}^{x}\frac{du}{g^2(u)}, \qquad (2.2.25)$$

and

$$g''(2x-x_0)\int_{x_0}^{2x-x_0}\frac{du}{g^2(u)} = [g''(2x-x_0)+g''(x_0)]\int_{x_0}^{x}\frac{du}{g^2(u)}, \qquad (2.2.26)$$

for all $x \in I$ and $2x - x_0 \in I$. Since $2x - x_0 \in I$ and g has no zeros in I, both sides of Equation (2.2.25) do not vanish. By comparing Equation (2.2.26) and Equation (2.2.25), we get

$$\frac{g''(2x-x_0)}{g(2x-x_0)} = \frac{g''(2x-x_0)+g''(x_0)}{g(2x-x_0)+g(x_0)}, \qquad (2.2.27)$$

for all $x \in I$ such that $2x - x_0 \in I$. Putting $y(x) := g(2x - x_0)$ and $\lambda := \frac{4g''(x_0)}{g(x_0)}$, Equation (2.2.27) yields the second order differential equation $y'' - \lambda y = 0$, whose general real-valued solution (depending on the sign of λ), has the following form:

$$
\begin{aligned}
g(x) &= Px + Q && \text{if } \lambda = 0,\\
g(x) &= Pe^{\sqrt{\lambda}x} + Qe^{-\sqrt{\lambda}x} && \text{if } \lambda = \mu^2, \mu > 0,\\
g(x) &= P\sin(\sqrt{-\lambda}x) + Q\cos(\sqrt{-\lambda}x) && \text{if } \lambda = -\mu^2, \mu > 0,
\end{aligned}
$$

where P, Q are real constants. Hence G has one of the following forms

$$G(x) = Ax^2 + Bx + C, \qquad (2.2.28)$$

$$G(x) = Ae^{\mu x} + Be^{-\mu x} + C, \mu > 0 \qquad (2.2.29)$$

$$G(x) = A\sin(\mu x) + B\cos(\mu x) + C, \mu > 0 \qquad (2.2.30)$$

where A, B are real constants.

Remark 2.2.1. *Altogether, we come to the following conclusion: on every interval $I \subset \mathbb{R}$ on which $G^t = 0$ either $\{F, G, 1\}$ are linearly dependent or G and thus also F, c.f. Equation (2.2.22) has one of the forms described in Equations (2.2.28)–(2.2.30).*

In the sequel, we call a function G (resp. the pair (F, G)) to be of quadratic, exponential or trigonometric type on I if G has (resp. both of F and G have) the form (2.2.28), (2.2.29) or (2.2.30), respectively.

Consider the set U_g and its representation, cf. Equation (2.2.5) and Equation (2.2.6). The following lemma plays a crucial role in the analysis of Equation (2.2.4).

Lemma 2.2.1. *Let $(p, q) \in \{I_\sigma\}_{\sigma\in\Sigma}$ be such that $p > -\infty$ and $f(p) = 0$. Then $\{F, G, 1\}$ are linearly dependent on $[p, q)$.*

Proof. We have $g(p) = 0$ by Equation (2.2.6) so by Remark 2.2.1, it is sufficient to consider the following cases.

Case 1: G is of quadratic type on (p, q). Then F is also of quadratic type on (p, q), and since $f(p) = g(p) = 0$, we have $F, G \in \operatorname{span}\{1, (x - p)^2\}$. Thus

$\{F, G, 1\}$ are linearly dependent on (p, q).

<u>Case 2:</u> G is of either exponential or trigonometric type on (p, q). First suppose that G is of exponential type on (p, q). Then so is F and since the set of functions satisfying Equation (2.2.4) is invariant with respect to the addition of constant functions, we can assume, without loss of generality, that $F, G \in \text{span}\{e^{\mu(x-p)}, e^{-\mu(x-p)}\}$ for some $\mu = 0$. Hence there are real constants u, v such that $F(x) = ue^{\mu(x-p)} + ve^{-\mu(x-p)}, x \in (p, q)$. Since $F'(p) = f(p) = 0$, we get $u = v$ and thus $F(x) = 2u\cosh(\mu(x-p))$. The same argument for G explains that $G(x) = 2w\cosh(\mu(x-p))$ for some real w, and consequently F and G are multiples of the same function $\cosh(\mu(x-p))$. If G is of trigonometric type, then in the same way as above, we can conclude that F and G are multiples of the same function $\cos(\mu(x-p))$, implying that $\{F, G, 1\}$ are linearly dependent on $[p, q)$. □

2.2.4 Proof of the Main Theorem: Theorem 2.2.1

Now, we are ready to prove Theorem 2.2.1.

Proof of Theorem 2.2.1. Consider the set U_g defined in Equation (2.2.5). If $U_g = \emptyset$ on R, then $g \equiv 0$ on R and thus G is identically constant on R. In this case F can be an arbitrary differentiable function on R and thus $\{F, G, 1\}$ are linearly dependent on R. If $U_g =$ R, then it follows (cf. Remark 2.2.1) that either $\{F, G, 1\}$ are linearly dependent or G has one of the forms (2.2.28)-(2.2.30) on the whole of R. Moreover, we get the same conclusion if $U_g \cap U_f = \emptyset$ (cf. Proposition 2.2.1).

Next, let us assume that $U_g \cap U_f = \emptyset$ and U_g is a proper subset of R. Consider the representation (2.2.6). It is clear (cf. Remark 2.2.1) that the index set Σ can be split into disjoint subsets as $\Sigma = \Sigma_{lr} \cup \Sigma_q \cup \Sigma_t \cup \Sigma_e$, where

$$\Sigma_{lr} := \{\sigma \in \Sigma : \{F, G, 1\} \text{ are in linear relationship on } I_\sigma\},$$

$$\Sigma_q := \{\sigma \in \Sigma : (F, G) \text{ are of quadratic type on } I_\sigma\},$$

$$\Sigma_t := \{\sigma \in \Sigma : (F, G) \text{ are of trigonometric type on } I_\sigma\},$$

$$\Sigma_e := \{\sigma \in \Sigma : (F, G) \text{ are of exponential type on } I_\sigma\}.$$

__Claim 1:__ If $\Sigma_{lr} = \varphi$, then $\Sigma_{lr} = \Sigma$.

Proof of Claim 1. Assume Σ_{lr} is a proper subset of Σ then there exists $\sigma_2 \in \Sigma$ such that $\sigma_2 \notin \Sigma_{lr}$. Since $\Sigma_{lr} = \emptyset$, there is $\sigma_1 \in \Sigma_{lr}$ and $A_1 \in$ R such that $f(x) = A_1 g(x)$ on $x \in I_{\sigma_1}$. Consider all $x, h \in$ R such that $x + h \in \sigma_2$ and $x \in I_{\sigma_1}$. Using Equation (2.2.4) for $a = x - h$ and $b = x + h$ and recalling that $g = 0$ on I_{σ_1}, we get

$$F(x+h) - A_1 G(x+h) = F(x-h) - A_1 G(x-h). \qquad (2.2.31)$$

Therefore,

$$\begin{aligned} f(x+h) - A_1 g(x+h) &= \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial h}\right) F(x+h) - \frac{A_1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial h}\right) G(x+h) \\ &= \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial h}\right)\left(F(x+h) - A_1 G(x+h)\right) = 0, \end{aligned}$$

and thus $f(x+h) = A_1 g(x+h)$ for all $x, h \in$ R such that $x + h \in I_{\sigma_2}$ and $x \in I_{\sigma_1}$. From this it follows that F and G are in linear relationship on I_{σ_2}, that is, $\sigma_2 \in \Sigma_{lr}$ which leads to a contradiction.

Claim 2: If $\Sigma_{lr} = \varphi$, then only one of the index sets $\Sigma_q, \Sigma_t, \Sigma_e$ is non-empty.

Proof of Claim 2. Let $\sigma \in \Sigma$ and $\boldsymbol{I}_\sigma = (p, q)$. Since U_g is a proper subset of R, one of p, q is finite. We can assume $p > -\infty$. Then $g(p) = 0$, and Lemma 2.2.1 yields $\boldsymbol{f}(p) = 0$. Hence using Equation (2.2.4) for $a = p - h$ and $b = p + h$ we get

$$G(p+h) = G(p-h) \quad \text{for all } h \in \mathbb{R}, \tag{2.2.32}$$

so the graph of G is symmetric with respect to the vertical line $y = p$.

If $\sigma \in \Sigma_q$ or $\sigma \in \Sigma_e$, then $q = +\infty$ since the functions of quadratic type have exactly one and the functions of exponential type have at most one critical point. Therefore, if $\sigma \in \Sigma_q$, then $G \in \operatorname{span}\{1, (x-p)^2\}$, $x \in \mathbb{R}$ and $\Sigma = \Sigma_q$. Similarly, it follows from Equation (2.2.32) that if $\sigma \in \Sigma_e$, then $\Sigma = \Sigma_e$. Next assume $\Sigma_{lr} = \Sigma_e = \Sigma_q = \emptyset$. Then $\Sigma = \Sigma_t$ and let $\sigma \in \Sigma_t$. Sine G is of trigonometric type on $\boldsymbol{I}_\sigma = (p, q)$, we must have $q < +\infty$, so $g(p) = g(q) = 0$ and it follows as in the proof of Lemma 2.2.1 that there are real constants u, v such that

$$G(x) = u + v\cos\left(\pi\frac{x-p}{q-p}\right), x \in (p, q). \tag{2.2.33}$$

Using Equation (2.2.32) we obtain that Equation (2.2.33) holds on the whole of R. □

Since $U_g = \emptyset$ at least one of $\Sigma_{lr}, \Sigma_e, \Sigma_q, \Sigma_t$ is non-empty. If $\Sigma_{lr} = \emptyset$, then Claim

1 and Proposition 2.2.2 imply that $\{F, G, 1\}$ are linearly dependent on R. If $\Sigma_{lr} = \emptyset$, then Claim 2 yields that one of the possibilities (b)–(d) holds. □

2.3 Another Functional Equation

As a consequence of Theorem 2.2.1, we can give a partial answer to the following still open question of Sahoo and Riedel (see [8], Section 2.7] for an equivalent formulation).

Problem: Find all functions $F, G, \varphi, \psi : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying

$$[F(x) - F(y)]\varphi\left(\frac{x+y}{2}\right) = [G(x) - G(y)]\psi\left(\frac{x+y}{2}\right) \tag{2.3.1}$$

for all $x, y \in \mathbb{R}$. We provide a partial solution to this problem under certain assumptions on the unknown functions. First let us change the variables

$$s = \frac{x+y}{2}, \quad t = \frac{x-y}{2}$$

and write Equation (2.3.1) equivalently as

$$[F(s+t) - F(s-t)]\varphi(s) = [G(s+t) - G(s-t)]\psi(s), \tag{2.3.2}$$

for $s, t \in \mathbb{R}$.

Theorem 2.3.1. *Let $F, G : \mathbb{R} \longrightarrow \mathbb{R}$ be three times differentiable and $\varphi, \psi : \mathbb{R} \longrightarrow \mathbb{R}$ be arbitrary functions satisfying Equation (2.3.2) on* R. *If either $\varphi = 0$ or $\psi = 0$ on* R*, then one of the following possibilities holds:*

a. There exists constants $A_0, A_1, A_2 \in \mathrm{R}$ *such that for all* $s \in \mathrm{R}$*, we have*

$$A_0 + A_1 F(s) + A_2 G(s) = 0 \quad \textit{and} \quad G^t(s)[A_1\psi(s) + A_2\varphi(s)] = 0,$$

b. There exists constants $A_0, A_1, A_2, B_0, B_1, B_2 \in \mathrm{R}$ *such that for all* $s \in \mathrm{R}$*, we have*

$$F(s) = A_0 + A_1 s + A_2 s^2, \quad G(s) = B_0 + B_1 s + B_2 s^2,$$

$$(A_1 + 2A_2 s)\varphi(s) = (B_1 + 2B_2 s)\psi(s),$$

c. There exists $\mu = 0$ *and constants* $A_0, A_1, A_2, B_0, B_1, B_2 \in \mathrm{R}$ *such that for all* $s \in \mathrm{R}$*, we have*

$$F(s) = A_0 + A_1 e^{\mu s} + A_2 e^{-\mu s}, \quad G(s) = B_0 + B_1 e^{\mu s} + B_2 e^{-\mu s},$$

$$(A_1 e^{\mu s} - A_2 e^{-\mu s})\varphi(s) = (B_1 e^{\mu s} - B_2 e^{-\mu s})\psi(s),$$

d. There exists $\mu = 0$ *and constants* $A_0, A_1, A_2, B_0, B_1, B_2 \in \mathrm{R}$ *such that for all* $s \in \mathrm{R}$*, we have*

$$F(s) = A_0 + A_1 \sin(\mu s) + A_2 \cos(\mu s), \; G(s) = B_0 + B_1 \sin(\mu s) + B_2 \cos(\mu s),$$

$$[A_1 \cos(\mu s) - A_2 \sin(\mu s)]\varphi(s) = [B_1 \cos(\mu s) - B_2 \sin(\mu s)]\psi(s).$$

Proof. Let f, g be the derivatives of F, G, respectively and the sets U_g, U_f (respectively Z_f, Z_g) be defined as in Section 2.2.1. Without loss of generality assume that φ does not vanish on R. By differentiating Equation (2.3.2) with respect to t and setting $t = 0$ in resulting equation, we get

$$f(s)\varphi(s) = g(s)\psi(s), s \in \mathrm{R}. \tag{2.3.3}$$

For any $s \in U_g$ and $t \in$ R. Equation (2.3.2) and Equation (2.3.3) give

$$\begin{aligned} F(s+t) - F(s-t) &= [G(s+t) - G(s-t)]\frac{\psi(s)}{\varphi(s)} \\ &= [G(s+t) - G(s-t)]\frac{f(s)}{g(s)}, \end{aligned}$$

and thus

$$[F(s+t) - F(s-t)]g(s) = [G(s+t) - G(s-t)]f(s), \tag{2.3.4}$$

for $s \in U_g, t \in$ R. On the other hand, observe that we have $Z_g \subset Z_f$ by Equation (2.3.3) since $\varphi = 0$ on R. So Equation (2.3.4) holds for all $s \in U_g \cup Z_g =$ R. Therefore, Theorem 2.2.1 can be applied to Equation (2.3.4) and the four characterizations follow immediately. □

2.4 The Cauchy's MVT for Divided Differences

In Chapter 1, we examined the Lagrange's MVT for divided differences. Ratz and Russel [33] have established the Cauchy's MVT for divided differences. Next, we state and prove their result.

Theorem 2.4.1. *[33] Let* $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ *be real valued functions with continuous* n^{th} *derivatives and* $g^{(n)}(t) = 0$ *on* $[a, b]$*. Further, let* $x_0, x_1, ..., x_n \in [a, b]$*. Then there exist a point* $c \in [min\{x_0, ..., x_n\}, max\{x_0, ..., x_n\}]$ *such that*

$$f[x_0, x_1, ..., x_n]g^{(n)}(c) = g[x_0, x_1, ..., x_n]f^{(n)}(c). \tag{2.4.1}$$

Proof. Without loss of generality we may assume $x_0 \leq x_1 \leq \cdots \leq x_n$. If $x_0 = x_1 = \cdots = x_n$, then from the definition of divided differences and the fact that f and g are n times continuously differentiable, Equation (2.4.1) holds with $x_0 = x_1 = \cdots = x_n = c$. Next suppose $x_0 < x_n$. For $x_0 \leq t \leq x_n$, define

$$F(t) = f[t, x_1, ..., x_n] \quad \text{and} \quad G(t) = g[t, x_1, ..., x_{n-1}]. \tag{2.4.2}$$

From the definition of divided differences and Equation (2.4.2) we see that

$$f[x_0, x_1, ..., x_n] = \frac{F(x_0) - F(x_n)}{x_0 - x_n} \tag{2.4.3}$$

and

$$g[x_0, x_1, ..., x_n] = \frac{G(x_0) - G(x_n)}{x_0 - x_n}. \tag{2.4.4}$$

Since $g^{(n)}(t) = 0$ on $[a, b]$, one can conclude that $g[x_0, x_1, ..., x_n] = 0$. Next we define

$$H(t) = g[x_0, x_1, ..., x_n]F(t) - f[x_0, x_1, ..., x_n]G(t). \tag{2.4.5}$$

Using Equation (2.4.3) and Equation (2.4.4) in Equation (2.4.5) it is easy to see that

$$H(x_0) = H(x_n). \tag{2.4.6}$$

The linearity of the divided difference and Equation (2.4.5) implies that:

$$\begin{aligned}
H(t) &= g[x_0, x_1, ..., x_n]F(t) - f[x_0, x_1, ..., x_n]G(t) \\
&= \left(\frac{G(x_0) - G(x_n)}{x_0 - x_n}\right) F(t) - \left(\frac{F(x_0) - F(x_n)}{x_0 - x_n}\right) G(t) \\
&= \left(\frac{G(x_0) - G(x_n)}{x_0 - x_n}\right) f[t, x_1, ..., x_{n-1}] - \left(\frac{F(x_0) - F(x_n)}{x_0 - x_n}\right) g[t, x_1, ..., x_{n-1}] \\
&= h[t, x_1, ..., x_{n-1}],
\end{aligned}$$

where

$$h(t) = g[x_0, x_1, ..., x_n]f(t) - f[x_0, x_1, ..., x_n]g(t) \tag{2.4.7}$$

with $x_0 \le t \le x_n$. Differentiating $H(t)$ with respect to t, we have from the properties of divided differences (see [34])

$$H'(t) = h[t, t, x_1, \cdots, x_{n-1}]. \tag{2.4.8}$$

Since f and g are n-times differentiable, so also h. Thus, using the MVT for divided differences (see Section 1.4), we have

$$h[t, t, x_1, \cdots, x_{n-1}] = \frac{h^{(n)}(\alpha(t))}{n!}, \tag{2.4.9}$$

for some $\alpha(t)$ in the interval $[x_0, x_n]$. Thus from Equation (2.4.8) and Equation (2.4.9), we have

$$H^t(t) = \frac{h^{(n)}(\alpha(t))}{n!}. \tag{2.4.10}$$

Since H is differentiable and H satisfies Equation (2.4.6), we obtain

$$H^t(\theta) = 0, \tag{2.4.11}$$

for some $\theta \in (x_0, x_n)$. From Equation (2.4.10) and Equation (2.4.11) and calling $\alpha(t)$ to be c, we have

$$\frac{h^{(n)}(c)}{n!} = 0. \tag{2.4.12}$$

Now using Equation (2.4.7) and Equation (2.4.12) we see that

$$g[x_0, x_1, \cdots, x_n]f^{(n)}(c) - f[x_0, x_1, \cdots, x_n]g^{(n)}(c) = 0.$$

Note that $c \in [x_0, x_n]$. $\square$

Chapter 3

Some Ostrowski Type Inequalities via the Cauchy's Mean Value Theorem

In this chapter, we use the Cauchy's MVT to derive some Ostrowski type inequalities. The Ostrowski's inequality and its different types have been used in numerical integration and in many other mathematics areas [35], [36], [37]. The next result is known in the literature as Ostrowski's inequality [38].

Theorem 3.0.1. *[38] Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on (a, b) such that $|f'(t)| \leq M$ for all $t \in (a, b)$. Then, for any $x \in [a, b]$, we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right](b-a)M. \qquad (3.0.1)$$

The constant $\frac{1}{4}$ is best possible constant in the sense that it cannot be replaced by a smaller constant.

The proof of Theorem 3.0.1 will be omitted since it is a direct consequence of Theorem 3.1.1 which is the generalization of the Ostrowski's inequality obtained in [39]. Next, we will state and prove the generalization of the Ostrowski's inequality obtained in [39].

3.1 A Generalization of the Ostrowski's Inequality and its Applications

Theorem 3.1.1. *[39] Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $g'(t) = 0$ for each $t \in (a, b)$ and*

$$\left\| \frac{f'}{g'} \right\|_{\infty} := \sup_{t \in (a,b)} \left| \frac{f'(t)}{g'(t)} \right| < \infty,$$

then for every $x \in [a, b]$, we have

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \le \left[2 \frac{\left| x - \frac{a+b}{2} \right|}{b-a} g(x) + \frac{\int_x^b g(t)dt - \int_a^x g(t)dt}{b-a} \right] \cdot \left\| \frac{f'}{g'} \right\|_{\infty}. \quad (3.1.1)$$

Proof. Let $x, t \in [a, b]$ with $t = x$. Applying Cauchy's MVT, there exists a c between t and x such that

$$f(x) - f(t) = \frac{f'(c)}{g'(c)} \left(g(x) - g(t) \right).$$

From this we get,

$$|f(x) - f(t)| = \left|\frac{f'(c)}{g'(c)}\right| |g(x) - g(t)| \le \left\|\frac{f'}{g'}\right\|_\infty |g(x) - g(t)|, \qquad (3.1.2)$$

for any $t, x \in [a, b]$. Using the properties of the intergal, we deduce by (3.1.2), that

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \frac{1}{b-a}\int_a^b |f(x) - f(t)|dt \le \left\|\frac{f'}{g'}\right\|_\infty \int_a^b |g(x) - g(t)|dt.$$

Since $g'(t) = 0$ on (a, b), it follows that either $g'(t) > 0$ or $g'(t) < 0$ for any $t \in (a, b)$. If $g'(t) > 0$ for all $t \in (a, b)$, then g is strictly monotonic increasing on (a, b) and

$$\begin{aligned}
&\int_a^b |g(x) - g(t)|dt \\
&= \int_a^x (g(x) - g(t))dt + \int_x^b (g(x) - g(t))dt \\
&= 2\left(x - \frac{a+b}{2}\right) g(x) + \int_x^b g(t)dt - \int_a^x g(t)dt.
\end{aligned}$$

If $g'(t) < 0$ for all $t \in (a, b)$, then

$$\int_a^b |g(x) - g(t)|dt = -\left[2\left(x - \frac{a+b}{2}\right) g(x) + \int_x^b g(t)dt - \int_a^x g(t)dt\right].$$

Inequality (3.1.1) is now proved. □

The following midpoint inequality is a natural consequence of the above result.

Corollary 3.1.1. *Under the same conditions as Theorem 3.1.1, one has the* *inequality*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{1}{b-a}\left| \int_{\frac{a+b}{2}}^b g(t)dt - \int_a^{\frac{a+b}{2}} g(t)dt \right| \cdot \left\| \frac{f'}{g'} \right\|_\infty$$

If in Theorem 3.1.1, we choose $g(t) = t$, then from (3.1.1) we recapture Ostrowski's inequality (3.0.1). If in Theorem 3.1.1 we choose $g(t) = t^p$, $p \in \mathbb{R}\setminus\{0\}$, or $g(t) = \ln t$ or $g(t) = (x-t)^p$ with $t \in (a,b) \subset (0,\infty)$, then we obtain the following theorems proved by S. S. Dragomir in [40].

Theorem 3.1.2. *[40] Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous on $[a,b]$ with $a > 0$ and differentiable on (a,b). Let $p \in \mathbb{R}\setminus\{0\}$ and assume that*

$$k_p(f') := \sup_{u\in(a,b)} u^{1-p}|f'(u)| < \infty.$$

Then we have the inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right|$$

$$\leq \frac{k_p(f')}{|p|(b-a)} \times \begin{cases} 2x^p(x-A) + (b-x)L_p^p(b,x) - (x-a)L_p^p(x,a), & \text{if } p \in (0,\infty) \\ (x-a)L_p^p(x,a) - (b-x)L_p^p(b,x) - 2x^p(x-A), & \text{if } p \in (-\infty,-1) \cup (-1,0) \\ (x-a)L_{-1}^{-1}(x,a) - (b-x)L_{-1}^{-1}(b,x) - \frac{2}{x}(x-A), & \text{if } p = -1 \end{cases}$$

for any $x \in (a,b)$, where $a \neq b$, $A = A(a,b) := \frac{a+b}{2}$ is the arithmetic mean,

$$L_p = L_p(a,b) = \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}}$$

is the p-Logarithmic mean $p \in \mathbb{R} \setminus \{-1, 0\}$ and $L = L(a, b) := \frac{b-a}{\ln b - \ln a}$ is the logarithmic mean.

Theorem 3.1.3. *[40] Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous on $[a, b]$ (with $a > 0$) and differentiable on (a, b). If $P(f') := \sup_{u\in(a,b)} |uf'(x)| < \infty$ then we have the following inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{P(f')}{b-a}\left[\ln\left(\frac{[I(x,b)]^{b-x}}{[I(a,x)]^{x-a}} \right) + 2(x - A)\ln x \right]$$

for any $x \in (a, b)$, where for $a = b$, $I = I(a, b) := \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$ is the identric mean.

If some local information around the point $x \in (a, b)$ is available, then we may state the following result as well.

Theorem 3.1.4. *[40] Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Let $p \in (0, \infty)$ and assume, for a given $x \in (a, b)$, we have that*

$$M_p(x) := \sup_{u\in(a,b)} \left(|x - u|^{1-p} |f'(u)| \right) < \infty.$$

Then we have the inequality

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \\ &\leq \frac{1}{p(p+1)(b-a)}\left[(x-a)^{p+1} + (b-x)^{p+1} \right] M_p(x). \end{aligned} \tag{3.1.3}$$

One may obtain many inequalities from Theorem (3.1.1) on choosing different functions g.

Proposition 3.1.1. *Let $f : [a, b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If there exists a constant $\Gamma < \infty$ such that*

$$|f'(t)| \leq \Gamma e^{-t} \text{ for any } t \in (a, b),$$

then for any $x \in (a, b)$, we have

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \Gamma \left[2\,\frac{x - A(a, b)}{b-a}\, e^x + \frac{(b-x)E(x, b) - (x-a)E(a, x)}{b-a} \right],$$

where $A = A(a, b) = \frac{a+b}{2}$ and E is the exponential mean, i.e, for $x, y \in \mathbb{R}$,

$$E(x, y) := \begin{cases} \frac{e^x - e^y}{x-a}, & \text{if } x = y \\ e^y, & \text{if } x = y. \end{cases}$$

In particular we have

$$\left| f(A) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{1}{2}\Gamma \left[E(A, b) - E(a, A) \right]$$

The proof is obvious by Theorem 3.1.1 on choosing $g(t) = e^t$ and we omit the details. Another example is considered in the following proposition.

Proposition 3.1.2. *Let $f : [a, b] \subset (0, \frac{\pi}{2}) \longrightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b).*

1. *If there exists a constant $\Gamma_1 < \infty$ such that*

$$|f'(t)| \leq \Gamma_1 \cos t,\ t \in (a, b)$$

then one has the inequality

$$f(x) - \frac{1}{b-a}\int_a^b f(t)dt$$
$$\le \Gamma_1 \; 2\;\frac{x - A(a,b)}{b-a}\;\sin x + \frac{(x-a)C(a,x) - (b-x)C(x,b)}{b-a},$$

for any $x \in (a,b)$*, where* C *is the cos-mean value, i.e.*

$$C(x,y) := \begin{cases} \frac{\cos x - \cos y}{x-y}, & \text{if } x = y, \\ -\sin y, & \text{if } x = y. \end{cases}$$

In particular we have:

$$f(A) - \frac{1}{b-a}\int_a^b f(t)dt \le \frac{1}{2}\Gamma_1[C(a,A) - C(A,a)]$$

2. *If there exists a constant* $\Gamma_2 < \infty$ *such that*

$$|f^{t}(t)| \le \Gamma_2 \sin t, \; t \in (a,b),$$

then one has the inequality

$$f(x) - \frac{1}{b-a}\int_a^b f(t)dt$$
$$\le \Gamma_2 \; 2\;\frac{x - A(a,b)}{b-a}\;\cos x + \frac{(b-x)S(x,b) - (x-a)S(a,x)}{b-a},$$

for any $x \in (a, b)$*, where* S *is the sin-mean value i.e,*

$$S(x, y) := \begin{cases} \frac{\sin x - \sin y}{x - y}, & \text{if } x = y, \\ \cos y, & \text{if } x = y. \end{cases}$$

In particular we have

$$\left| f(A) - \frac{1}{b - a} \int_a^b f(t)dt \right| \leq \frac{1}{2} \Gamma_2 [S(A, b) - S(a, A)]$$

The following result also holds.

Theorem 3.1.5. *Let* $f : [a, b] \longrightarrow \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* $(a, b) \setminus \{x\}$, $x \in (a, b)$*. If* $g'(t) = 0$ *for* $t \in (a, x) \cup (x, b)$*, then we have the inequality*

$$\left| f(x) - \frac{1}{b - a} \int_a^b f(t)dt \right| \leq \frac{1}{b - a} \left| g(x)(x - a) - \int_a^x g(t)dt \right| \cdot \left\| \frac{f'}{g'} \right\|_{(a,x), \infty} + \frac{1}{b - a} \left| g(x)(b - x) - \int_x^b g(t)dt \right| \cdot \left\| \frac{f'}{g'} \right\|_{(x,b), \infty} \quad (3.1.4)$$

Proof. we obviously have:

$$\begin{aligned} & \left| f(x) - \frac{1}{b - a} \int_a^b f(t)dt \right| \\ = & \left| \frac{1}{b - a} \int_a^b (f(x) - f(t))dt \right| \\ \leq & \frac{1}{b - a} \int_a^b |f(x) - f(t)| dt \\ = & \frac{1}{b - a} \left[\int_a^x |f(x) - f(t)| dt + \int_x^b |f(x) - f(t)| dt \right]. \end{aligned} \quad (3.1.5)$$

Applying Cauchy's Mean Value theorem on the interval (a, x) we deduce (see the proof of Theorem 3.1.1) that

$$|f(x) - f(t)| \leq \left\| \frac{f'}{g'} \right\|_{(a,x),\infty} |g(x) - g(t)|,$$

for any $t \in (a, x)$, and similarly

$$|f(x) - f(t)| \leq \left\| \frac{f'}{g'} \right\|_{(x,b),\infty} |g(x) - g(t)|,$$

for any $t \in (x, b)$. Consequently,

$$\int_a^x |f(x) - f(t)| dt \leq \left\| \frac{f'}{g'} \right\|_{(a,x),\infty} \int_a^x |g(x) - g(t)| dt,$$

and

$$\int_x^b |f(x) - f(t)| dt \leq \left\| \frac{f'}{g'} \right\|_{(x,b),\infty} \int_x^b |g(x) - g(t)| dt.$$

Since g' has a constant sign in either (a, x) or (b, x), it follows that g is strictly increasing or strictly decreasing in (a, x) and (x, b). Thus

$$\int_a^x |g(x) - g(t)| dt = \begin{cases} g(x)(x-a) - \int_a^x g(t)dt, & \text{if } g \text{ is increasing on } [a, x] \\ \int_a^x g(t)dt - g(x)(x-a), & \text{if } g \text{ decreasing on } [a, x] \end{cases}$$

$$= \left| g(x)(x-a) - \int_a^x g(t)dt \right|.$$

Consequently, using (3.1.5), we deduce Inequality (3.1.4). □

The following particular case may be of interest.

Corollary 3.1.2. *Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b) \setminus \{\frac{a+b}{2}\}$. If $g'(t) = 0$ on $(a, \frac{a+b}{2}) \cup (\frac{a+b}{2}, b)$ then we have the inequality*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{1}{2}\left| g\left(\frac{a+b}{2}\right) - \frac{2}{b-a}\int_a^{\frac{a+b}{2}} g(t)dt \right| \cdot \left\| \frac{f'}{g'} \right\|_{(a, \frac{a+b}{2}), \infty}$$
$$+\frac{1}{2}\left| g\left(\frac{a+b}{2}\right) - \frac{2}{b-a}\int_{\frac{a+b}{2}}^{b} g(t)dt \right| \cdot \left\| \frac{f'}{g'} \right\|_{(\frac{a+b}{2}, b), \infty}$$

Proposition 3.1.3. *Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b) \setminus \{x\}$, $x \in (a, b)$. Assume that, for $p > 0$, we have*

$$|f'(t)| \leq \begin{cases} M_{1,p}(x)(x - t)^{1-p}, & \text{for any } t \in (a, x) \\ M_{2,p}(x)(t - x)^{1-p}, & \text{for any } t \in (x, b). \end{cases}$$

Then we have the inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right|$$
$$\leq \frac{1}{p(p+1)(b-a)}\left[M_{1,p}(x)(x - a)^{p+1} + M_{2,p}(x)(b - x)^{p+1} \right]. \quad (3.1.6)$$

The proof follows by Theorem 3.1.5 applied for $g(x) = |x - t|^p$, $p > 0$. We omit the details.

Remark 3.1.1. *If f is as in Proposition 3.1.3 and*

$$|f'(t)| \leq \begin{cases} M_1\left(\frac{a+b}{2}\right)\left(\frac{a+b}{2} - t\right)^{1-p}, & \text{for any } t \in (a, \frac{a+b}{2}), \\ M_2\left(\frac{a+b}{2}\right)\left(t - \frac{a+b}{2}\right)^{1-p}, & \text{for any } t \in (\frac{a+b}{2}, b), \end{cases}$$

then by (3.1.6), we get

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{(b-a)^{p+1}}{2^{p+1}p(p+1)}\left[M_1\left(\frac{a+b}{2}\right) + M_2\left(\frac{a+b}{2}\right)\right]$$

Remark 3.1.2. *If f is as in Proposition 3.1.3 and $|f'(t)| \leq M_p(x)|x-t|^{1-p}$, $t \in (a, b)$ then by (3.1.6) we get,*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{1}{p(p+1)(b-a)}\left[(x-a)^{p+1} + (b-a)^{p+1}\right] M_p(x),$$

which is the result obtained in (3.1.3).

3.2 Some Inequalities of Midpoint Type

We will derive some inequalities of midpoint type:

1. **Inequality of Midpoint Type I.** Let $0 < a < b$. Consider the function $g : [a, b] \longrightarrow \mathbb{R}$, $g(t) = t^p$, $t \in \mathbb{R} \setminus \{0, -1\}$. Then $g'(t) = pt^{p-1}$, $g(\frac{a+b}{2}) = A^p(a, b)$,

$$\frac{2}{b-a}\int_a^{\frac{a+b}{2}} g(t)dt = L_p^p(a, A(a, b)),$$

$$\frac{2}{b-a}\int_{\frac{a+b}{2}}^b g(t)dt = L_p^p(A(a, b), b),$$

   and by Corollary 3.1.2, we may state the following proposition.

   Proposition 3.2.1. *Let $f : [a, b] \subset (0, \infty) \longrightarrow \mathbb{R}$ be continuous on $[a, b]$*

and differentiable on $(a,b) \setminus \{\frac{a+b}{2}\}$*. If*

$$|f'(t)| \le \begin{cases} M_1(\frac{a+b}{2})t^p, & t \in (a, \frac{a+b}{2}) \\ M_2(\frac{a+b}{2})t^p, & t \in (\frac{a+b}{2}, b) \end{cases}$$

then we have the inequality

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \frac{1}{2}M_1\left(\frac{a+b}{2}\right)\left[A^p(a,b) - L_p^p(a, A(a,b))\right]$$
$$+\frac{1}{2}M_2\left(\frac{a+b}{2}\right)\left[L_p^p(A(a,b),b) - A^p(a,b)\right].$$

The particular case $p = 1$ *is of interest and so we may state the following corollary.*

Corollary 3.2.1. *Let* $f : [a,b] \subset (0,\infty) \longrightarrow \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* $(a,b) \setminus \{\frac{a+b}{2}\}$*. If*

$$|f'(t)| \le \begin{cases} N_1(\frac{a+b}{2})t, & t \in (a, \frac{a+b}{2}) \\ N_2(\frac{a+b}{2})t, & t \in (\frac{a+b}{2}, b) \end{cases}$$

then we have the inequality:

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \frac{1}{8}\left[N_1\left(\frac{a+b}{2}\right) + N_2\left(\frac{a+b}{2}\right)\right](b-a).$$

2. **Inequality of Midpoint Type II.** Let $0 < a < b$. Consider the function $g : [a,b] \longrightarrow \mathbb{R}$ defined by $g(t) = \frac{1}{t}$. Then $g'(t) = -\frac{1}{t^2}$, $g(\frac{a+b}{2}) =$

$A^{-1}(a, b)$

$$\frac{2}{b-a}\int_a^{\frac{a+b}{2}} g(t)dt = L^{-1}(a, A(a, b)),$$

$$\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b} g(t)dt = L^{-1}(A(a, b), b)$$

and by Corollary 3.1.2, we may state the following proposition.

Proposition 3.2.2. *Let* $f : [a, b] \subset (0, \infty) \longrightarrow \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* $(a, b) \setminus \{\frac{a+b}{2}\}$*. If*

$$|f'(t)| \leq \begin{cases} M_1(\frac{a+b}{2})t^{-2}, & t \in (a, \frac{a+b}{2}) \\ M_2(\frac{a+b}{2})t^{-2}, & t \in (\frac{a+b}{2}, b) \end{cases}$$

then we have the inequality

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{1}{2} M_1\left(\frac{a+b}{2}\right) \cdot \frac{[A(a, b) - L(a, A(a, b))]}{L(a, A(a, b))A(a, b)}$$
$$+\frac{1}{2} M_2\left(\frac{a+b}{2}\right) \cdot \frac{[L(a, A(a, b)) - A(a, b)]}{L(A(a, b), b)A(a, b)}$$

3. **Inequality of Midpoint Type III.** Let $0 < a < b$. Consider the function $g : [a, b] \longrightarrow \mathbb{R}$, $g(t) = \ln t$. Then $g'(t) = \frac{1}{t}$, $g(\frac{a+b}{2}) = \ln A(a, b)$

$$\frac{2}{a+b}\int_a^{\frac{a+b}{2}} g(t)dt = \ln I(a, A(a, b)),$$

$$\frac{2}{a+b}\int_{\frac{a+b}{2}}^{b} g(t)dt = \ln I(A(a, b), b),$$

and by Corollary 3.1.2 we may state the following proposition.

Proposition 3.2.3. *Let $f : [a, b] \subset (0, \infty) \longrightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b) \setminus \{\frac{a+b}{2}\}$. If*

$$|f'(t)| \leq \begin{cases} M_1(\frac{a+b}{2})t^{-1}, & t \in (a, \frac{a+b}{2}) \\ M_2(\frac{a+b}{2})t^{-1}, & t \in (\frac{a+b}{2}, b) \end{cases}$$

then we have the inequality:

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \ln G\left(\left(\frac{A(a,b)}{I(a, A(a,b))}\right)^{M_1(\frac{a+b}{2})}, \left(\frac{I(A(a,b),b)}{A(a,b)}\right)^{M_2(\frac{a+b}{2})} \right)$$

3.3 The Case of Weighed Integrals

Theorem 3.3.1. *Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) and $w : [a, b] \longrightarrow [0, \infty)$ an integrable function such that $\int_a^b w(s)ds > 0$. If $g'(t) = 0$ for each $t \in (a, b)$ and $\left\| \frac{f'}{g'} \right\|_\infty := \sup_{t\in(a,b)} \left|\frac{f'(t)}{g'(t)}\right| < \infty$, then for any $x \in (a, b)$ one has the inequality*

$$\left| f(x) - \frac{1}{\int_a^b w(t)dt}\int_a^b f(t)w(t)dt \right| \leq \left[g(x)\cdot\frac{\int_a^x w(t)dt - \int_x^b w(t)dt}{\int_a^b w(t)dt} + \frac{\int_x^b w(t)g(t)dt - \int_a^x g(t)w(t)dt}{\int_a^b w(t)dt} \right] \cdot \left\| \frac{f'}{g'} \right\|_\infty$$

Proof. Let $x, t \in [a, b]$ with $t = x$. Applying Cauchy's MVT, there exists a c

between t and x such that

$$f(x) - f(t) = \frac{f'(c)}{g'(c)}[g(x) - g(t)],$$

from where we get,

$$|f(x) - f(t)| = \left|\frac{f'(c)}{g'(c)}\right| \cdot |g(x) - g(t)| \leq \left\|\frac{f'}{g'}\right\|_{\infty} |g(x) - g(t)|, \qquad (3.3.1)$$

for ant $t, x \in [a, b]$. Using the properties of integral, we deduce by (3.3.1), that

$$\begin{aligned} \left| f(x) - \frac{1}{\int_a^b w(s)ds} \int_a^b w(s)f(s)ds \right| &\leq \frac{1}{\int_a^b w(s)ds} \int_a^b w(t)|f(x) - g(t)|dt \\ &\leq \left\|\frac{f'}{g'}\right\|_{\infty} \frac{1}{\int_a^b w(s)ds} \int_a^b w(t)|g(x) - g(t)|dt. \end{aligned}$$

Since $g'(t) = 0$ on (a, b), it follows that either $g'(t) > 0$ or $g'(t) < 0$ for any $t \in (a, b)$. If $g'(t) > 0$ for all $t \in (a, b)$, then g is strictly monotonic increasing on (a, b) and

$$\begin{aligned} \int_a^b w(t)|g(x) - g(t)|dt &= \int_a^x w(t)(g(x) - g(t))dt + \int_x^b w(t)(g(t) - g(x))dt \\ &= g(x)\int_a^x w(t)dt - \int_a^x w(t)g(t)dt \\ &\quad + \int_x^b w(t)g(t)dt - g(x)\int_x^b w(t)dt \\ &= g(x)\left[\int_a^x w(t)dt - \int_x^b w(t)dt\right] \\ &\quad + \int_x^b w(t)g(t)dt - \int_a^x w(t)g(t)dt. \end{aligned}$$

If $g'(t) < 0$ for all $t \in (a, b)$, then

$$\int_a^b w(t)[g(x) - g(t)]dt = -g(x)\left[\int_a^x w(t)dt - \int_x^b w(t)dt\right] - \int_x^b w(t)g(t)dt + \int_a^x w(t)g(t)dt$$

and Inequality (3.1.1) is proved. □

Corollary 3.3.1. *If $x_0 \in [a, b]$ is a point for which $\int_a^{x_0} w(t)dt = \int_{x_0}^b w(t)dt$ and f, g, w, are as in Theorem 3.3.1, then we have the inequality*

$$\left| f(x_0) - \frac{1}{\int_a^b w(t)dt} \int_a^b w(t)f(t)dt \right| \leq \frac{\int_{x_0}^b w(t)g(t)dt - \int_a^{x_0} g(t)w(t)dt}{\int_a^b w(t)dt} \cdot \left\| \frac{f'}{g'} \right\|_\infty$$

Chapter 4

Pompeiu's Mean Value Theorem and Associated Functional Equations

In this chapter, we examine a mean value theorem due to Pompeiu [41]. We then state and prove one of the generalizations of the Pompeiu's MVT, namely the one proposed by Boggio in [42, 43]. Functional equations arising from the Pompeiu's MVT are known as Stamate type functional equations. We deal with some Stamate type functional equations and their generalizations.

4.1 Pompeiu's Mean Value Theorem

In 1946, Pompeiu derived a variant of Langrange's MVT, now known as Pompeiu's MVT.

Theorem 4.1.1. *[41] For every real valued function f differentiable on an interval $[a, b]$ not containing 0 and for all pairs $x_1 = x_2$ in $[a, b]$, there exists a point $r \in (x_1, x_2)$ such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(r) - r f^{t}(r). \tag{4.1.1}$$

Proof. Define a real valued F on the interval $[\frac{1}{b}, \frac{1}{a}]$ by

$$F(t) = t f(\frac{1}{t}) \tag{4.1.2}$$

Since f is differentiable on $[a, b]$ and 0 is not in $[a, b]$, we see that F is differentiable on $[\frac{1}{b}, \frac{1}{a}]$ and

$$F^{t}(t) = f \; \frac{1}{t} \; - \frac{1}{t} \, f^{t} \, \frac{1}{t} \; . \tag{4.1.3}$$

Applying the Lagrange's MVT to F on the interval $[x, y] \subset [\frac{1}{b}, \frac{1}{a}]$, we get

$$\frac{F(x) - F(y)}{x - y} = F^{t}(c), \tag{4.1.4}$$

for some $c \in (x, y)$. Let $x_2 = \frac{1}{x}$, $x_1 = \frac{1}{y}$ and $r = \frac{1}{c}$. Then since $c \in (x, y)$, we have $x_1 < r < x_2$. Now using Equation (4.1.2) and Equation (4.1.3) on Equation (4.1.4) we have

$$\frac{x f(\frac{1}{x}) - y f(\frac{1}{y})}{x - y} = f \; \frac{1}{c} \; - \frac{1}{c} f^{t} \; \frac{1}{c} \; ,$$

that is $\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(r) - r f^{t}(r)$. □

Let us discuss the geometrical interpretation of this theorem. The equation of the secant line joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is given by

$$y = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1).$$

This line intersects the y-axis at the point $(0, y)$ where y is

$$\begin{aligned} y &= f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(0 - x_1) \\ &= \frac{x_2 f(x_1) - x_1 f(x_1) - x_1 f(x_2) + x_1 f(x_1)}{x_2 - x_1} \\ &= \frac{x_2 f(x_1) - x_1 f(x_2)}{x_2 - x_1} \\ &= \frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2}. \end{aligned}$$

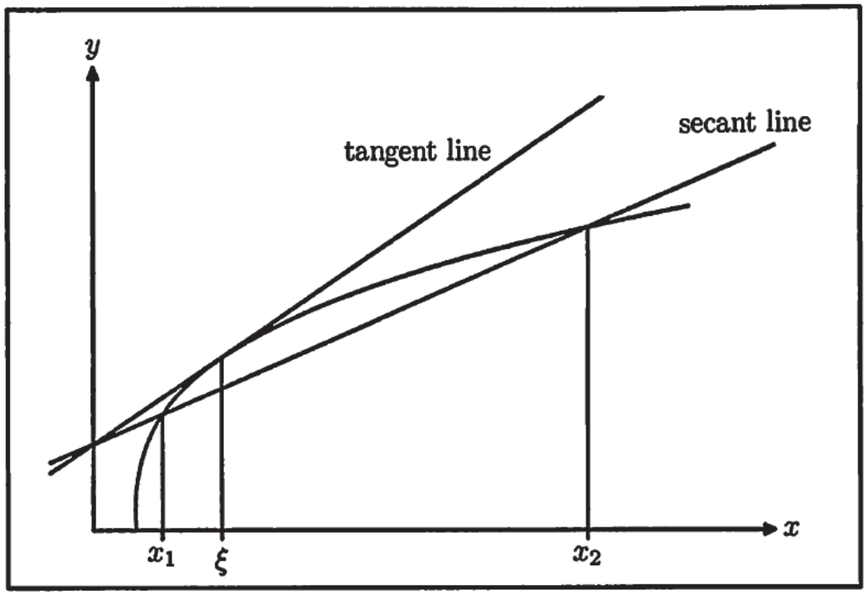
The equation of the tangent line at the point $(r, f(r))$ is

$$y = (x - r)f'(r) + f(r).$$

This tangent line intersects the y-axis at the point $(0, y)$ where $y = -rf'(r) + f(r)$. If this tangent line intersects the y-axis at the same point as the secant line joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$, then we have

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(r) - rf'(r),$$

which is Equation (4.1.1) in Theorem 4.1.1. Hence the geometric meaning of this is that the tangent at the point $(r, f(r))$ intersects on the y-axis at the same point as the secant line connecting the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.



4.2 Stamate Type Equations

The algebraic expression (4.1.1) yields a functional equation. It turns out that the exact form of the right-hand side is not essential [10, 19]. The relevant fact is that the right-hand side of Equation (4.1.1) depends only on r and not directly on x_1 and x_2. Thus we have the following functional equation

$$\frac{xf(y) - yf(x)}{x - y} = h(r(x, y)) \quad \forall x, y \in \mathrm{R} \quad \text{with} \quad x = y.$$

Similar to divided difference, a variant of it was defined in [44] recursively as $f\{x_1\} = f(x_1)$ and

$$f\{x_1, \cdots, x_n\} = \frac{x_n f\{x_1, \cdots, x_{n-1}\} - x_1 f\{x_2, \cdots, x_n\}}{x_1 - x_n}.$$

An easy computation shows that

$$f\{x_1, x_2\} = \frac{x_2 f(x_1) - x_1 f(x_2)}{x_2 - x_1}$$

and

$$f\{x_1, x_2, \cdots, x_n\} = \sum_{i=1}^{n} \left(\prod_{j=i}^{n} \frac{x_j}{x_i - x_j} \right) f(x_i).$$

The following results were established in [19].

Theorem 4.2.1. *The functions $f, h : \mathrm{R} \longrightarrow \mathrm{R}$ satisfy the functional equation*

$$f\{x, y\} = h(x + y) \quad \text{for all} \quad x, y \in \mathrm{R} \quad \text{with} \quad x = y \tag{4.2.1}$$

if and only if

$$f(x) = ax + b \quad \text{and} \quad h(x) = b, \tag{4.2.2}$$

where a, b are arbitrary constants.

Proof. We write Equation (4.2.1) as

$$xf(y) - yf(x) = (x - y)h(x + y), \tag{4.2.3}$$

which is now true for all $x, y \in \mathrm{R}$, also for $x = y$. Substituting $y = 0$ in Equation (4.2.3), we get $xf(0) = xh(x)$, that is

$$h(x) = f(0) = b, \quad 0 = x \in \mathrm{R}. \tag{4.2.4}$$

Letting Equation (4.2.4) into Equation (4.2.3), we have

$$xf(y) - yf(x) = (x - y)b, \tag{4.2.5}$$

for all $x, y \in R$ with $x + y = 0$. Putting $x = 1$ and $y = -1$ (so that $x + y = 0$) in Equation (4.2.5) we obtain $f(y) - yf(1) = (1 - y)b$, which is

$$f(y) = y[f(1) - b] + b = ay + b, \tag{4.2.6}$$

for all $y = -1$. Letting $y = 2$ in Equation (4.2.6), we see that

$$f(2) = 2f(1) - b. \tag{4.2.7}$$

Next putting $x = -1$ and $y = 2$ in Equation (4.2.3) and then using Equation (4.2.4) and Equation (4.2.7) we get $f(-1) = -[f(1) - b] + b$, that is

$$f(-1) = -a + b. \tag{4.2.8}$$

Together with Equation (4.2.8), we see that Equation (4.2.6) holds for all $y \in R$. Next, substituting $x = 1$ and $y = -1$ in Equation (4.2.3) we obtain $h(0) = b$ so that Equation (4.2.4) holds for all $x \in R$. Hence we have the asserted solution. □

The following lemma sets the path for a generalization of Theorem 4.2.1.

Lemma 4.2.1. *If $f, g, h : R \longrightarrow R$ satisfy the functional equation*

$$\frac{xf(y) - yg(x)}{x - y} = h(x + y),$$

for all $x, y \in \mathrm{R}$ *with* $x = y$, *then* $f(x) = g(x)$ *for all* $x \in \mathrm{R}$.

Proof. Interchanging x with y in the above functional equation, we have

$$\frac{yf(x) - xg(y)}{y - x} = h(y + x).$$

Next comparing with this resulting equation with the functional equation in the lemma, we obtain

$$xf(y) - yg(x) = xg(y) - yf(x), \qquad x, y \in \mathrm{R}, \quad x = y,$$

hence

$$\frac{f(x) - g(x)}{x} = \frac{g(y) - f(y)}{y}, x, y \in \mathrm{R} \setminus \{0\}, \quad x = y.$$

Let α be a fixed non zero real number and put $c = \frac{g(\alpha) - f(\alpha)}{\alpha}$. Thus from the above equation, we have

$$f(x) = g(x) + cx, \quad x \in \mathrm{R} \setminus \{0, \alpha\}.$$

Let $u, v \in \mathrm{R} \setminus \{0, \alpha\}$ with $u = v$. Putting $x = u$ and $y = v$ in the above equation, we see that $c = -c$ and hence $c = 0$. Thus

$$f(x) = g(x), \quad x \in \mathrm{R} \setminus \{0\}.$$

Next letting $x = \alpha$ and $y = 0$ in the functional equation of the lemma, we get $f(0) = h(\alpha)$. Further putting $x = 0$ and $y = \alpha$ in the functional equation of the lemma, we get $h(\alpha) = g(0)$. Thus we have $f(0) = g(0)$, and $f(x) = g(x)$

for all $x \in \mathbb{R}$. □

The following corollary follows from Lemma 4.2.1 and Theorem 4.2.1.

Corollary 4.2.1. *The functions $f, h : \mathbb{R} \longrightarrow \mathbb{R}$ satisfy the functional equation*

$$\frac{xf(y) - yg(x)}{x - y} = h(x + y) \quad \text{for all} \quad x, y \in \mathbb{R} \quad \text{with} \quad x = y$$

if and only if $f(x) = g(x) = ax + b$ and $h(x) = b$, where a, b are arbitrary constants.

Next, we present a theorem similar to Theorem 1.3.4.

Theorem 4.2.2. *Let s and t be the real parameters. The functions f, h : $\mathbb{R} \longrightarrow \mathbb{R}$ satisfy*

$$\frac{xf(y) - yf(x)}{x - y} = h(sx + ty), \tag{4.2.9}$$

for all $x, y \in \mathbb{R}, x = y$ if and only if

$$f(x) = ax + b, \tag{4.2.10}$$

$$h(x) = \begin{cases} \text{arbitrary with } \ b = h(0), & \text{if } s = 0 = t, \\ b, & \text{if } s = -t, x = 0, \\ b, & \text{otherwise,} \end{cases} \tag{4.2.11}$$

where a, b are arbitrary constants.

Proof. Rewriting Equation (4.2.9), we have

$$xf(y) - yf(x) = (x - y)h(sx + ty), \tag{4.2.12}$$

for $x, y \in \mathbb{R}$ with $x = y$. To establish the theorem, we consider several cases.

<u>Case 1:</u> Suppose $s = 0 = t$. Then from Equation (4.2.12), we have

$$x(f(y) - b) = y(f(x) - b).$$

Letting $y = 1$ in the above equation, we get $x(f(1) - b) = f(x) - b$, therefore

$$f(x) = x(f(1) - b) + b = ax + b,$$

where $a = f(1) - b$. Thus, we have the asserted solution

$$\begin{aligned} f(x) &= ax + b, \\ h(x) &= \text{arbitrary with } h(0) = b. \end{aligned}$$

<u>Case 2:</u> Suppose $t = 0$ but $s = 0$. Then Equation (4.2.12) yields

$$xf(y) - yf(x) = (x - y)h(sx). \tag{4.2.13}$$

Letting $y = 0$ in Equation (4.2.13), we have $xf(0) = xh(sx)$, that is

$$h(x) = b, \quad x \in \mathbb{R} \setminus \{0\}, \tag{4.2.14}$$

where $b = f(0)$. Using Equation (4.2.14) in Equation (4.2.13), we have

$$x(f(y) - b) = y(f(x) - b), \quad x = 0. \tag{4.2.15}$$

Letting $x = 1$ in Equation (4.2.15), we have

$$f(y) = y(f(1) - b) + b = ay + b, \tag{4.2.16}$$

for $y \in$ R. Letting $x = 0$ in Equation (4.2.13), we get $h(0) = f(0) = b$ and hence Equation (4.2.14) holds for all $x \in$ R.

<u>Case 3:</u> Suppose $t = 0$ but $s = 0$. This case can be handled similar to the previous case. Thus, we have the solution (4.2.10)–(4.2.11) as asserted in the theorem.

<u>Case 4:</u> Suppose $s = 0 = t$. Letting $y = 0$, we get $xf(0) = xh(sx)$. Hence,

$$h(x) = b, x \in \mathrm{R} \setminus \{0\}, \tag{4.2.17}$$

where $b = f(0)$. Letting Equation (4.2.17) into Equation (4.2.16), we get

$$xf(y) - yf(x) = (x - y)b, \tag{4.2.18}$$

with $sx + yt = 0$. Hence putting $x = 1$ in Equation (4.2.18), we get

$$f(y) = y(f(1) - b) + b = ay + b, \tag{4.2.19}$$

for $y = -\frac{s}{t}$. Letting $x = \frac{-s}{t}$ and $y = \frac{2s}{t}$ in Equation (4.2.12), we have

$$\frac{-s}{t} f\left(\frac{2s}{t}\right) - \frac{2s}{t} f\left(\frac{-s}{t}\right) = \frac{-3s}{t} h\left(\frac{s}{t}\right),$$

that is

$$f\left(\frac{2s}{t}\right) + 2f\left(\frac{-s}{t}\right) = 3b.$$

Therefore by Equation (4.2.19) we have

$$f\left(\frac{-s}{t}\right) = \frac{-s}{t} a + b.$$

Thus Equation (4.2.19) holds for all $y \in \mathbb{R}$. Next we show that $h(x) = b$ holds for all $x \in \mathbb{R}$ except the case when $s = -t$. If $s = -t$, then $h(x)$ can be defined arbitrarily at $x = 0$ and $h(x) = b$ for all $x \in \mathbb{R} \setminus \{0\}$. If $s = -t$, then we show that $h(0) = b$. Letting $x = 1$ and $y = \frac{-s}{t}$ in Equation (4.2.19), we obtain

$$f\left(\frac{-s}{t}\right) + \frac{s}{t} f(1) = \left(1 + \frac{s}{t}\right) h(0),$$

that is $h(0) = b$, and hence we have $h(x) = b$ for all $x \in \mathbb{R}$. □

We close this section by pointing out the connection between the two functional equations

$$\frac{F(x) - F(y)}{x - y} = F'(r(x, y))$$

and

$$\frac{xf(y) - yf(x)}{x - y} = f(r) - rf'(r(x, y)).$$

It was remarked in [20] that if one defines

$$F(x) = xf\left(\frac{1}{x}\right),$$

$$r(x, y) = \frac{1}{r\left(\frac{1}{x}, \frac{1}{y}\right)},$$

for all $x \in \mathbb{R}\setminus\{0\}$, then one obtains

$$\frac{F(x) - F(y)}{x - y} = F'(r(x, y)).$$

Thus to solve the first functional equation if 0 is not in the domain, one may apply the above transformation and then solve the second functional equation to obtain the solution of the first functional equation. If 0 is in the domain, then we can solve the second equation on an interval not containing 0 and then check whether the solutions obtained admit extensions to the whole domain satisfying the first equation.

4.3 An Equation of Kuczma

T. Boggio [42, 43] gave the following generalization of Pompeiu's MVT.

Theorem 4.3.1. *[42, 43] For all real valued function f and g differentiable on an interval $[a, b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a, b]$, there exists a point $r \in (x_1, x_2)$ such that*

$$\frac{g(x_1)f(x_2) - g(x_2)f(x_1)}{g(x_1) - g(x_2)} = f(r) - \frac{g(r)}{g'(r)f'(r)}. \qquad (4.3.1)$$

Here it is assumed that neither $g(x)$ *nor* $g^t(x)$ *is ever zero in* $[a, b]$.

Obviously r depends on x_1 and x_2 and one may ask for what f and g the mean value r depends on x_1 and x_2 in a given manner. From this point on view Equation (4.3.1) becomes a functional equation. The right side of Equation (4.3.1) can be replaced by an unknown function h to get

$$g(x_1)f(x_2) - g(x_2)f(x_1) = [g(x_1) - g(x_2)]h(r).$$

Replacing x_1 by x and x_2 by y and assuming r to be arithmetic mean of x and y, one obtains

$$g(x)f(y) - g(y)f(x) = [g(x) - g(y)]h\left(\frac{x+y}{2}\right), \quad x, y \subset [a, b],$$

where $[a, b]$ is a proper interval not containing 0. Kuczma [20] gave the solution of a similar equation and left the solution of this equation to the reader. His proof is a bit involved since he worked on a proper interval $[a, b]$. We will solve the above functional equation following his method but we have replaced the proper interval $[a, b]$ by the whole real line R. Further, we also assume that $g(h) = 0$ for some $k \in$ R. This is done to make the proof less technical and easily readable.

Theorem 4.3.2. *Let* $g : \mathrm{R} \longrightarrow \mathrm{R}$ *be a continuous and strictly increasing function with* $g(k) = 0$ *for some* $k \in$ R*. The functions* $f, g, h : \mathrm{R} \longrightarrow \mathrm{R}$ *satisfy the functional equation*

$$g(x)f(y) - g(y)f(x) = [g(x) - g(y)]h\left(\frac{x+y}{2}\right), \quad x, y \subset \mathrm{R} \qquad (4.3.2)$$

if and only if

$$\begin{cases} f(x) = \alpha g(x) + \beta, \\ h(x) = \beta, \\ g(x) = arbitrary, \end{cases}$$

where α, β *are arbitrary constants.*

Proof. Since g is strictly increasing, we note that $g = 0$. Thus, there exists a positive $\delta \in \mathbb{R}$ such that $g(\delta) = k$, where k is a non zero constant. Note that if g satisfies Equation (4.3.2) so also cg, where c is an arbitrary constant. Hence, we assume without loss of generality $g(\delta) = 1$. Furthermore, we observe that the functions

$$f_\delta(x) = f(\delta x), \quad g_\delta(x) = g(\delta x), \quad h_\delta(x) = h(\delta x) \tag{4.3.3}$$

satisfy Equation (4.3.2) if f, g and h do, that is

$$g_\delta(x) f_\delta(y) - g_\delta(y) f_\delta(x) = [g_\delta(x) - g_\delta(y)] h_\delta\left(\frac{x+y}{2}\right), \tag{4.3.4}$$

for all $x, y \in \mathbb{R}$. Moreover, we see that $g_\delta(1) = 1$. Since g is continuous and strictly increasing so also g_δ. For arbitrary constants α and β, if we define

$$\begin{cases} F(x) = f_\delta(x) - \alpha g_\delta(x) - \beta, \\ H(x) = h_\delta(x) - \beta, \\ G(x) = g_\delta(x), \end{cases} \tag{4.3.5}$$

then from Equation (4.3.4) and Equation (4.3.5), we get the following equation

$$F(x)G(y) - F(y)G(x) = [G(y) - G(x)]H\left(\frac{x+y}{2}\right). \tag{4.3.6}$$

If we choose $\alpha = f_\delta(1) - h_\delta(1)$ and $\beta = h_\delta(1)$, then from Equation (4.3.5) we see that

$$\begin{cases} F(1) = f_\delta(1) - \alpha g_\delta(1) - \beta = f_\delta(1) - f_\delta(1) + h_\delta(1) - h_\delta(1) = 0, \\ H(1) = h_\delta(1) - \beta = h_\delta(1) - h_\delta(1) = 0, \\ G(1) = g_\delta(1) = 1. \end{cases} \tag{4.3.7}$$

Letting $y = 1$ in Equation (4.3.6) and using Equation (4.3.7) we get,

$$F(x)G(1) - F(1)G(x) = [G(1) - G(x)]H\left(\frac{x+1}{2}\right),$$

which is

$$F(x) = [1 - G(x)]H\left(\frac{1+x}{2}\right). \tag{4.3.8}$$

Substituting Equation (4.3.8) in Equation (4.3.6), we see that

$$G(y)[1 - G(x)]H\left(\frac{1+x}{2}\right) - G(x)[1 - G(y)]H\left(\frac{1+y}{2}\right) = [G(y) - G(x)]H\left(\frac{x+y}{2}\right), \tag{4.3.9}$$

for all $x, y \in \mathbb{R}$. Since g is continuous and monotonic so also G (see Equation (4.3.3) and Equation (4.3.5)). Hence, continuity and monotonicity of G imply that

$$G(x) = 0 \text{ for all } x \in \mathbb{R} \setminus \{x_0\}, \tag{4.3.10}$$

for some $x_0 \in \mathbb{R}$. Dividing Equation (4.3.9) by $G(x)G(y)$, we obtain:

$$\frac{1 - G(x)}{G(x)} H\left(\frac{1+x}{2}\right) - \frac{1 - G(y)}{G(y)} H\left(\frac{1+y}{2}\right) = \frac{G(y) - G(x)}{G(y)G(x)} H\left(\frac{x+y}{2}\right), \tag{4.3.11}$$

for all $x, y \in \mathbb{R} \setminus \{x_0\}$. Substituting

$$\begin{cases} z = \frac{1}{2}(x+1), \\ w = \frac{1}{2}(y+1), \end{cases}$$

in Equation (4.3.11), we obtain

$$\frac{1 - G(2z-1)}{G(2z-1)} H(z) - \frac{1 - G(2w-1)}{G(2w-1)} H(w) = -\frac{G(2z-1) - G(2w-1)}{G(2z-1)G(2w-1)} H(z+w-1), \tag{4.3.12}$$

for all $z, w \in \mathbb{R} \setminus \{\frac{1}{2}(x_0 - 1)\}$. Putting $z = 2 - w$ in Equation (4.3.12) and using Equation (4.3.7), we get

$$\frac{G(3-2w) - 1}{G(3-2w)} H(2-w) = \frac{G(2w-1) - 1}{G(2w-1)} H(w), \tag{4.3.13}$$

for all $w \in \mathbb{R} \setminus \{\frac{1}{2}(x_0 - 1), \frac{1}{2}(3 - x_0)\}$. Next we replace w by $2 - w$ in Equation

(4.3.12) to obtain

$$
\begin{aligned}
&\frac{G(2z-1)-1}{G(2z-1)}H(z) - \frac{G(3-2w)-1}{G(3-2w)}H(2-w) \\
&= \frac{G(2z-1)-G(3-2w)}{G(2z-1)G(3-2w)}H(z-w+1),
\end{aligned}
\tag{4.3.14}
$$

for $z \in \mathbb{R} \setminus \{\frac{1}{2}(x_0-1), \frac{1}{2}(3-x_0)\}$. Now using Equation (4.3.13) in Equation (4.3.14), we see that

$$
\begin{aligned}
&\frac{G(2z-1)-1}{G(2z-1)}H(z) - \frac{G(2w-1)-1}{G(2w-1)}H(w) \\
&= \frac{G(2z-1)-G(3-2w)}{G(2z-1)-G(3-2w)}H(z-w+1),
\end{aligned}
\tag{4.3.15}
$$

for $z \in \mathbb{R} \setminus \{\frac{1}{2}(x_0+1)\}$ and $w \in \mathbb{R} \setminus \{\frac{1}{2}(x_0-1), \frac{1}{2}(3-x_0)\}$. Comparing Equation (4.3.15) with Equation (4.3.12), we get

$$
\frac{G(2z-1)-G(2w-1)}{G(2z-1)-G(2w-1)}H(z+w-1) = \frac{G(2z-1)-G(3-2w)}{G(2z-1)-G(3-2w)}H(z-w+1),
$$

for $z \in \mathbb{R} \setminus \{\frac{1}{2}(x_0+1)\}$ and $w \in \mathbb{R} \setminus \{\frac{1}{2}(x_0-1), \frac{1}{2}(3-x_0)\}$. By Equation (4.3.10), we see that $G(2z-1)=0$ for $z \in \mathbb{R} \setminus \{\frac{1}{2}(x_0+1)\}$. Hence the above equation reduces to

$$
\begin{aligned}
&\frac{G(2z-1)-G(2w-1)}{G(2w-1)}H(z+w-1) \\
&= \frac{G(2z-1)-G(3-2w)}{G(3-2w)}H(z-w+1).
\end{aligned}
\tag{4.3.16}
$$

We introduce two new variables

$$t = z + w - 2 \quad and \quad s = z - w \tag{4.3.17}$$

so that

$$z = 1 + \frac{1}{2}(t + s) \quad and \quad w = 1 + \frac{1}{2}(t - s).$$

Thus using Equation (4.3.17) in Equation (4.3.16), we get

$$\frac{G(1 + t + s) - G(1 + t - s)}{G(1 + t - s)} H(1 + t)$$
$$= \frac{G(1 + t + s) - G(1 - t + s)}{G(1 - t + s)} H(1 + s), \tag{4.3.18}$$

for $t, s \in$ R with $t - s = \pm(x_0 - 1)$. We define

$$w(t) = H(1 + t), \quad \text{for} \quad t \in \text{R} \tag{4.3.19}$$

and

$$z(t, s) = \frac{G(1 + t + s) - G(1 + s - t)}{G(1 + s + t) - G(1 + t - s)} \cdot \frac{G(1 + t - s)}{G(1 + s - t)}, \tag{4.3.20}$$

for all $t, s \in$ R with $t - s = \pm(x_0 - 1)$ and $s = 0$. Since g is continuous and strictly increasing so also G. Note that in (4.3.20),

$$G(1 + s + t) - G(1 + t - s) = 0,$$

for $s = 0$ because of the strict monotonicity of G. Further $G(1 + s - t) = 0$ for all $t, s \in \mathbb{R}$ with $t - s = \pm(x_0 - 1)$ since $G(x) = 0$ for all $\in \mathbb{R} \setminus \{x_0\}$ by the continuity of G. Thus the definition of $z(t, s)$ in (4.3.20) makes sense. Then the use of Equation (4.3.19) and Equation (4.3.20) in Equation (4.3.18) yields

$$w(t) = z(t, s)w(s), \tag{4.3.21}$$

for $t, s \in \mathbb{R}$ with $t - s = \pm(x_0 - 1)$ and $s = 0$. From Equation (4.3.21) one obtains either

$$w(t) = 0, \quad \text{for all} \quad t \in \mathbb{R} \setminus \{0\}$$

or

$$w(t) = 0 \quad \text{for all} \quad t \in \mathbb{R}.$$

Now we consider 2 cases.

Case 1: First, we treat the case when $w(t) = 0$ for $t \in \mathbb{R} \setminus \{0\}$. We fix $s = r \in \mathbb{R} \setminus \{0\}$ and obtain from Equation (4.3.21),

$$w(t) = c_0 z(t, r), \tag{4.3.22}$$

for all $t \in \mathbb{R} \setminus \{0\}$, where $c_0 = w(r)$ is a real constant. Substituting Equation (4.3.22) in Equation (4.3.21), we obtain

$$z(t, r) = z(t, s)z(s, r), \qquad t, s, r \in \mathbb{R} \setminus \{0\}. \tag{4.3.23}$$

Since we have unrestricted choice of $r \in \mathbb{R} \setminus \{0\}$, we can consider r as an independent variable. The functional equation (4.3.23) is the well known Sincov equation (see [45]) and hence we have

$$z(t, s) = \frac{\varphi(t+1)}{\varphi(s+1)}, \quad \text{for all} \quad t, s \in \mathrm{R}, s = -1, \tag{4.3.24}$$

where $\varphi : \mathrm{R} \longrightarrow \mathrm{R}$ is an arbitrary function satisfying

$$\varphi(t) = 0, \quad t \in \mathrm{R} \setminus \{-1\}, \quad \text{with } \varphi(-1) = 0. \tag{4.3.25}$$

By Equation (4.3.24) and Equation (4.3.22), we observe that

$$w(t) = c\varphi(t+1), \qquad t \in \mathrm{R} \tag{4.3.26}$$

where c is a real constant. Therefore, by Equation (4.3.26) and Equation (4.3.19), we obtain

$$H(t) = c\varphi(t), \quad t \in \mathrm{R}.$$

This with Equation (4.3.8) and Equation (4.3.5) yields

$$\begin{aligned} f(x) &= \alpha g(x) + \gamma[1 - g(x)]\varphi \frac{1+x}{2} + \beta, \\ h(x) &= \gamma\varphi(x) + \beta, \\ g(x) &= \text{arbitrary}, \end{aligned}$$

where α, β and γ are arbitrary constants and $\varphi : \mathrm{R} \longrightarrow \mathrm{R}$ an arbitrary function satisfying Equation (4.3.25). Letting the above form of f, g, h into Equation

(4.3.2) we get

$$g(y)[1-g(x)]\varphi\left(\frac{1+x}{2}\right)-g(x)[1-g(y)]\varphi\left(\frac{1+y}{2}\right)$$
$$=[g(y)-g(x)]\varphi\left(\frac{x+y}{2}\right). \tag{4.3.27}$$

Substituting $x = k$ in Equation (4.3.27), we get

$$g(y)\varphi\left(\frac{1+k}{2}\right)=g(y)\varphi\left(\frac{k+y}{2}\right).$$

Thus if $y = k$, we have $\varphi\left(\frac{1+k}{2}\right) = \varphi\left(\frac{k+y}{2}\right)$, that is $\varphi(y) = \gamma_0$, where γ_0 is an arbitrary constant. However, letting $x = k - 1$ and $y = k + 1$ in Equation (4.3.27), we see that

$$\gamma_0[g(\gamma+1)-g(\gamma-1)]=[g(\gamma+1)-g(\gamma-1)]\varphi(k).$$

Thus $\varphi(t) = \gamma_0$ for all $t \in \mathbb{R}$. Hence we have the asserted solution.

Case 2: Next, we consider the case when $w(t) = 0$ for all $t \in \mathbb{R}$. Hence from Equation (4.3.19) we have $H(t) = 0$ for all $t \in \mathbb{R}$. From Equation (4.3.8) and the fact that G is continuous with $G(1) = 1$ we have $f(x) = \alpha g(x)$. Thus we get the asserted solution with $\beta = 0$. □

4.4 Equations Motivated by Simpson's Rule

Simpson's rule is an elementary numerical method for evaluating a definite integral $\int_a^b f(t)dt$. The method consists of partitioning the interval $[a, b]$ into subintervals of equal lengths and then approximating the graph of f over each subinterval with a quadratic function. If $a = x_0 < x_1 < \cdots < x_{2n} = b$ is a partition of $[a, b]$ into $2n$ subintervals, each of length $\frac{b-a}{2n}$, then

$$\int_a^b f(t)dt \approx \frac{b-a}{6n}[f(x_0) + 4f(x_1) + \cdots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(x_{2n})].$$

The approximation formula is called Simpson's rule. It is well known that the error bound for Simpson's rule approximation is

$$\left|\int_a^b f(t)dt - \frac{b-a}{6n}[f(x_0) + \cdots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(x_{2n})]\right| \leq \frac{K(b-a)^5}{180n^4},$$

where $K = \sup\{|f^{(4)}(x)| \mid x \in [a, b]\}$. It is easy to note from this inequality that if f is four times continuously differentiable and $f^{(4)}(x) = 0$, then

$$\int_a^b f(t)dt = \frac{b-a}{6n}[f(x_0) + 4f(x_1) + \cdots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(x_{2n})].$$

This is obviously true if $n = 1$ and it reduces to

$$\int_a^b f(t)dt = \frac{b-a}{6}[f(x_0) + 4f(x_1) + f(x_2)].$$

Letting $a = x$ and $b = y$ and $x_1 = \frac{x+y}{2}$ in the above formula, we obtain

$$\int_x^y f(t)dt = \frac{y-x}{6}\left[f(x) + 4f\left(\frac{x+y}{2}\right) + f(y)\right]. \qquad (4.4.1)$$

This integral equation (4.4.1) holds for all $x, y \in \mathbb{R}$ if f is a polynomial of degree at most three. However, it is not obvious that if (4.4.1) holds for all

$x, y \in \mathrm{R}$ then the only solution f is a cubic polynomial. The integral equation (4.4.1) leads to functional equation

$$g(y) - g(x) = \frac{y-x}{6}\ f(x) + 4f(\frac{x+y}{2}) + f(y)\ ,$$

where g is an anti-derivative of f. The above equation is a special case of the functional equation

$$f(x) - g(y) = (x - y)\ h(x+y) + \psi(x) + \varphi(y)\ , \qquad (4.4.2)$$

for all $x, y \in \mathrm{R}$. In this section, we determine the general solution of the above functional equation (4.4.2). The following two functionals equations.

$$g(x) - g(y) = (x-y)f(x+y) + (x+y)f(x-y) \qquad (4.4.3)$$

and

$$xf(y) - yf(x) = (x-y)[g(x+y) - g(x) - g(y)] \qquad (4.4.4)$$

are instrumental in solving the functional equation (4.4.2). The functional equation (4.4.3) can be considered as a variant of

$$\frac{f(x) - f(y)}{x-y} = h(x+y)$$

and obtained by adding an extra term $(x+y)h(x-y)$ to the equation

$$f(x) - f(y) = (x-y)h(x+y).$$

Equation (4.4.4) is another variant of

$$xf(y) - yf(x) = (x - y)g(x + y)$$

and it is obtained by replacing the term $g(x+y)$ by the Cauchy difference of g, that is, $g(x+y) - g(x) - g(y)$. The following theorem is needed to establish the next result.

Theorem 4.4.1. *The functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ satisfy the functional equation*

$$g(x) - g(y) = (x - y)f(x + y) + (x + y)f(x - y), \tag{4.4.5}$$

for all $x, y \in \mathbb{R}$, if and only if,

$$f(x) = ax^2 + A(x), \quad \text{and} \quad g(x) = 2ax^4 + 2xA(x) + b, \tag{4.4.6}$$

where $A : \mathbb{R} \longrightarrow \mathbb{R}$ is additive, a and b are arbitrary constants.

Proof. Letting $y = 0$ in Equation (4.4.5), we see that

$$g(x) = 2xf(x) + b, \tag{4.4.7}$$

where $b = g(0)$. Substitution of Equation (4.4.7) into Equation (4.4.5) yields

$$2xf(x) - 2yf(y) = (x - y)f(x + y) + (x + y)f(x - y). \tag{4.4.8}$$

Letting $x = y$ in Equation (4.4.5), we get $f(0) = 0$. Next, we let $x = 0$ to obtain

$$g(y) = yf(y) - yf(-y) + b.$$

This with Equation (4.4.7) yields $f(-y) = -f(y)$ for $y = 0$. In Equation (4.4.8), we replace y by $y + z$ and obtain

$$\begin{aligned} & 2xf(x) - 2(y+z)f(y+z) \\ = \; & (x - y - z)f(x + y + z) + (x + y + z)f(x - y - z). \end{aligned} \tag{4.4.9}$$

Similarly replacing y by z and x by $x + y$ in Equation (4.4.8), we get

$$\begin{aligned} & 2(x+y)f(x+y) - 2zf(z) \\ = \; & (x + y - z)f(x + y + z) + (x + y + z)f(x + y - z). \end{aligned} \tag{4.4.10}$$

Adding Equation (4.4.9) and Equation (4.4.10), one obtains

$$\begin{aligned} & 2xf(x) - 2zf(z) + 2(x+y)f(x+y) - 2(y+z)f(y+z) \\ = \; & (x + y + z)[f(x - y - z) + f(x + y - z)] \\ & +2(x - y)f(x + y + z) \end{aligned} \tag{4.4.11}$$

Using Equation (4.4.8) twice to replace the terms on the left side of Equation

(4.4.11), we get

$$\begin{aligned} & (x-z)f(x+z)+(x+z)f(x-z) \\ & +(x-z)f(x+2y+z)+(x+2y+z)f(x-y) \\ = \;& (x+y+z)[f(x-y-z)+f(x+y-z)] \\ & +2(x-z)f(x+y+z) \end{aligned} \tag{4.4.12}$$

Now letting $z = -x$ in Equation (4.4.12), we see that

$$2xf(2y)+2yf(2x) = 4xf(y)+y\;f(2x-y)+f(2x+y)\;. \tag{4.4.13}$$

Substitution of $u = 2x$ in Equation (4.4.13) yields

$$\frac{u}{y}f(2y)+2f(u)-\frac{2u}{y}f(y) = f(u-y)+f(u+y), \tag{4.4.14}$$

for $y = 0$. Interchanging u and y (so now $u = 0$ as well) and using that f is odd, we have

$$\frac{y}{u}f(2u)+2f(y)-\frac{2y}{u}f(u) = f(u+y)-f(u-y). \tag{4.4.15}$$

Adding Equation (4.4.14) and Equation (4.4.15), we obtain

$$\begin{aligned} f(u+y)-f(u)-f(y) \;=\; & \frac{u}{2y}\;f(2y)-2f(y) \\ & +\frac{y}{2u}\;f(2u)-2f(u)\;, \end{aligned} \tag{4.4.16}$$

for all $u, y \in R \setminus \{0\}$. Define

$$h(x) = \frac{f(2x) - 2f(x)}{2x}, \quad \text{for } x = 0. \tag{4.4.17}$$

Note that h is even since f is odd. Equation (4.4.17) and Equation (4.4.16) yield

$$f(u + y) - f(u) - f(y) = uh(y) + yh(u), \tag{4.4.18}$$

where $u, y \in R \setminus \{0\}$. Let

$$H(u, v) = f(u + v) - f(u) - f(v) \tag{4.4.19}$$

be the Cauchy difference of f. Hence H satisfies

$$H(u + v, w) + H(u, v) = H(u, v + w) + H(v, w), \tag{4.4.20}$$

for all $u, v, w \in R$. From Equation (4.4.19), Equation (4.4.18) and Equation (4.4.20), after some simplification, we see that

$$w[h(u + v) - h(u) - h(v)] = u[h(v + w) - h(v) - h(w)], \tag{4.4.21}$$

for $u, v, w, u + v, v + w \in R \setminus \{0\}$. Letting $w = v$ in Equation (4.4.21), we get

$$v \; h(u + v) - h(u) - h(v) \; = u \; h(2v) - 2h(v) \; .$$

Interchanging u and v, we obtain

$$v[h(v+u) - h(v) - h(u)] = v[h(2u) - 2h(u)]. \qquad (4.4.22)$$

Thus we have $u^2[h(2v) - 2h(v)] = v^2[h(2u) - 2h(u)]$. Hence

$$h(2u) - 2h(u) = 6au^2, \quad \text{for all } u = 0, \qquad (4.4.23)$$

where a is a constant. Using Equation (4.4.23) in Equation (4.4.22), we have

$$h(u+v) - h(u) - h(v) = 6auv,$$

$u, v \in R \setminus \{0\}$, which can be rearranged as

$$h(u+v) - 3a(u+v)^2 = h(u) - 3au^2 + h(v) - 3av^2. \qquad (4.4.24)$$

Letting

$$A_0(u) = h(u) - 3au^2, \qquad (4.4.25)$$

in Equation (4.4.24) we get

$$A_0(u+v) = A_0(u) + A_0(v),$$

for all $u, v, u+v \in R \setminus \{0\}$. Since

$$A_0(1) = A_0(u+1-u) = A_0(u) + A_0(1) + A_0(-u),$$

we get $A_0(-u) = -A_0(u)$ for all $u = 0, 1$. But $A_0(2) = 2A_0(1)$ and $A_0(-2) = 2A_0(-1)$, that is $A_0(-1) = -A_0(1)$. Hence $A_0(-u) = -A_0(u)$ for all $u = 0$. Changing u to $-u$ in Equation (4.4.25) and using the fact that h is even, we obtain

$$h(x) = 3ax^2, \quad \text{for all } x = 0. \tag{4.4.26}$$

Letting Equation (4.4.26) into Equation (4.4.18) we obtain

$$f(u+v) - f(u) - f(v) = 3au^2v + 3auv^2,$$

for all $u, v \in R \setminus \{0\}$. This in turn gives a Cauchy equation (see [46])

$$f(u+v) - a(u+v)^3 = f(u) - au^3 + f(v) - av^3,$$

and hence

$$f(u) = au^3 + A(u),$$

for all $u \in R$ since $f(0) = 0$. Here $A : R \longrightarrow R$ is an additive function. From Equation (4.4.7), we get the form of g as asserted in (4.4.6). □

Now we present the general solution of the functional equation (4.4.27) without any regularity assumption on the unknown functions.

Theorem 4.4.2. *The functions $f, g : R \longrightarrow R$ satisfy the functional equation*

$$xf(y) - yf(x) = (x-y)[g(x+y) - g(x) - g(y)] \tag{4.4.27}$$

for all $x, y \in \mathbb{R}$ *if and only if*

$$\begin{cases} f(x) = 3ax^2 + 2bx^2 + cx + d, \\ g(x) = -ax^3 - bx^2 - A(x) - d, \end{cases}$$

where $A : \mathbb{R} \longrightarrow \mathbb{R}$ *is additive and* a, b, c, d *are arbitrary constants.*

Proof. Letting $x = 0$ in Equation (4.4.27) yields $yf(0) = -yg(0)$ for all $y \in \mathbb{R}$. Choosing $y = 0$, we see that $f(0) = -g(0)$. Substituting $y = -x$ in Equation (4.4.27), we obtain

$$x[f(x) + f(-x)] = 2x[g(0) - g(x) - g(-x)],$$

for all $x \in \mathbb{R}$. Hence

$$f(x) + f(-x) = 2[g(0) - g(x) - g(-x)], \tag{4.4.28}$$

for all $x, y \in \mathbb{R} \setminus \{0\}$. But in view of $f(0) = -g(0)$, we see that Equation (4.4.28) holds for all $x \in \mathbb{R}$. Next replacing x by $-x$ in Equation (4.4.27), we get

$$yf(-x) + xf(y) = (y + x)[g(y - x) - g(-x) - g(y)]. \tag{4.4.29}$$

Substituting Equation (4.4.27) from Equation (4.4.29), we have

$$
\begin{aligned}
& y[\boldsymbol{f}(x) + \boldsymbol{f}(-x)] \\
= \; & y[g(x+y) + g(y-x) - g(x) - g(-x) - 2g(y)] \\
& +x[g(y-x) - g(x-y) + g(x) - g(-x)], \qquad (4.4.30)
\end{aligned}
$$

for all $x, y \in$ R. Using Equation (4.4.28) in Equation (4.4.30), we obtain

$$
\begin{aligned}
& 2y[g(0) - g(x) - g(-x)] \\
= \; & y[g(x+y) + g(y-x) - g(x) - g(-x) - 2g(y)] \\
& +x[g(y-x) - g(x+y) + g(x) - g(-x)],
\end{aligned}
$$

that is

$$
\begin{aligned}
& y[g(x+y) + g(y-x) + g(x) - g(-x) - 2g(y) - 2g(0)] \\
= \; & x[g(y+x) - g(y-x) - g(x) - g(-x)]. \qquad (4.4.31)
\end{aligned}
$$

If $g(x)$ satisfies Equation (4.4.31), so does $g(x) - g(0)$. Thus without loss of generality we may suppose that $g(0) = 0$. Then Equation (4.4.31) reduces to

$$
\begin{aligned}
& y[g(x+y) + g(y-x) - g(x) - g(-x) - 2g(y)] \\
= \; & x[g(y+x) - g(y-x) - g(x) + g(-x)]
\end{aligned}
$$

which is

$$
\begin{aligned}
&(y-x)g(x+y)+(x+y)g(y-x)+y[g(x)+g(-x)] \\
=\;& 2yg(y)-x[g(x)-g(-x)]. \qquad (4.4.32)
\end{aligned}
$$

Replacing y by $-y$ in Equation (4.4.32), we obtain

$$
\begin{aligned}
&-(y+x)g(x-y)+(x-y)g(-y-x) \\
=\;& y[g(x)+g(-x)]-2yg(-y)-x[g(x)-g(-x)]=0. \qquad (4.4.33)
\end{aligned}
$$

Subtracting Equation (4.4.33) in Equation (4.4.32), we have

$$
\begin{aligned}
&(y+x)[g(y-x)+g(x-y)]+(y-x)[g(x+y)+g(-y-x)] \\
&=2y[g(y)+g(-y)]-2y[g(x)-g(-x)]=0. \qquad (4.4.34)
\end{aligned}
$$

Defining

$$\varphi(x)=g(x)+g(-x) \qquad (4.4.35)$$

we see that φ is even and using it in Equation (4.4.34) we obtain

$$(y+x)\varphi(y-x)+(y-x)\varphi(x+y)=2y\varphi(y)-2y\varphi(x). \qquad (4.4.36)$$

Interchanging x and y in Equation (4.4.36) and using the fact that φ is even, we get

$$(y+x)\varphi(y-x)-(y-x)\varphi(x+y)=2x[\varphi(x)-\varphi(y)]. \qquad (4.4.37)$$

Adding Equation (4.4.36) and Equation (4.4.37), we have

$$(y+x)\varphi(y-x)=(x-y)[\varphi(x)-\varphi(y)]. \qquad (4.4.38)$$

Letting $2u=x+y$ and $2v=x-y$ in Equation (4.4.38), we get

$$u\varphi(2v)=v[\varphi(u+v)-\varphi(u-v)].$$

Again interchanging u with v in the above and using the fact that φ is even, we get

$$v\varphi(2u)=u[\varphi(u+v)-\varphi(u-v)].$$

Hence from the above two equations, we see that $u^2\varphi(2v)=v^2\varphi(2u)$ that is $\varphi(u)=-2bu^2$, where b is a constant. Therefore in view of Equation (4.4.35), we get

$$g(x)+g(-x)=-2bx^2. \qquad (4.4.39)$$

Now adding Equation (4.4.33) to Equation (4.4.32), we get

$$\begin{aligned}&(y+x)[g(y-x)+g(x-y)]+(y-x)[g(x+y)+g(-y-x)]\\ =\;& 2y[g(y)+g(-y)]-2x[g(x)-g(-x)] \qquad (4.4.40)\end{aligned}$$

Defining

$$\psi(x) = g(x) - g(-x), \tag{4.4.41}$$

we observe that ψ is odd and using **it** in Equation (4.4.40), we see that

$$2x\psi(x) - 2y\psi(y) = (x - y)\psi(x + y) + (x + y)\psi(x - y), \tag{4.4.42}$$

for all $x, y \in$ R. The general solution of Equation (4.4.42) can be obtained from Theorem 4.4.1. Hence, we have

$$\psi(x) = -2ax^3 - 2A(x), \tag{4.4.43}$$

where A is additive and a is an arbitrary constant. Using Equation (4.4.43), Equation (4.4.41) and Equation (4.4.39), we have $g(x) = -ax^3 - bx^2 - A(x)$. Now removing the assumption $g(0) = 0$, we obtain

$$g(x) = -ax^3 - bx^2 - A(x) - d, \tag{4.4.44}$$

where d is an arbitrary constant. Letting Equation (4.4.44) into Equation (4.4.27), and simplifying we have,

$$y[\boldsymbol{f}(x) - 3ax^3 - 2bx^2 - d] = x[\boldsymbol{f}(y) - 3ay^3 - 2by^2 - d],$$

for all $x, y \in$ R. Hence $\boldsymbol{f}(x) - 3ax^3 - 2bx^2 - d = cx$ for $x = 0$. Since

$f(0) = -g(0)$, we get

$$f(x) = 3ax^3 + 2bx^2 + cx + d,$$

for all $x \in \mathbb{R}$. □

The following theorem is established using Theorem 4.4.2.

Theorem 4.4.3. *The functions $f, g, h, k : \mathbb{R} \longrightarrow \mathbb{R}$ satisfy the functional equation*

$$f(x) - g(y) = (x - y)\left[h(x + y) + k(x) + k(y)\right], \qquad (4.4.45)$$

for all $x, y \in \mathbb{R}$ if and only if

$$\begin{cases} f(x) = 3ax^4 + 2bx^3 + cx^2 + dx + \alpha, \\ g(y) = 3ay^4 + 2by^3 + cy^2 + dy + \alpha, \\ h(x) = ax^3 + bx^2 + A(x) + d - 2\beta, \\ k(x) = 2ax^3 + bx^2 + cx - A(x) + \beta, \end{cases}$$

where $A : \mathbb{R} \longrightarrow \mathbb{R}$ is an additive function and $a, b, c, d, \alpha, \beta$ are arbitrary constants.

Proof. Letting $x = y$ in Equation (4.4.45), we see that $f(x) = g(x)$. Hence Equation (4.4.45) reduces to

$$f(x) - f(y) = (x - y)[h(x + y) + k(x) + k(y)]. \qquad (4.4.46)$$

Putting $y = 0$ in Equation (4.4.46), we obtain

$$f(x) = f(0) + x[h(x) + k(x) + k(0)]. \qquad (4.4.47)$$

Letting Equation (4.4.47) into Equation (4.4.46) and rearranging, we obtain

$$\begin{aligned} & y[h(x) + k(x)] - x[h(y) + k(y)] \\ = \ & (x - y)[h(x + y) - h(x) - h(y) - k(0)]. \end{aligned} \qquad (4.4.48)$$

Defining

$$\varphi(x) = h(x) + k(x) \quad \text{and } g(x) = -h(x) - k(0), \qquad (4.4.49)$$

and using Equation (4.4.49) in Equation (4.4.48), we have

$$x\varphi(y) - y\varphi(x) = (x - y)[g(x + y) - g(x) - g(y)], \qquad (4.4.50)$$

for all $x, y \in$ R. The general solution of Equation (4.4.50) can be obtained from Theorem 4.4.2. Therefore

$$\begin{cases} \varphi(x) = 3ax^3 + 2bx^2 + cx + d_0, \\ g(x) = -ax^3 - bx^2 - A(x) - d_0, \end{cases} \qquad (4.4.51)$$

where a, b, c, d_0 are constants. From Equation (4.4.49) and Equation (4.4.51), we obtain $k(x) = \varphi(x) + g(x) + \beta$ where $\beta = k(0)$. Now using Equation

(4.4.51), we obtain

$$k(x) = 2ax^3 + bx^2 + cx - A(x) + \beta. \tag{4.4.52}$$

Again from Equation (4.4.49) and Equation (4.4.52), we have

$$h(x) = ax^3 + bx^2 + A(x) + d - 2\beta,$$

where $d = d_0 + \beta$. Using Equation (4.4.47), Equation (4.4.49) and Equation (4.4.51), we get

$$f(x) = 3ax^4 + 2bx^3 + cx^2 + dx + \alpha,$$

where $\alpha = f(0)$. The proof is now complete. □

Remark 4.4.1. *It follows easily from Theorem 4.4.3 that the solution of Equation (4.4.7) is*

$$g(x) = 3ax^4 + 2bx^3 + cx^2 + dx + \alpha$$

and

$$f(x) = 12ax^3 + 6bx^2 + 2cx + d$$

as predicted. Note that the general solution is obtained without any regularity assumption on f and g.

Theorem 4.4.4. *The functions $f, g, h, \varphi, \psi : \mathbb{R} \longrightarrow \mathbb{R}$ satisfy the functional equation*

$$f(x) - g(y) = (x - y)[h(x + y) + \varphi(x) + \psi(x)] \tag{4.4.53}$$

for all $x, y \in \mathbb{R}$ if and only if:

$$\begin{cases} f(x) = 3ax^4 + 2bx^3 + cx^2 + dx + \alpha, \\ g(y) = 3ay^4 + 2by^3 + cy^2 + dy + \alpha, \\ h(x) = ax^3 + bx^2 + A(x) + d - 2\beta, \\ \varphi(x) = 2ax^3 + bx^2 + cx - A(x) + \beta + \gamma, \\ \psi(y) = 2ay^3 + by^2 + cy - A(y) + \beta - \gamma, \end{cases} \tag{4.4.54}$$

where $A : \mathbb{R} \longrightarrow \mathbb{R}$ is an additive function and $a, b, c, d, \alpha, \beta, \gamma$ are arbitrary constants.

Proof. First letting $x = y$ in Equation (4.4.53) we see that $f = g$. Interchanging x with y in Equation (4.4.53) and using the fact $f = g$, we get

$$f(x) - g(y) = (x - y)[h(x + y) + \varphi(x) + \psi(x)]. \tag{4.4.55}$$

Adding Equation (4.4.55) to Equation (4.4.53) and using $f = g$ we get

$$\psi(x) - \varphi(x) = \psi(y) - \varphi(y),$$

for all $x, y \in \mathbb{R}$. Thus

$$\psi(x) = \varphi(x) - 2\gamma, \tag{4.4.56}$$

where γ is an arbitrary constant. Putting Equation (4.4.56) into Equation

(4.4.53), we have

$$f(x) - f(y) = (x - y)[h(x + y) + \varphi(x) + \varphi(y) - 2\gamma].$$

From Theorem 4.4.4 and Equation (4.4.56), we have the asserted solution (4.4.54). □

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