WADIA SABRA: UNIVERSAL SCALE AND NEW UNIT OF INTERVAL MEASUREMENT

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by

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To my uncle and spiritual father, Rev. Dr. Charbel Abi Khalil.

My mind is sculpted by his words.

Acknowledgments

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Abstract

In his writings, Wadia Sabra (1876-1952) announced the creation of a “Universal Scale” and a “New Unit of Interval Measurement”. However, it is noteworthy that this particular work lacks precise information related to its numbers. This study aims to reconstruct and reveal the related numbers and tables, by researching Wadia Sabra’s archive. Also, this study reviews the acoustical and musical characteristics of this particular work of Sabra, its validity, and its possible new input to the musical field.

Chapter I: Introduction

Wadia Sabra (1876-1952), known for composing the Lebanese national anthem and founding the Lebanese National Conservatory, was an active person on the Lebanese and Arab musical scene during the first half of the 20th century. After his death, his reputation has faded and his archive was kept hidden. However, 50 years later, his archive was rediscovered, classified and stored by the CPML[1]. Thus, more than 950 documents, scores, press articles, correspondences, concert programs and conferences, constitute a corpus worth studying.

Problematic

Beside his numerous compositions, Wadia Sabra had developed many musicological works. Using his double knowledge in both western and oriental music, Sabra always tried to reveal the relation between these two types of music. In his musicological studies, he worked on 3 dimensions. First, he created a measurement instrument, a sort of a Monochord with a keyboard, along with the construction of an experimental "Commatic Piano". His purpose was to play authentically different oriental scales. Second, he invented a new unit of interval measurement along with a "Universal scale" which he described suitable to combine all Western, Arabic, Greek, Indian, and Chinese scales. Finally, he even went farther, by proving that Arabic music was the origin of the western Art. In the context of the second dimension, Sabra published a leaflet[2] and declared that he has discovered a New Unit of music Intervals

[1] Centre du Patrimoine Musical Libanais, founded in 2013, located at Jamhour, Lebanon, brings together the musical work of Lebanese composers and performers. Whether Western or Eastern, classical or contemporary, folk or liturgical, this music is now available to the public. The archive of Wadia Sabra, came to public and was stored at the CPML on February 2016.

[2] "Au monde scientifique et artistique, Nouvelle unité de mesure des intervalles musicaux, Gamme Universelle" Translated: "To Scientific and Artistic world, A new unit of music interval measurement, Universal Scale" (n.d.)

Measurement, along with a Universal Scale. In this leaflet he described briefly his discovery, he talked about its importance and characteristics, and promised to reveal the exact numbers of the Universal Scale in a form of 6 tables. Although this promise was re-declared in different writings and following occasions, Wadia Sabra never published the numbers, neither did anyone later. This suggests that either the work was complete, but didn't get the opportunity to be published, or the work was incomplete.

The problematic is that Sabra promised in different writings to reveal the numbers of his Universal scale which he never did. The aim of this thesis is to reveal the numbers which Sabra mentioned but never wrote down.

Research Questions

This research suggests many questions: How did this theory about a Universal Scale evolve during Sabra's lifetime? More importantly, why didn't he publish its numbers? What were the opinions of his peers? Can his archive be useful to reveal the numbers of his scale, and what are the risks that could obstruct such research? Moreover, after nearly a century, is this theory still valid among the recent units of measurement in addition to the expansion of new musical styles?

Statement of the hypothesis

Sabra's universal scale is a new system of pitch[3] quantification, and the new unit of interval measurement should be accurate enough, so all the intervals of the different musical systems can be exactly included in a finite octave division. Based on Sabra's announcements and the available sources[4], we should be able to discover and reconstruct this scale from his manuscripts.

Null Hypothesis

Wadia Sabra's work was inconclusive; therefore, it will be demonstrated it by numbers.

Purpose of the study

This research will expose Sabra's theory, which he called revolutionary. While assembling and studying Sabra's personal calculation notes in addition to the related published material, articles, correspondences and reactions, this research will aim to publish the main tables that Sabra described in his writings, and discover by numbers the new unit of interval measurement. Finally, we will also review its validity and criticize its raison d'être.

[3] Pitch is the position of a single sound relatively to a complete range of sound. Sounds are higher or lower in pitch depending on the frequency of the relative sound waves
[4] Unless another source provides new clarifications about the subject

Significance of the study

This study, while aiming to evaluate a particular work of Wadia Sabra in its scientific and historical context, is not meant to judge the scientific and musical quality of the author. This research could revive Sabra's theory as a starting point for new researches, in order to offer new opportunities and tools for modern composers, both Oriental and Western. The addition that Sabra's theory could bring to the Oriental music is the possibility to review its acoustical bonds with other musical systems, which could be used to import and develop new possibilities within this same type. Also, Sabra's theory could add to the Western music a new quantified microtonal system, which could offer new tools for contemporary composers.

Thesis Plan

Following the introduction in chapter 1, where we exposed the subject to be studied in this paper, we will start in chapter 2, by writing a brief overview about Wadia Sabra's life, then, after having an apercu about his musicological activity in general, we will gather from the literature, all available particular information about his "Universal Scale". This information will be categorized in order to have a better understanding on the development of his work. Next, in chapter 3, we will develop the research methods used in this study and the process of collecting unpublished data from Sabra's archive. As for chapter 4, we will discuss the analyzed data and then present the results obtained. Finally, the last chapter will be dedicated to the conclusion and it will answer the proposed research questions.

General context

As a generalized definition, a scale is a series of sounds arranged by definite frequency intervals, suitable for musical purposes (Oslon, 2013). The interval divisions, whether a division of a third, fourth, fifth, octave, etc... have constantly evolved. Ancient Greece was marked by the works of Pythagoras and Aristoxenus (Thomas, 2017). All divisions took into consideration the harmonic properties of the sound in addition to conventional ratios and proportions (Thomson, 2001). The octave division remained an open musicological subject, and Sabra's work fits perfectly in this context.

Chapter II: Literature Review

I- Life

Wadia Sabra was born on February 23, 1876 at Aïn al-Jadidé, a region of Mount Lebanon, and died on April 11, 1952. He was raised in a well-educated family. His father was Girges Sabra, director of the British school of Beirut[5], and his mother was Sarah Sarkis, a teacher at the same school. She was a poet, spoke five languages, and was also a musician. The young Wadia Sabra used to sing as an Alto in his school choir, he had two music teachers: Mr. Walker, and Mr. Day (Kayali, 2018)[6]. Mr. Day encouraged Sabra in pursuing a musical career. In 1893, Sabra moved to France, where he was tutored by many renowned professors at that time[7]. His studies were focused on Piano, Pipe Organ, Singing, Theory, Harmony and Composition. In Sabra's correspondences we can find many positive feedbacks and attestations about his musical progress and talent. After returning to Lebanon, Wadia Sabra founded Dar al-Mousiqa al-Loubnania[8] in 1910, where he started and developed his career. His constant musical activities were mainly divided on Administration, Composition, Performance, and Musicology.

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[5] Later Lebanese Evangelical School for Boys and Girls - Loueizeh

[6] Kayali's book, "Figures musicales du liban – Wadia Sabra" (2018), will be mentioned many times in this section as it is the only book containing an important information about Wadia Sabra's biography, which was gathered from the recently found archive.

[7] Paul Vidal (1863-1931), Piano and Accompaniment. Max d'Ollone (1875-1959), Harmony. Charles-Marie Widor (1844-1937), Pipe Organ. Louis-Albert Bourgault Ducoudray (1840-1910), History of Music. Florentin Numa Augez (1847-1903), Singing. Alfred Giraudet (1845-1911), Singing. Albert Lavignac (1846-1916), Harmony and Composition. Charles Lenepveu (1840-1910), Composition. Alexandre Guilmant (1837-1911), Pipe Organ

[8] Later Lebanese National Conservatory

It was Albert Lavignac[9], who incited Sabra toward the musicological field, in particular toward research in the oriental music. Once, Lavignac told him that he wasn't able to find two measures of Oriental music, everything that existed then was counterfeit. Thus, Sabra proceeded his studies on the Oriental modes using ancient and contemporary references. Alternatively, Wadia Sabra had an active composition career: he composed different Hymns (Lebanese, Ottoman, Syrian…), patriotic songs, and operas in French, Arabic and Turkish, he also composed sacred vocal music, and arranged other Oriental popular songs.

In her book, "Figures musicales du liban – Wadia Sabra" (2018), Zeina Kayali concluded the part dedicated to Sabra's life, with a section entitled "Rêves Brisés", which literally translates to: Broken Dreams. In this section, she expressed Wadia Sabra's wish to make of his country the lighthouse of the Middle East in the musicological field. While she considered him a musical polyglot, who had an ambition to promote Western music in the Orient, and the Oriental music in the West, she mentioned Sabra's dream about founding a Lebanese musicological journal, but this dream was never realized. Moreover, in the same section, Kayali dedicated an important part to talk about a specific dream of Wadia Sabra, which was about organizing a Universal Musical Congress under the aegis of the UNESCO[10]. He also promised that during this Congress, he will disclose the secret of his discovery. Unfortunately, this never happened.

[9] Albert Lavignac (1846-1916), was a French musicologist and composer.
[10] The United Nations Educational, Scientific and Cultural Organization, was born on 1945, it has 195 Members and more than 50 field offices around the world to date.

II- Musicological Work

This section covers Sabra's musicological development. We will start by the "Monochord Keyboard" and "Commatic Piano", in addition to Sabra's researches on the Oriental scales and his contribution to the Congress of Cairo – 1932. Then, we will present the published material related to his theory about the "Universal Scale" and the "New Unit-Interval".

The "Monochord Keyboard" and "Commatic Piano"

Mohamed Saifallah Abderrazzak[11] considered the Monochord Keyboard as one of the most important realizations of Wadia Sabra. He considered that this measurement instrument was indispensable for the elaboration of his scale. Sabra collaborated for the elaboration of this instrument with Gustave Lyon, a French acoustician and proprietary of Pleyel piano factory[12]. Later, Sabra has built a Commatic Piano[13], which the first model was entirely made by Bechara Ferzane in Lebanon, other model was developed with the help of Gustave Lyon, and the Pleyel Piano Factory[14] (Kheirallah, 1995). The piano and its plans have disappeared, we still have just a photo of it from the Congress of Cairo in 1932, where it was demonstrated.

If we sort Sabra's realizations based on a timeline, we notice that the "Commatic Piano" appeared many years before the "Universal Scale". Also, while demonstrating his "Commatic

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[11] Abderrazzak wrote the musicological postface of Kayali's book "Figures musicales du Liban - Wadia Sabra" (2018).
[12] Pleyel, is a French piano factory that began in 1807 with Ignace Pleyel
[13] Considered as an oriental piano, suitable to play authentically the Arabic Scales as noted by Sabra.
[14] Further to our request, Pleyel representatives replied that they weren't able to find any information about Sabra's Commatic piano in their archive, and they confirmed that they had no knowledge about this specific project. However, they noted that only a small amount of Pleyel's archive remained from the beginning of the 20th century.

Piano" Sabra mentioned that it contains 90 notes in the octave[15], without giving any information about the exact numerical proportions. The secrecy around the numerical proportion, would suggest a certain link between Sabra's 90 divisions of the octave and the later "Universal Scale", which are both based on microtones. Moreover, Sabra's work about the "Universal Scale" may have been developed and generalized from this particular development of the "Commatic Piano"

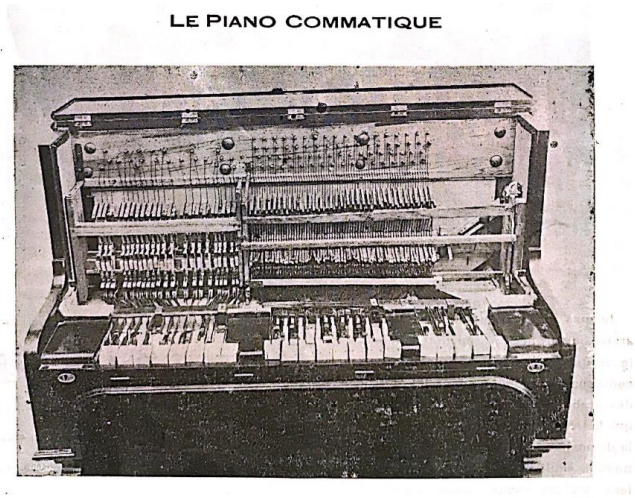


Figure 1: Sabra's Commatic Piano

Figure 2: The Monochord Keyboard

[15] The 90 divisions of the octave are mentioned in the next paragraph, in the context of Sabra's participation in the Congress of Cairo-1932.

Research on the Oriental Scales and contribution to the Congress of Cairo 1932

Sabra has performed his researches on the Oriental scales aiming to understand its composition, and its links with the natural harmony. To do so, he studied the ancient writings such as Al-Farabi's[16], in addition to more contemporary works by Kamil al-Khula'î, Alexandre Chalfoun and R.P. Collangette. (Kayali, 2018). He always criticized the Oriental instruments for not being developed well to play the Oriental Music[17], and considered that their imprecision is "disgusting for the educated ears" (Sabra, 1929). In another writing he considered that Arabic music, which was too long underestimated, should finally be rehabilitated (Sabra, 1941).

Sabra always opposed the use of the 24 equal quarter tone system in the Arabic music, and this particular point was expressed clearly during his participation in the international convention for Arabic Music, Cairo 1932, were he headed the Lebanese delegation[18]. One principal objective of the convention was to set the scale and to determine a standard musical notation. This objective was problematic and was divided by two antagonist opinions (Maalouf, 2002). Further to his attendance in the Congress, Sabra wrote a report Entitled "Congrès de Musique Arabe du Caire - Etude Détaillée sur les Travaux des Commissions - Considérations & Conclusions", translated to: "Congress of Arabic Music of Cairo- Detailed Study on the work of the commissions – Considerations and Conclusions".

[16] Abū Naṣr al-Fārābī, (878-950), was a member of the eastern group of Moslem Philosophers. He is also considered an important musical theorist. Among his works is Kitab Mausiqi al-Kabir (The Grand Book of Music)

[17] In the sources, the terms Oriental and Arabic music were both meant Arabic music.

[18] The Lebanese Delegation to the Congress of Cairo was formed by: Wadia Sabra, Eduard Kadhagi and Bechara Ferzan

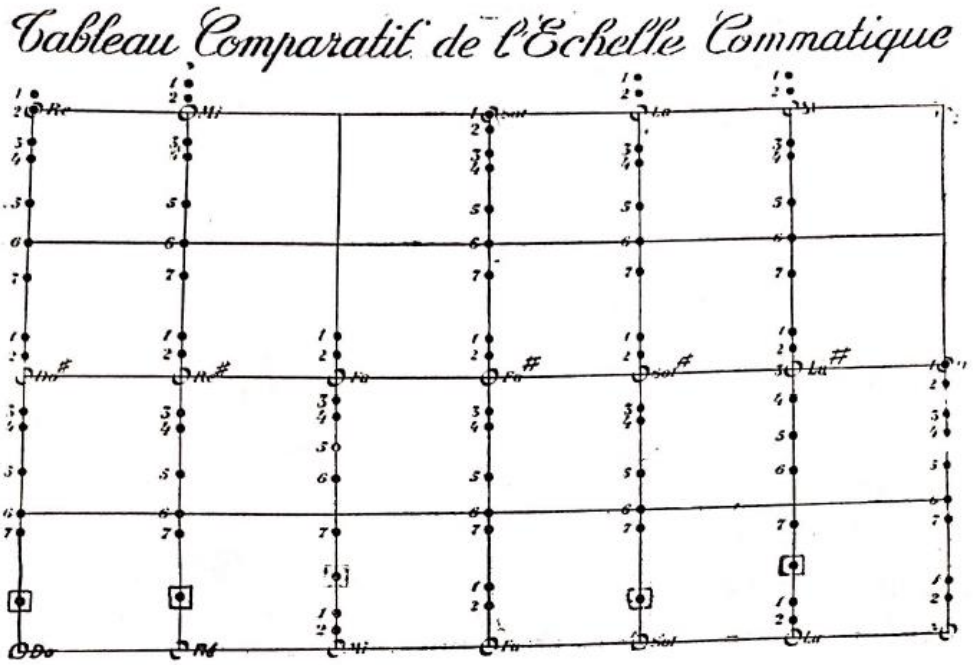
In the above-mentioned report, Wadia Sabra considered that setting the Oriental Scale was the most important topic of the congress, however the committee responsible of the subject failed to take any decision and kept all questions suspended (SABRA, 1932). After explaining the opinions of the participants, he mentioned his approval to the "Arabic Just Scale"[19], which he proposed during the Congress. His proposition consisted of a system of 90 commas. He did not present a tangible reason for choosing this particular 90 divisions, neither the logic behind it. He considered that these 90 divisions of the octave will reproduce authentically the Oriental Music. He also added that this division gradually shrinks by a certain numerical proportion which will be revealed at a later stage (Sabra, 1932).

The Commatic Scale, as described by Sabra is translated and presented as follows:

"… the Arabic musical scale is composed by:

· *Tones physically major (8/9th of the length of a string)*	…	*12*
· *Tones physically minor (9/10th of the length of a string)*	…	*12*
· *Each of these 24 tones has a diminished tone of around 1/50th of a whole tone*	…	*24*
· *The tones emanating from the use of the 3 types of cigah[20] (the fourth being among the diminished)*	…	*36*
· *Notes omitted from the physical scale*	…	*5*
· *The octave of the fundamental*	…	*1*
Total	…	*90 "*

[19] Sabra used the term in French, "La gamme juste arabe"
[20] Sabra mentioned the different types of the Cigah at the congress of Cairo-1932



Légende du Tableau Comparatif

1.- Les rondes représentent les notes de la gamme tempérée.

2.- Les noirs représentent les 7 commas contenus dans chaque demi-ton.

3.- Le signe carré indique les 5 notes empruntées à la gamme physique.

4.- Les notes *do*, *sol*, *ré*, *la*, *mi*, précédées du chiffre 1 représentent les notes justes selon l'accord du quatuor à cordes.

5.- La distance qui sépare les notes *do*, et *ré*, représente l'intervalle théorique de ces deux notes.

6.- La position du *do dièze* indique celle du demi-ton tempéré.

7.- Le point noir no. 6 représente approximativement les trois quarts du ton.

8.- La distance entre le *do* fondamental et le signe placé immédiatement au-dessus représente la distance du comma connu figurant la 9ème partie du ton.

تفسير جدول المقابلة

١ ـ علامات « الروند » تمثل علامات سلم البيانو العادي

٢ ـ العلامات السوداء تمثل السبع كومايات الموجودة في كل نصف برج

٣ ـ العلامة (المربعة) تدل على الخمس علامات المأخوذة من السلم الطبيعي

٤ ـ العلامات دو ، صول ، دي ، لا ، مي ، التي يتقدمها رقم ١ تمثل العلامات المضبوطة وفقاً لموزنة الآلات الوترية في الاوركسترا

٥ ـ المسافة التي تفصل بين العلامتين (دو) و (ري) تمثل البعد النظري بين هاتين العلامتين

٦ ـ موقع دو (دييز) يدل على موقع نصف الصوت في البيانو العادي

٧ ـ العلامة السوداء رقم ٦ تمثل تقريباً ثلثة ارباع البرج

٨ ـ المسافة بين دو الاساسية والعلامة الموضوعة فوقها تماماً تمثل مسافة الكوما المعروفة بتسع البرج

Figure 3 Sabra's Commatic Scale – A comparative Table

The Universal Scale and New Unit of Interval Measurement

In order to promote his theory, Wadia Sabra published many writings, and held multiple conferences treating his subject. During our research, we started by reading Sabra's publications and writings presented in his conferences. Then, in order to have a better understanding of the theory, we categorized his approach into four axes:

1. The problematic

Wadia Sabra presented the problematic residing in the 12 divisions of the octave [21] (Sabra, 1940). He detailed his statement by explaining the old musicological debate related to temperaments calculations. His approach was as follows:

First, he mentioned the existence of the Pythagorean comma[22],

$$Pythagorian\ Comma = \frac{12\ perfect\ fifths}{7\ octaves} = \frac{(\frac{3}{2})^{12}}{2^7} = \frac{531441}{524288} = 1.013643265$$

Then he added[23]:

"Nous n'ignorons pas, par ailleurs, que la tierce naturelle dont le rapport égale 5/4 est la 8^e quinte d'une série de 12 autres quintes descendantes dont la dernière note est inferieure d'un comma a la note fondamentale. Ces notes de la série descendante sont d'un

[21] "Exposé d'un Nouveau Système perfectionné de partage des 12 demi-tons de l'octave"
[22] The Pythagorean Comma, is the difference between twelve just perfect fifths and seven octaves.
[23] Idem.

comma plus bas que les mêmes notes obtenues par quintes ascendantes ; et alors que la fondamentale et la quinte de l'accord parfait majeur sont prises dans la série ascendante, la tierce majeure est prise dans la série descendante. »

Its English translation:

"We are also aware that the natural third whose ratio equals 5/4 is the 8th fifth in a series of 12 other descending fifths, whose last note is one comma less than the fundamental. These notes of the descending series are one comma lower than the same notes obtained by ascending fifths; and while the fundamental and the fifth of a perfect major chord are selected from the ascending series, the major third is taken from the descending series.[24] "

It is important to note that the ratio of the 8th fifth in a series of 12 descending fifths is not equal to 5/4

$$(\frac{3}{2})^{-8} * 2^5 = \frac{8192}{6561} = 1.24859015393995 < \frac{5}{4}$$

Difference = $\frac{\frac{5}{4}}{\frac{8192}{6561}} = 1.00112915039062 = \frac{1}{12.0078}\ of\ the\ pythagorian\ comma$

Sabra added: *"…the ancients and the moderns [Musicians] have both agreed, for practical purposes, not to exceed the number of 12 notes per octave"*. According to him, that's where the problem of the octave division resides.

[24] The note which Sabra referred to as a Major third is a Diminished Fourth

Moreover, Sabra always had a dream about unifying the different musical systems[25].
He considered that every science has a universal system, but unfortunately, music doesn't, even though it should be treated as a universal language (Sabra, 1934). This statement was in the context of a comparative study between different musical systems.[26]

These statements indicate that Sabra started from an existing problematic that was debated for centuries, and it is the composition of the scale. However, in addition to this challenge, he enlarged his subject to include the unification of different music systems. Sabra emphasized the need to define in an exact manner, by a mathematical formula, all the degrees that constitute these different scales (Sabra, n.d.)[27]. Hence, a new unit of interval measurement, able to represent all the degrees in the different music systems, shall be considered.

[25] When sabra talk about the different musical systems, he means the Greek, Oriental, Modern Western, Indian, and Chinese, however he focused his studies on the first three.
[26] "Etude comparative des grands systemes musicaux en usage chez le Hindous, les Chinois, les Arabes, les peules de l'Occidents ainsi que ceux de l'Anciènne Grèce."(1934) Translated: "A comparative study of the major Music systems used by the Hindus, Chinese, Arabs, and the peoples of the west, as well as the ancient Greece."(1934)
[27] Sabra wrote this statement in his leaflet "Au monde scientifique et artistique, Nouvelle unité de mesure des intervalles musicaux, Gamme Universelle" (n.d) Translated: "To Scientific and Artistic world, A new unit of music interval measurement, Universal Scale" (n.d.). Based on an article in Grove's dictionary of Music and Musicians, written by H.G. Farmer about Wadia Sabra, we estimate that this leaflet was published in the late 1930's

2. Publishing the theory (Numbers, formula, properties, etc…)

In the context of the second axis, it is already known that Sabra didn't publish neither numbers, nor formulas. However, he did mention in his writings many of the proprieties and clues related to his theory. In a leaflet entitled, « Au monde scientifique et artistique, Nouvelle unité de mesure des intervalles musicaux, Gamme Universelle », Sabra cited explicitly the characteristics of the new interval measurement unit. We will write its English translation below.

"The measurement unit which we propose to the scientific and artistic world, has the following properties:

1) *It is contained a whole number of times in the octave interval.*
2) *It is contained a whole number of times in any musical interval belonging to ancient or modern scale, recognized and defined until today.*[28]
3) *The sequence on N intervals equal to the unit, forms a scale of which all the degrees can be obtained by a regular progression of perfect chords starting from the fundamental and ending in the octave.*[29]
4) *This Interval-Unit admits neither multiple, nor submultiple, integer or fractional, having the properties of the unit which have just been enumerated*[30] *".*

[28] It is mathematically impossible, refer to discussion at p.38
[29] Sabra did not explain how can "the sequence of N intervals equal to the unit" form a scale. However, he mentioned that the degrees of the universal scale can be generated, from a sequence of chords containing the fundamental, a pure major third, a perfect fifth and a harmonic seventh. Refer to p. 53
[30] If property 1 and 2 are mathematically impossible, then this property should be invalid as well

In the same leaflet, he also published the nomenclature of the 6 tables of the universal scale.

Below is its English translation:

"We have also drawn up the following tables:

1) *Table giving each degree of the UNIVERSAL SCALE the number of vibrations[31], the length of the string, and the logarithm of the interval it forms with the fundamental. (The fundamental being given by a string of length equal to 100cm, its number of vibrations taken as a unit)[32]*
2) *Table locating each degree of all recognized musical scales, in the UNIVERSAL SCALE.*
3) *Table of chords whose regular sequence generates the totality of the degrees of the UNIVERSAL SCALE*
4) *Table of order allowing to locate the place of each interval in the previous table*
5) *Table of square roots of the ratios defining the intervals.[33]*
6) *Table of the Commatic Scale reproducing, for the first time, the ratio of each comma with the fundamental."*

In another writing[34], Sabra mentioned the existence of 28 types of intervals in his universal system. Another property was also found in a small fascicule[35], he said that the universal

[31] Frequency ratio
[32] Defining the ratios, the string length and the logarithm of each degree of the Universal Scale will give us a clear view on how Sabra divided the Octave. Refer to p.42 & p.49
[33] Sabra did not mention the acoustical value of this table
[34] "Exposé d'un nouveau système perfectionné de partage des 12 demi-tons de l'octave" (Sabra, 1940) Translated: "Presentation of a new and improved system for dividing the 12 semitones of the octave"(Sabra, 1940)
[35] "La Gamme Universelle" (Sabra, n.d.) Translated : "The Universal Scale" (Sabra, n.d.)

scale was based on the Pythagorean system[36] and on all the intervals of both ancient and modern scales. All these intervals represent a small part of the totality of the universal scale intervals.

Presenting only this much information, Sabra justified himself by considering that the scientific utility of this discovery would require, in accordance to its importance, certain precautions before its disclosure. (Sabra, n.d.)

3. Proving its validity by experiments and direct application

Proving a validity of a certain theory without the disclosure of its essence may be unusual. However, Wadia Sabra, mentioned a hint of application in his leaflet[37] to the scientific and artistic world. We will write its English translation:

"Here is an example, taken randomly among so many others, which allows to realize both the exactitude and the facility with which one finds, by the use of the tables of the Universal Scale, the solution of the problems concerning the definition of the intervals using string length, number of vibrations and logarithm.

Let's find the string length, number of vibrations and logarithm of the tempered major third (1000000/12599211) of the Chinese seventh (32768/59049)

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[36] In his writing sabra referred to this system as "Le système de justesse absolue" Translated: "The absolute just system"

[37] "Au monde scientifique et artistique, Nouvelle unité de mesure des intervalles musicaux, Gamme Universelle" Translated: "To Scientific and Artistic world, A new unit of music interval measurement, Universal Scale" (n.d.)

1) *Using the classical method*

a- String Length

$$\frac{32768}{59049} x \frac{10000000}{12599211} x2 = \frac{655360000000}{743970810339}$$

Let 88.0894, string length

b- Number of vibrations

$$\frac{59049}{32768} x \frac{12599211}{10000000} x \frac{1}{2} = \frac{743970810339}{655360000000}$$

Let 1.135209, number of vibrations

c- Logarithm of the number of vibrations

$$Log.1.135209 = 0.055759$$

Logarithm of the number of vibrations of the requested note

2) *Using the tables of the universal scale*

While representing by a and b the order numbers which in the universal scale define respectively the tempered major third and the Chinese seventh, and while searching in the tables, the number corresponding with (a+b), we find, exactly, the same results already found, and without any calculation."

Besides the above text, no other proofs or application were found in his writings.

4. Assessing the added value of the theory, in the application field

In this axis, Sabra has dedicated the biggest part of his writings. The advantages brought by his theory were thoroughly described on different occasions. In a booklet where he compared different temperaments with his new improved system, he started by analyzing the following systems: Just intonation, Pythagorean, Meantone and Equal temperaments, in addition to some contemporary theories at his time: "Gamme Transpositrice"[38], "Gamme Rationnelle"[39], and the "Theory of Fifths" by Sir James Jeans. Then, he introduced his new improved system of dividing the 12 semitones of the octave.

"…il ne reste plus qu'à énumérer les avantages qu'aura le nouveau système sur le système tempéré ainsi que sur tous les autres systèmes employés jusqu'ici :

1) *Le nouveau système perfectionné garde à chacune des notes dont sont formés les accords l'importance émanant du rang qu'elles occupent dans la série des harmoniques…*
2) *Il réhabilite le plus important des harmoniques, la quinte juste, ainsi qu'en partie la septième harmonique.*
3) *Il fait réapparaitre les modes anciens ainsi que les véritables couleurs des gammes.*
4) *Par l'emplie de ses 28 espèces d'intervalles distincts et varies, il met au choix du compositeur, de intervalles ayant plusieurs formes allant de la consonnance parfaite, à la consonnance moyenne et jusqu'à la dissonance, ceci sans augmenter le nombre des 12 demi-tons dans l'octave.*
5) *Il donne la solution du problème de la dualité de la 6eme degré, problème non résolu dans la Gamme Physique*
6) *Il fait ressortir des tierces majeures et mineures plus consonantes que celles de la Gamme Tempéré.*
7) *Autant que la Gamme Tempéré a fait réaliser à l'orchestre moderne de la justesse quant à l'ensemble, le Nouveau Système Perfectionne lui fera réaliser encore plus de justesse et rendra l'accordage des instruments à claviers et à sons fixes plus rationnel.* "

[38] Prudent Pruvost (18..-1960)
[39] Lucien Rouzet (1886-1948)

We will present below the English translation:

"…It only remains to enumerate the advantages that the new system will have on the temperate system as well as on the other systems used until now:

1) *The new improved system keeps each of the notes from which the chords are formed the importance emanating from the rank they occupy in the series of harmonics…*
2) *It rehabilitates the most important of the harmonics, the perfect fifth, as well as in part the harmonic seventh.*
3) *It makes reappear the old modes as well as the real colors of the scales.*
4) *Using its 28 distinct and varied species of intervals, it puts at the choice of the composer, intervals having several forms, ranging from the perfect consonant, to the middle consonant and up to the dissonance, this without increasing the number of the twelve semitones in the octave*
5) *It gives the solution to the problem of duality of the sixth degree[40], an unsolved problem in the Pythagorean scale[41].*
6) *It brings out major and minor thirds more consonant than in the Tempered Scale[42]*
7) *As much as the Tempered Scale has made the modern orchestra achieve overall accuracy, the new improved system will make it achieve even more accuracy and will make the tuning of keyboards and fixed-sounding instruments more rational."*

[40] Sabra did not define this issue.
[41] Sabra uses the term "Gamme Physique"
[42] Sabra referred to the 12 Tone Equal Temperament as the Tempered Scale

Also, in his leaflet which was addressed to the "Scientific and Artistic World", he enumerated many advantages to using his universal system. Sabra considered that the discovery of his unit of interval measurement opens new horizons to theorists, because it is not by coincidence, as he stated, that this unit answers both to musical and mathematical conditions. He added that the existence of a law, unknown back then, showed that all intervals found in ancient and modern modes, Oriental and Western, are integer multiples of the same Unit-Interval (Sabra, n.d.). Sabra also justified the universal property of his scale by stating that any series of N+1 equidistant degree of a unit, create a scale "A". This scale is characterized by the fact that a random degree of a random scale "B", find its exact place in the scale "A" (Sabra, n.d.). In the same context, he assured that the Universal Scale is the solution to the most complex problems in music, such as Ancient modes, Color of scales, origin of tonalities, and designation of monodic music systems in relation with the modern polyphonic system. Moreover, he compared his new scale to Mendeleev Periodic Table,[43] where it can predict the music trends of the next centuries. He believed that if the law, created various existing music systems through the centuries and over the globe, this same law will also govern future music systems, which will necessarily coincide with the Universal Scale.

All of the above phrases were repeatedly stated on different occasions. While reading his conferences, we noticed that he always mentioned the advantages of his theory. Moreover, he also repeated that the "Secret" of the Universal Scale, as he described it, will be revealed, in the right time.

[43] Dmitri Mendeleev (1834-1927), developed the periodic classification of chemical elements

Sabra's Theory in other's Literature

After expressing Wadia Sabra's approach on presenting his own theory, an overview on how this theory was treated by other authors will be performed.

In the musicological Postface, the third part of Kayali's book, "Figures musicales du Liban - Wadia Sabra – Compositeur" (2018), Mohamed Saifallah Ben Abderrazak dedicated an important part to talk about Wadia Sabra's universal scale in the context of his musicological realizations. He considered that the Universal Scale in addition to the unit of interval measurement, were Sabra's most important achievements and the outcome of his researches. He also confirms that Sabra never provided any mathematical formula nor any numerical value for his Unit-Interval.

It is to be noted that Abderrazak's judgment on Sabra's work was based only on his articles, conferences and correspondences, without tackling the numbers and calculation manuscripts. Although his work was not meant to be focused on Sabra's Scale only, but considering the Universal Scale as Sabra's most important achievements, is based only on Sabra's intention as reflected in his writings, but will this hold true?

Marc-Henri Mainguy, while mentioning Wadia Sabra in his book "La Musique au Liban", witnessed that the latter researched patiently to create new bonds between Oriental and Western music conception. He considered that for Sabra, the New Interval Unit was his life discovery, unfortunately, it was abandoned then, despite the valued review of European musical authorities. In the next chapter, an apercu on what was considered "Valued Reviews" will be performed and commented.

Shereen Khairallah also wrote about Sabra's universal scale and considered that it was the result of 30 years of work and it solved the problem of the musical scale. However, she did not define the problem of the musical scale, neither presented any evidence how Sabra supposedly solved it. Also, while developing the subject, she only rephrased Wadia Sabra's publications. Finally, due to the fact that the secret of this universal unit was never disclosed, she wondered if the time had come to search for it. (Kheirallah, 1995)

Theophile Fakhr was a music history teacher at the conservatory. It is mentioned in an untitled and undated document found in the CPML archives, that Sabra trusted him to complete his tables, however, we were not able to find any trace of him elsewhere. We wished to find any link to him in the Lebanese National Conservatory, nevertheless, we failed due to the lack of a proper archive, as we have been informed.

After consulting the literature that tackled this subject, we noticed that the level of scientific information about the topic did not exceed what Sabra had already revealed in his writings.

Chapter III: Methodology

After completing the Literature Review based on the published material that we found on the subject, we will be exposing and analyzing in this chapter, more material from Sabra's Archive. The compiled materials from Wadia Sabra's correspondences, personal calculation notes and other documents will be used in this case study. We will proceed first by exposing the reactions found in Sabra's correspondences, so we may later collect some clues. Second, we will be searching in his personal calculation manuscripts and other key documents for useful hints that define the needed parameters.

I- Reactions and Interactions

The announcements performed by Wadia Sabra, have been the interests of musical and scientific authorities back then. The reactions of recognized musicologists and scientists at the time, are clearly expressed in different letters and reviews. Although the reviews were positive, they came with a large spectrum of assumptions. Each person conceived Sabra's announcements based on his field of interest. Some reviews were cited by Sabra himself while promoting his theory during his conferences. Others were also mentioned in the literature about Wadia Sabra, in the purpose of showing the importance of his work in general. For this study, we have chosen to talk about the reactions and interactions, not to agree with all the content, but in order to show the expectations that were considered further to Sabra's announcements, and to estimate later how Sabra's peers influenced the work propaganda.

A Mystery

Henri Rabaud (1873-1949) was a French conductor, composer and pedagogue, and the director of the "Convervatoire de Paris" from 1922 till 1941. Further to a brochure sent to him by Wadia Sabra exposing his new scale, he expressed in a letter[44], his interest about knowing more on the subject. He wrote that Sabra cited the advantages of his system, however, didn't describe exactly what this system is and how exactly are the intervals formed. He also considered that "the existence of pure fifths and pure thirds in a 12 notes division is as mysterious as squaring the circle[45]". Below this letter, Sabra added a personal comment stating that he surely wouldn't send Mr. Rabaud the secret of his discovery, as he reserved it for a public communication at a right time.

A Needed Tool

In a correspondence between Wadia Sabra and Vladimir Belaiev[46] (1909-1990) who was a Soviet orientalist writer, Belaiev shows his great interest and appreciation to Sabra's work. He considered that this work has two particular significances for the Soviet Union. First, because it is related to the study of the different musical oriental cultures in the U.S.S.R, and second, because it is related to the development of the western music. Moreover, he shares with Sabra the same concerns and interest in creating a universal music system; his main

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[44] Dated June 2nd 1940, Ref SI-4-072

[45] Squaring the Circle, is an ancient geometrical challenge that consist on constructing a square with an area equal to that of a given circle using only a finite number of steps with compass and straightedge. This challenge was proven insoluble by Ferdinand von Lindemann (1852-1939), who proved that the number π is transcendental

[46] Ref SII-4-049/051/087/088/115

drive was that the different populations of the USSR possess diverse musical systems, as he mentioned for instance the complexity of Azerbaijani and Armenian music systems.

Music Destinies

Prudent Pruvost (18..-1960), French musicologist and composer, expressed his interest and appreciation to the fact that Sabra was dedicated to study a "problem which is a fundament of every harmony"[47]. He considered that "during this period of renovation, this work would have a great influence on the destinies of music".

Possible Dispelled Ambiguity

Eugène Borrel (1864-1962), was a French violinist and musicologist, and also member of the "Société Française de Musicologie"[48]. In a letter to Wadia Sabra in 1937, he expressed his interest in Sabra's work and urged him to reveal his theory the soonest. In the same letter, Borrel mentioned that the "idea of attracting the attention of the Academies on Sabra's new Unit of measurement isn't an easy task, because scientific institutions never like to change their habitude, and will generally not see an interest in the subject". However, he concluded his letter by saying: "We live, in the acoustic and music fields, on ambiguities that would first have to be dispelled – and your unit[49] could be for this matter, a big help"

[47] Ref. SII-4-106
[48] Founded in 1927 by Lionel de la Laurencie (1861-1933). E. Borrel was elected Secretary-General in 1935
[49] Sabra's new unit of interval measurement

Quantum Theory

D.S. Aisberg (n.d.) a French pianist and pedagogue, expressed his admiration to the value of Sabra's work. In a letter sent to Wadia Sabra in 1937[50], Mr. Aisberg assumed that the Universal Scale, as announced by Sabra, was not just a simplification of the calculation, however, it may be based on the physiological particularities of the ear. He considered that this scale was similar to Planck's Quantum Theory[51] which will put an end to the concept of continuity of the musical frequencies as perceived by our brain. He added that the new unit of interval measurement could be considered as a pitch perception Quantum. Although this idea was based on a personal assumption, and was never related to Sabra's work, we have found in Sabra's archive some undated papers explaining Planck's Quantum theory. It may have triggered Sabra to learn more on this subject.

Extensive discussion

In a rich correspondence between Wadia Sabra and Lucien Rouzet[52] (1886-1948), who was a French physicist and acoustician, we found an extensive discussion in the 1930's about both Sabra's theory and the Rational Scale or "La Gamme Rationelle", invented by Rouzet. Perhaps, we can say that Rouzet was the closest to understand the work of Sabra, as himself achieved another work in the very same direction. Rouzet first created a geometrical representation of his scale as follows:

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[50] Ref. SII-4-211
[51] Max Planck (1858-1947), was a German physicist who originated the quantum theory which won him the Nobel Prize for Physics in 1918 (Stuewer, 1998). Planck suggested that the radiation energy is not continuous, however it is emitted in discrete packets called Quanta (Squires, 1999)
[52] Ref. SII-4-117/184/191/196/197/251

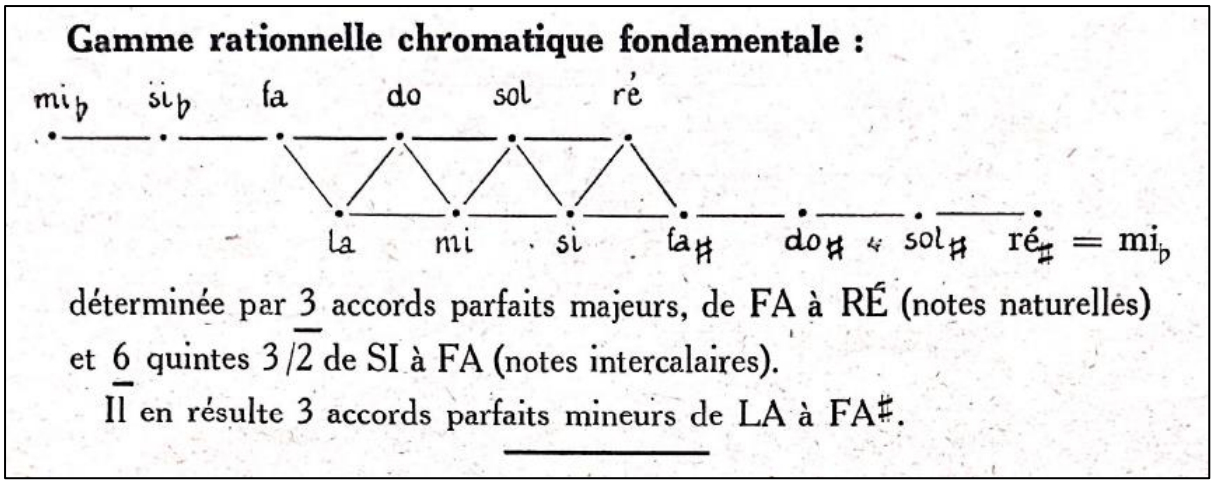

Figure 4 : La Gamme Rationnelle" by Lucien Rouzet[53]

All consecutive notes in the same line have 3/2 ratio between each other, whereas the ratio between 2 consecutive notes in different lines is 5/4. Rouzet considers that his scale is a natural deduction of the consonance. Sabra considered that it has the quality of Simplicity, however, he highlighted some remarks related to the existence of shorter fifth between the second and sixth degree. Rouzet justified this issue by stating that it was to show the colors of the scales. In a letter sent to Sabra, while introducing him to his rational scale and unit of measurement, Rouzet wondered if they both had found the same numbers. Rouzet wrote that his unit was contained 113 times in the 9/8 interval and 51 times in the 16/15 interval. Apart from this letter, we did not find any evidence about an explicit reply by Sabra on this specific point.

[53] Ref. SII-5-1-040

Expression of interest

Sir James Jeans (1877-1946), was and English physicist and mathematician. In the 1930's, he published a book entitled "Science & Music". We have found extracts of this book in Sabra's archive. In a letter sent by Wadia Sabra to Sir James Jeans[54], Sabra highlighted his "deep interest" in Jeans's approximation which is: "306 fifths = 179 – 0.0014 Octaves"[55], and told Jeans that his universal Scale expresses the numbers of this approximation. Also, Sabra asked him about his disposition to correspond with him on "this important topic". In a brief reply[56], Sir James wrote that he is interested in the whole subject but "Unhappily too busy to discuss the various questions raised".

A loss

In a condolence letter sent to Adele Misk, Alexis Chottin (1891-1975), a French Ethnomusicologist, considered that "Sabra's death was premature as he was not able to publish the results of his studies on the Universal Scale". He also considered that "No one is able to continue his work, which is a loss for both Science and Art". We are not sure what did he mean by the word "continue", is it to continue the publication or the elaboration of the work?

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[54] Not dated. Ref. SII-4-265

[55] Sir James Jeans, used the method of continued fraction to elaborate an increasingly good approximation to the ratio of the intervals of fifths and octave. He suggested the following:

- 12 fifths = 7 + 0.019 Octaves
- 41 fifths = 24 - 0.016 Octaves
- 53 fifths = 31 + 0.003 Octaves
- 306 fifths = 179 – 0.0014 Octaves

[56] Dated Dec 2, 1939. Ref. SII-4-266

II- Calculation Manuscripts

While searching in Sabra's Archive, we found in a box referenced: "Informations"[57], a pack of folders, each containing pages of handwritten calculations done by Sabra himself. Also, we found some key documents that gave us important hints and helped us understand Sabra's approach. These hints were mostly located in (and not limited to) correspondences with H.P. Mulholland, whom Sabra consulted in 1947, and also deducted from copied extracts of books by Hermann von Helmholtz, L. Rouzet, Max Planck, A.J. Ellis, S.J. Jeans, etc…

In one of his letters[58] sent to the general manager of the UNESCO, Mr. Julian Huxley, Sabra asked him to organize a Universal Music Congress under the patronage of the UNESCO, and mentioned the elaboration of 27 Tables of the universal scale while requesting their support to print them. However, we found in his archive 21 folders (Tables) enumerated several times by the author, which caused some ambiguity. In the following paragraphs, we will elaborate the description of these tables, while considering the enumeration found in the center of each folder cover.

01- "Justesse Absolue"

A calculation of the logarithm of 26 upward pure fifths for each note starting from the note C (Do) consecutively up to B (Si), and 26 downward pure fifths for each note starting from C (Do) consecutively down to Db (Re b). At the end of this document, Sabra has also drawn a

[57] Ref. SII-5-1-001 to SII-5-1-118
[58] Dated June 7, 1948

table entitled, “Composition de la Véritable Gamme Commatique”, translated to, “The Composition of the real Commatic scale”[59]

02- “Logarithmes”

In this sheet, Sabra calculated the logarithm of every step ratio of his scale. The lines counted 613. The rows were divided in 51 blocks of 12 lines each. We have deducted here that he may had divided the octave to 612 parts.

03- “Tableau Indicateur”

This table shows the equivalent rank of each note going downward fifths in black and upward fifths in red.

04- “Longueurs”

It contains the string length of each of the 612 divisions based on a string of 100cm. The calculation is rounded to 5 decimals. In addition, 2 other columns were made for the cumulative variation ($L(0) - L(n)$), and step length ($L(n) - L(n-1)$).

05- “Parties Retenues”

This table shows the length of the retained part of the string at each step from 1 to 613

06- “Accords”

[59] Refer to p. 55

This folder was divided to 12 tables, representing each note from 1 to 12 (C, C#, D, D#, E, F, F#, G, G#, A, A#, B). Each table contains 2 sections: 4 rows on the left for the descending fifths and 4 other rows on the right for the ascending fifths (as shown in the Table No3 "Tableau Indicateur"). In each section, left and right, we find the reference numbers and String Lengths of the chord, which is made from the fundamental, pure 3rd, pure 5th, and minor seventh.

07- "Racine Carré"

This table contains 2 columns. The first column is dedicated to the retained string length of the ranks $n = \{1,2,3, \dots, 306\}$, while the second shows the retained string length of the ranks $N = n - 1$

08- "Gammes"

This folder entitled "Gammes", which means "Scales", contains draft calculation pages with ratios, and string lengths of several scales: Collangettes, Pruvost, 53 divisions, 36 divisions, 34 divisions and 17 divisions.

09- N/A

10- N/A

11- "Rapports"

In this folder we find a draft of the emplacement of several ratios in Sabra's scale.

12- "Nombre de Vibration"

Here, Sabra has drawn a table containing the ratio of each step of the octave division. The numbers are expressed with 11 decimal places, going from $R(1) = 1$ till $R(613) = 2$

13- "306"

It's a draft folder showing Sir. J. Jeans's approximation of the octave and fifths.

14- N/A

15- "Schismes"

Calculation of a Schism.

16- "Schismes à 16d"

Sabra has drawn a table of logarithms for his octave division, expressed in 16 decimal places. He also mentioned in this table 25 red lines that separate the lines with constant addition factor of 0.0004939605507023

17- "Justesses Absolues 16dec"

Similar to table No 01 but expressed in 16 Decimal places.

18- "Les 306 Quintes"

This table contains the logarithm, retained string length, and reference numbers of 306 upward and downward Fifths.

19- "Les 306 Schismes- 16 Decimals"

Here, sabra has drawn a table showing the Logarithm relative to each step, expressed with 16 Decimal places, in addition to the reference numbers, retained string length and note name.

20- "Farmer"

Tables Comparing Sabra's Division to the 1200 division used by Farmer.

21- "Gustave Lyon"

This folder contains a single page entitled "Divisions en collaboration avec Gustave Lyon", translated to: "Divisions in collaboration with Gustave Lyon".

22- "Comma de Pythagore"

Table of the 613 decimal ratios used to show and calculate Pythagoras comma and Jeans's approximation.

23- N/A

24- N/A

25- "8192-10935"

Calculation of a scale generated by the ratio $\frac{8192}{10935}$

26- "Divers"

This folder contains a number of oriental scales and other divisions, in addition to a table similar to the one presented in No 01[60].

27- N/A

[60] This folder is dated 1942

III- Key Documents

In addition to the calculation manuscripts, we found a couple of key documents[61], consisting of 2 letters from H.P. Mulholland[62], who apparently Sabra consulted him. Based on the content, there are reasons to believe that Wadia Sabra had met him and trusted him with his calculation sheets. On May 20 1947, one week later to Sabra's request, Mulholland replied with a first letter containing general comments, then, on June 20 1947, he sent his second review commenting on the mathematical aspect of Sabra's work. Due to the importance of these documents, we will be exposing below their content while categorizing them into different related aspects of the work.

Interval Approximation

In the first letter, Mulholland commented on three interval approximations used by Sabra.

1- Tempered Fourth[63]

$$\text{Let } A = \frac{10935}{8192} = 1.3348388671875$$

The interval A = 1.3348388671875 is considered equal to a

Tempered Fourth $= 2^{\frac{5}{12}} = 1.33483985417003$ with an accuracy up to 5 decimal places.

He wrote, *"The group of intervals generated by the Octave, the true Fifth, and the true Third*

[61] Ref. SII-04-249/250

[62] We have found many publications related to the Mathematical Field written by H.P. Mulholland in the 1st half of the 20th century, however we did not find any biographical information about him. Based on the content and context of the letter, we believe that H.P. Mulholland was related to the American University of Beirut and Sabra may have met him in Lebanon.

[63] It refers to the 6th tone in a 12 Tones Equal Temperament

contains an excellent Approximation to a tempered fourth (and hence to all the other tempered intervals within the octave)"

2- Approximation of 12 A minus 5 Octaves

In this point, he calculated the difference between 12 steps of interval A and 5 Octaves. He found that the ratio of the difference is 0.999991. Thus, the error is 1/100000, and less than $1/100^{th}$ of a comma. Mulholland considered that if the interval "A" "were to replace the tempered fourth in tuning all the 12 semitones, no appreciable discrepancy would be encountered".

3- Fifths and octave approximation

He also mentioned the approximation given by J. Jeans, which is:

$$306\ Fifths = 179 - 0.0014\ Octaves$$

However, he proposed another approximation:

$$665\ Fifths = 389 - 0.00008\ Octaves$$

Unit of intervals measurement

In his second letter, Mulholland commented on Sabra's choice of the interval measurement unit. We have clearly understood through the first paragraph of this letter, that the Schism was proposed as a structural unit for Sabra's scale, which is the excess of the Pythagorean

Comma over the Syntonic Comma[64]. Mulholland pointed out that this choice has the advantage of belonging to the group of intervals generated by the octave, the fifth, and the third. He also mentioned that it could be approximated to 1/12th of the Pythagorean Comma, which exceeds a Schism by a millionth of an octave. Moreover, he noted that Sabra adapted the best approximation of the fifths as given by Jeans.

$$\frac{179}{306}\ of\ an\ octave, or\ 7\frac{1}{51}\ tempered\ semitones$$

He suggested dividing the tempered semitone into 51, and octave into 612 equal parts, each of which would exceed a Schism by about six millionths of an octave. For this, he proposed calling it a "Tempered Schism" in order to differentiate it from the "Natural Schism", which was considered by Sabra.

In another paragraph, entitled "Desiderata in choosing a unit", Mulholland observed that in the declaration made by Sabra in his leaflet to the Scientific world, the new unit is contained a whole number of times in the octave, and in any musical interval belonging to ancient or modern scale, recognized and defined until today. Mulholland demonstrated that no unit can fulfill exactly the above-mentioned properties announced by Sabra. He gave the following example:

[64] The syntonic comma, is a small comma type interval with a ratio = 81/80. It is the correction amount of the Pythagorean major third (81:64) to a just major third (5:4)

"If the unit is contained a whole number of times in the octave it cannot be so contained in the true fifth. For, otherwise, we should have N fifths equal to n octaves, and thus,

$$(\frac{3}{2})^N = 2^n \implies 3^N = 2^{n+N}$$

... This equality cannot be true if N and n are whole numbers... Thus, the proprieties enumerated in your leaflet can only hold approximately"

Mulholland suggested that this point should be stated explicitly by Sabra, as he may be "criticized for asserting a mathematical impossibility." He continued by dressing a table comparing different unit choices to other intervals. For its importance, it is reproduced here below.

Interval	**Ratio**	**Octaves**	**Semitones**	**Tempered Schisms**	**Natural Schisms**
Octave	2	1	12	612	614.21
Semitone	$2^{\frac{1}{12}}$	$1/12 = 0.0833333$	1	51	51.14
Tempered Schism	$2^{\frac{1}{612}}$	$1/612 = 0.0016340$	$1/51 = 0.0196078$	1	1.003626
Natural Schism	$\frac{3^8 \times 5}{2^{15}}$	0.0016281	0.0195372	0.996397	1
Fifth	$3/2$	0.5849626	7.01955	357.99705	359.29
Third	$5/4$	0.3219281	3.8631371	197.0099	197.73
7th Harmonic	$7/4$	0.807345	9.68814	494.095	495.88
11th Harmonic	$11/8$	0.45943	5.5132	281.17	282.19
Mesotonic Fifth	$5^{\frac{1}{4}}$	0.580499	6.965784	355.255	356.54

Table 1 : Mulholland's Unit Comparision

Mulholland concluded that the Tempered Schism is the most convenient unit in terms of size. His opinion was based on the fact that the Tempered Schism is contained a whole number of times in the octave and the Tempered Semitone (by definition), and nearly an exact number of times in both the true fifth (error = 0.003 of a Schism) and the true third (error = 0.02 of a Schism). He also mentioned that the Tempered Schism offers no great advantages over the Natural Schism for the remaining intervals (error ranging from $1/10^{th}$ to $1/4^{th}$ of a Schism)

In the other paragraphs, he continued by citing the advantages of considering 612 as a divisor, and discussed the possibility of finding a larger divisor. Then after explaining a mathematical method of finding fractions approximating closely to a given ratio, he discussed the negligibility of certain fractions of a Schism. Here, he considered that negligibility is always related to the envisaged application. He also advised that "if a table of musical intervals is expressed in schisms, it should give not only the nearest whole number of schisms, but also the requisite fractional correction".

IV- Parameters and Tools

Based on the aforementioned observations, we will be discussing our findings based on the following criteria:

1- Finding the octave divisor of sabra's scale
2- Checking the regularity of the octave division
3- Elaborating and calculating the numbers (string length ratios and logarithm)
4- Comparing the results with Sabra's calculations

As for the mathematical, musical and acoustical tools, we have proceeded in accordance with the calculation methods used by Sabra:

1- Defining the physical proprieties of the fundamental note

In accordance with Sabra's work, all divisions will be done on a string where the fundamental note is defined by:

a) $String\ Length$: $L_1 = 100.00000$ *(5 decimal places)*

b) $Number\ of\ Vibrations$[65]: $R_1 = 1.00000000000$ *(11 decimal places)*

c) $Logarithm\ of\ the\ \ R_1$: $\quad l_1 = 0.00000000000$ $(11\ decimal\ places)$

2- Defining the physical proprieties of the other notes and their relation with the fundamental

All the other notes (Ranks of Sabra's Universal Scale) will be defined by their rank number n

$$n = (1,2,3,4, \dots, 613)$$

Where, $n = 1$, refers to the fundamental, and $n = 613$ refers to the octave

[65] Sabra used the term Number of vibrations to refer to frequency ratio

After determining the divisor *d*, we can get the Unit ratio *u*, as follows,

$$u = 2^{\frac{1}{d}}$$

As for the physical proprieties, we will have:

a) String Length: $L_n = L_{n-1} \times u^{-1}$

b) Ratio, (or number of vibrations as assigned by Sabra): $R_n = R_{n-1} \times u = L_n^{-1}$

c) Logarithm of the interval formed with the fundamental: $l_n = LOG(R_n)$

3- Rounding the numbers to a certain decimal

To round the numbers, we will proceed with Sabra's method: 5 decimals for the string length expression, 11 decimals for Ratios and Logarithms and 13 decimals for the Unit ratio.

Chapter IV: Results and discussion

After we searched the Literature and the Archive, we were able to understand Sabra's logic in developing his theory. We will proceed by answering the criteria mentioned in the last chapter, revealing some main tables as defined in the leaflet "La Gamme Universelle", then discuss the results.

Parameters

For the octave divisor, it seems clear from the manuscripts and from our key document that 612 is the number of divisions of the octave. The main question remains whether this division is regular or not. We have first proceeded by dividing the octave to 612 equal parts, and comparing it to Sabra's Numbers, as follows:

1- Unit ratio calculation:

$$u = 2^{\frac{1}{612}} = 1.0011298906275$$

2- Dressing the table

The Columns show in order, 1) the rank N, 2) the calculated string length based on the unit ratio u, 3) the String length as calculated by Sabra, 4) the difference between Sabra's calculation and ours, 5) the calculated ratios based on the unit ratio u, 6) Sabra's Ratios and 7) the logarithms of our calculated ratio

We have found that our calculation and Sabra's numbers coincide exactly each 51 step (Highlighted in Green) and diverge to a maximum difference at each 25 steps up and down (Highlighted in Yellow).

N	String Length	String Length Sabra	Difference	Ratios	Ratios Sabra	Logarithms
1	100.00000	100.00000		1.00000000000	1.00000000000	0.00000000000
2	99.88680	99.88713	0.00033	1.00113323506	1.00112989070	0.00049187908
25	97.31839	97.32619	0.00780	1.02755504977	1.02747266820	0.01180509787
26	97.20823	97.21635	0.00812	1.02871951118	1.02863359994	0.01229697695
27	97.09819	97.09008	-0.00811	1.02988529220	1.02997130812	0.01278885602
52	94.38743	94.38743	0.00000	1.05946309436	1.05946309445	0.02508583297
53	94.28059	94.28090	0.00031	1.06066371508	1.06066017170	0.02557771205
77	91.75235	91.76001	0.00766	1.08989035654	1.08979933666	0.03738280992
102	89.19083	89.19053	-0.00030	1.12119147482	1.12119522063	0.04967978687
103	89.08987	89.08987	0.00000	1.12246204831	1.12246204833	0.05017166594
104	88.98903	88.98932	0.00029	1.12373406165	1.12373030768	0.05066354502
128	86.60269	86.60992	0.00723	1.15469860966	1.15460217741	0.06246864289
153	84.18493	84.18465	-0.00028	1.18786098929	1.18786495763	0.07476561984
154	84.08964	84.08964	0.00000	1.18920711500	1.18920711508	0.07525749892
155	83.99446	83.99473	0.00027	1.19055476620	1.19055078908	0.07574937799
179	81.74205	81.74887	0.00682	1.22336056204	1.22325839576	0.08755447586
204	79.46000	79.45973	-0.00027	1.25849487938	1.25849908368	0.09985145281
205	79.37005	79.37005	0.00000	1.25992104989	1.25992104988	0.10034333189
206	79.28021	79.28047	0.00026	1.26134883660	1.26134462294	0.10083521097
230	77.15422	77.16066	0.00644	1.29610536657	1.29599712519	0.11264030884
255	75.00025	75.00000	-0.00025	1.33332887914	1.33333333381	0.12493728578
256	74.91535	74.91535	0.00000	1.33483985417	1.33483985430	0.12542916486
257	74.83055	74.83080	0.00025	1.33635254149	1.33634807731	0.12592104394
281	72.82389	72.82997	0.00608	1.37317580229	1.37306112443	0.13772614181
306	70.79081	70.79057	-0.00024	1.41261274009	1.41261745925	0.15002311875
307	70.71068	70.71070	0.00002	1.41421356237	1.41421356236	0.15051499783
308	70.63064	70.63080	0.00016	1.41581619876	1.41581146922	0.15100687691
332	68.73660	68.74240	0.00580	1.45482908459	1.45470758776	0.16281197478
357	66.81763	66.81740	-0.00023	1.49661106475	1.49661606443	0.17510895173
358	66.74199	66.74190	-0.00009	1.49830707688	1.49830707706	0.17560083080
359	66.66644	66.66660	0.00016	1.50000501098	1.50000000015	0.17609270988
383	64.87871	64.88410	0.00539	1.54133772372	1.54120900234	0.18789780775
408	63.06744	63.06720	-0.00024	1.58560418971	1.58560948683	0.20019478470
409	62.99605	62.99600	-0.00005	1.58740105197	1.58740105221	0.20068666378
410	62.92474	62.92490	0.00016	1.58919995049	1.58919464173	0.20117854285
434	61.23735	61.24240	0.00505	1.63299043423	1.63285405861	0.21298364072
459	59.52774	59.52750	-0.00024	1.67988912126	1.67989473337	0.22528061767
460	59.46036	59.46030	-0.00006	1.68179283051	1.68179283061	0.22577249675
461	59.39305	59.39320	0.00015	1.68369869710	1.68369307260	0.22626437583
485	57.80036	57.80520	0.00484	1.73009309851	1.72994861347	0.23806947370
510	56.18670	56.18650	-0.00020	1.77978052659	1.77978647231	0.25036645064
511	56.12310	56.12310	0.00000	1.78179743628	1.78179743645	0.25085832972
512	56.05957	56.05980	0.00023	1.78381663160	1.78381067272	0.25135020880
536	54.55627	54.56080	0.00453	1.83296978767	1.83281671127	0.26315530667

561	53.03319	53.03300	-0.00019	1.88561178398	1.88561808343	0.27545228361
562	52.97315	52.97310	-0.00005	1.88774862536	1.88774862570	0.27594416269
563	52.91319	52.91340	0.00021	1.88988788829	1.88988157491	0.27643604177
587	51.49426	51.49850	0.00424	1.94196384312	1.94180166428	0.28824113964
612	50.05666	50.05650	-0.00016	1.99773609542	1.99774276925	0.30053811659
613	50.00000	50.00000	0.00000	2.00000000000	2.00000000000	0.30102999566

Table 2: 1st elaboration of the numbers[66]

The convergence with Sabra's numbers each 51 step, is due to the fact that 612 is a multiple of 12. Thus, the 12 tones equal tempered scale is shown in Sabra's scale. Moreover, the above 612 equal division coincides with Mulholland suggestion to use the Tempered Schism. However, the shown difference suggests that an irregular division was considered by Sabra.

Going deeper in observing the differences, revealed that Sabra had divided the Octave following two consecutive steps. The first division is an Equal Tempered division with a unit ratio $u = 2^{\frac{1}{612}}$, where he got the following main ranks that are similar to the 12 tones equal tempered scale: $N = \{1, 52, 103, 154, 205, 256, 307, 358, 409, 460, 511, 562, 613\}$. All these ranks are positioned eventually at a difference of 51 steps. As for the remaining ranks, he used the following ratio $u' = \frac{1}{12}\ of\ the\ pythagorean\ comma = 1.0011298906275$, in order to go 25 steps up and down from each note. In the same logic of the first table, we have drawn another table where we have calculated the remaining String Lengths, Ratios and Logarithms based on the Unit ratio u' and then compared it to Sabra's numbers found in his calculation manuscripts No 02, 04 and 12.

[66] The numbers which converged with Sabra's Calculation are highlighted in green
However, the numbers having maximum divergence with Sabra's Calculation are highlighted in yellow

Our numbers converged with Sabra's Calculation, and showed a constant residue each 51 steps, starting from n=27 (Written in Red)

N	String Length	String Length Sabra	Difference	Ratios	Ratios Sabra	Difference	Logarithms	l(n)-l(n-1)
1	100.00000	100.00000		1.00000000000	1.00000000000		0.00000000000	
2	99.88714	99.88713	-0.00001	1.00112989063	1.00112989070	0.00000000007	0.00049042825	0.00049042825
3	99.77440	99.77440	0.00000	1.00226105791	1.00226105797	0.00000000006	0.00098085650	0.00049042825
4	99.66180	99.66179	-0.00001	1.00339350328	1.00339350340	0.00000000012	0.00147128476	0.00049042825
5	99.54932	99.54931	-0.00001	1.00452722820	1.00452722830	0.00000000010	0.00196171301	0.00049042825
25	97.32619	97.32619	0.00000	1.02747266821	1.02747266820	-0.00000000001	0.01177027804	0.00049042825
26	97.21635	97.21635	0.00000	1.02863359995	1.02863359994	-0.00000000001	0.01226070629	0.00049042825
27	97.09008	97.09008	0.00000	1.02997130797	1.02997130812	0.00000000015	0.01282512668	0.00056442039
51	94.49408	94.49407	-0.00001	1.05826736798	1.05826736800	0.00000000002	0.02459540472	0.00049042825
52	94.38743	94.38743	0.00000	1.05946309436	1.05946309445	0.00000000009	0.02508583297	0.00049042825
53	94.28090	94.28090	0.00000	1.06066017178	1.06066017170	-0.00000000008	0.02557626122	0.00049042825
77	91.76001	91.76001	0.00000	1.08979933677	1.08979933666	-0.00000000011	0.03734653926	0.00049042825
78	91.64084	91.64084	0.00000	1.09121658904	1.09121658920	0.00000000016	0.03791095965	0.00056442039
102	89.19053	89.19053	0.00000	1.12119522034	1.12119522063	0.00000000029	0.04968123769	0.00049042825
103	89.08987	89.08987	0.00000	1.12246204831	1.12246204833	0.00000000002	0.05017166594	0.00049042825
104	88.98932	88.98932	0.00000	1.12373030766	1.12373030768	0.00000000002	0.05066209420	0.00049042825
128	86.60992	86.60992	0.00000	1.15460217756	1.15460217741	-0.00000000015	0.06243237224	0.00049042825
129	86.49743	86.49742	-0.00001	1.15610370404	1.15610370424	0.00000000020	0.06299679262	0.00056442039
153	84.18465	84.18465	0.00000	1.18786495752	1.18786495763	0.00000000011	0.07476707066	0.00049042825
154	84.08964	84.08964	0.00000	1.18920711500	1.18920711508	0.00000000008	0.07525749892	0.00049042825
155	83.99474	83.99473	-0.00001	1.19055078898	1.19055078908	0.00000000010	0.07574792717	0.00049042825
179	81.74888	81.74887	-0.00001	1.22325839579	1.22325839576	-0.00000000003	0.08751820521	0.00049042825
180	81.64270	81.64270	0.00000	1.22484920768	1.22484920781	0.00000000013	0.08808262560	0.00056442039
204	79.45973	79.45973	0.00000	1.25849908358	1.25849908368	0.00000000010	0.09985290364	0.00049042825
205	79.37005	79.37005	0.00000	1.25992104989	1.25992104988	-0.00000000001	0.10034333189	0.00049042825
206	79.28047	79.28047	0.00000	1.26134462288	1.26134462294	0.00000000006	0.10083376014	0.00049042825
230	77.16066	77.16066	0.00000	1.29599712521	1.29599712519	-0.00000000002	0.11260403818	0.00049042825
231	77.06045	77.06049	0.00004	1.29768253169	1.29768253050	-0.00000000119	0.11316845857	0.00056442039
255	75.00000	75.00000	0.00000	1.33333333333	1.33333333381	0.00000000048	0.12493873661	0.00049042825
256	74.91535	74.91535	0.00000	1.33483985417	1.33483985430	0.00000000013	0.12542916486	0.00049042825
257	74.83080	74.83080	0.00000	1.33634807721	1.33634807731	0.00000000010	0.12591959311	0.00049042825
281	72.82997	72.82997	0.00000	1.37306112455	1.37306112443	-0.00000000012	0.13768987115	0.00049042825
282	72.73538	72.73537	-0.00001	1.37484675052	1.37494675076	0.00010000024	0.13825429154	0.00056442039
306	70.79057	70.79057	0.00000	1.41261745915	1.41261745925	0.00000000010	0.15002456958	0.00049042825
307	70.71068	70.71070	0.00002	1.41421356237	1.41421356236	-0.00000000001	0.15051499783	0.00049042825
308	70.63087	70.63080	-0.00007	1.41581146902	1.41581146922	0.00000000020	0.15100542608	0.00049042825
332	68.74234	68.74240	0.00006	1.45470758776	1.45470758776	0.00000000000	0.16277570412	0.00049042825
333	68.65306	68.65300	-0.00006	1.45659939258	1.45659939263	0.00000000005	0.16334012451	0.00056442039

357	66.81740	66.81740	0.00000	1.49661606441	1.49661606443	0.00000000002	0.17511040255	0.00049042825
358	66.74199	66.74190	-0.00009	1.49830707688	1.49830707706	0.00000000018	0.17560083080	0.00049042825
359	66.66667	66.66660	-0.00007	1.50000000000	1.50000000015	0.00000000015	0.17609125906	0.00049042825
383	64.88413	64.88410	-0.00003	1.54120900232	1.54120900234	0.00000000002	0.18786153710	0.00049042825
384	64.79986	64.79980	-0.00006	1.54321329970	1.54321330010	0.00000000040	0.18842595748	0.00056442039
408	63.06723	63.06720	-0.00003	1.58560948667	1.58560948683	0.00000000016	0.20019623552	0.00049042825
409	62.99605	62.99600	-0.00005	1.58740105197	1.58740105221	0.00000000024	0.20068666378	0.00049042825
410	62.92495	62.92490	-0.00005	1.58919464154	1.58919464173	0.00000000019	0.20117709203	0.00049042825
434	61.24246	61.24240	-0.00006	1.63285405865	1.63285405861	-0.00000000004	0.21294737007	0.00049042825
435	61.16292	61.16290	-0.00002	1.63497753776	1.63497753801	0.00000000025	0.21351179046	0.00056442039
459	59.52754	59.52750	-0.00004	1.67989473319	1.67989473337	0.00000000018	0.22528206850	0.00049042825
460	59.46036	59.46030	-0.00006	1.68179283051	1.68179283061	0.00000000010	0.22577249675	0.00049042825
461	59.39325	59.39320	-0.00005	1.68369307246	1.68369307260	0.00000000014	0.22626292500	0.00049042825
485	57.80519	57.80520	0.00001	1.72994861362	1.72994861347	-0.00000000015	0.23803320304	0.00049042825
486	57.73011	57.73010	-0.00001	1.73219836136	1.73219836174	0.00000000038	0.23859762343	0.00056442039
510	56.18652	56.18650	-0.00002	1.77978647223	1.77978647231	0.00000000008	0.25036790147	0.00049042825
511	56.12310	56.12310	0.00000	1.78179743628	1.78179743645	0.00000000017	0.25085832972	0.00049042825
512	56.05976	56.05980	0.00004	1.78381067250	1.78381067272	0.00000000022	0.25134875797	0.00049042825
536	54.56083	54.56080	-0.00003	1.83281671127	1.83281671127	0.00000000000	0.26311903601	0.00049042825
537	54.48997	54.48990	-0.00007	1.83520023598	1.83520023630	0.00000000032	0.26368345640	0.00056442039
561	53.03301	53.03300	-0.00001	1.88561808316	1.88561808343	0.00000000027	0.27545373444	0.00049042825
562	52.97315	52.97310	-0.00005	1.88774862536	1.88774862570	0.00000000034	0.27594416269	0.00049042825
563	52.91337	52.91340	0.00003	1.88988157484	1.88988157491	0.00000000007	0.27643459094	0.00049042825
587	51.49857	51.49850	-0.00007	1.94180166431	1.94180166428	-0.00000000003	0.28820486898	0.00049042825
588	51.43168	51.43170	0.00002	1.94432692078	1.94432692100	0.00000000022	0.28876928937	0.00056442039
612	50.05649	50.05650	0.00001	1.99774276917	1.99774276925	0.00000000008	0.30053956741	0.00049042825
613	50.00000	50.00000	0.00000	2.00000000000	2.00000000000	0.00000000000	0.30102999566	0.00049042825

Table 3: 2nd elaboration of the numbers

Following this division, we find that the residue between the 25th upper rank of a main rank n and the 25th lower rank of another main rank $n + 51$, is constant and its logarithm is equal to:

$l_{n+26} - l_{n+25} = 0.00056442039$ (written in Red).

This constant residue suggests an un-equal but periodic division of the Octave.

In order to get a graphical representation of this scale, we have drawn the following diagram.

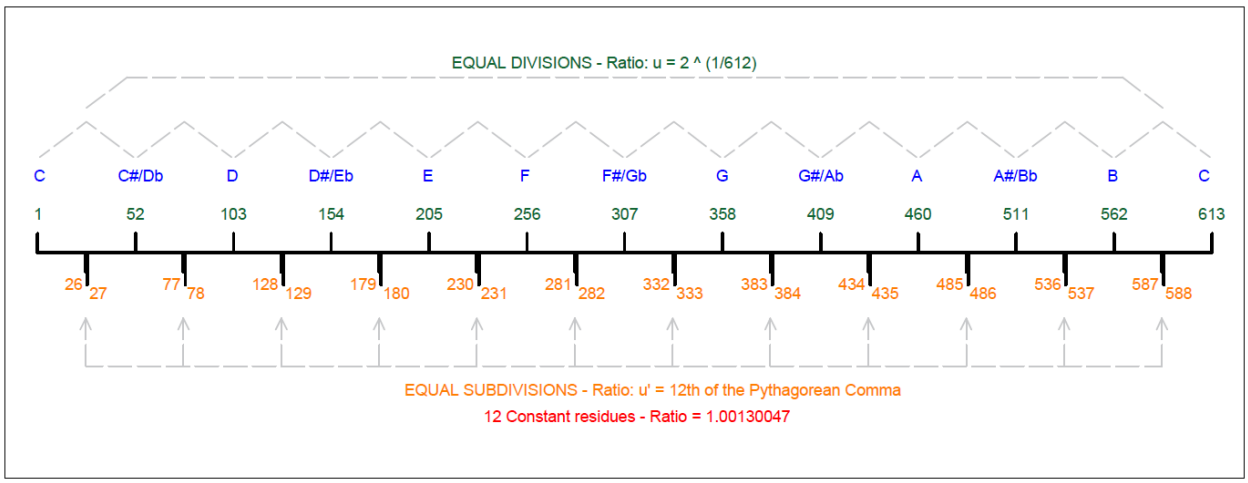

Figure 5: Sabra's division of the Octave

This diagram shows the equally spaced 12 notes (in green) which represent the chromatic scale, and the equal subdivisions (in orange) based on the unit ratio $u' = \frac{1}{12}\ of\ the\ pythagorean\ comma.$

Tables

In this section, we will choose and dress some main tables as described by Wadia Sabra. These tables will be suitable to present Sabra's work by numbers. The remaining tables described by him, can be deductible one from another.

Table No 1

> *"Table giving each degree of the UNIVERSAL SCALE the frequency ratio, the length of the string, and the logarithm of the interval it forms with the fundamental. (The fundamental being given by a string of length equal to 100cm, its ratio is taken as a unit)"*

In accordance with the above, we dressed a table that contains all described information. Due to the higher accuracy of our modern computers, we have considered our calculation which already converged with Sabra's numbers. The columns show in order, 1) the rank N, 2) the calculated string length, 3) the calculated ratios and 4) the logarithms of our calculated ratios. We have also kept the highlighted ranks in green and yellow in order to show the periodicity of this scale. For the full table, refer to Appendix A

N	String Length	Ratios	Logarithms
1	100.00000	1.00000000000	0.00000000000
2	99.88714	1.00112989063	0.00049042825
26	97.21635	1.02863359995	0.01226070629
27	97.09008	1.02997130797	0.01282512668
51	94.49408	1.05826736798	0.02459540472
52	94.38743	1.05946309436	0.02508583297
53	94.28090	1.06066017178	0.02557626122
77	91.76001	1.08979933677	0.03734653926
78	91.64084	1.09121658904	0.03791095965
102	89.19053	1.12119522034	0.04968123769
103	89.08987	1.12246204831	0.05017166594
104	88.98932	1.12373030766	0.05066209420
128	86.60992	1.15460217756	0.06243237224
129	86.49743	1.15610370404	0.06299679262
153	84.18465	1.18786495752	0.07476707066
154	84.08964	1.18920711500	0.07525749892
155	83.99474	1.19055078898	0.07574792717
179	81.74888	1.22325839579	0.08751820521
180	81.64270	1.22484920768	0.08808262560
204	79.45973	1.25849908358	0.09985290364
205	79.37005	1.25992104989	0.10034333189
206	79.28047	1.26134462288	0.10083376014
230	77.16066	1.29599712521	0.11260403818
231	77.06045	1.29768253169	0.11316845857
255	75.00000	1.33333333333	0.12493873661
256	74.91535	1.33483985417	0.12542916486
257	74.83080	1.33634807721	0.12591959311
281	72.82997	1.37306112455	0.13768987115
282	72.73538	1.37484675052	0.13825429154
306	70.79057	1.41261745915	0.15002456958
307	70.71068	1.41421356237	0.15051499783
308	70.63087	1.41581146902	0.15100542608
332	68.74234	1.45470758776	0.16277570412
333	68.65306	1.45659939258	0.16334012451
357	66.81740	1.49661606441	0.17511040255
358	66.74199	1.49830707688	0.17560083080
359	66.66667	1.50000000000	0.17609125906
383	64.88413	1.54120900232	0.18786153710
384	64.79986	1.54321329970	0.18842595748
408	63.06723	1.58560948667	0.20019623552
409	62.99605	1.58740105197	0.20068666378
410	62.92495	1.58919464154	0.20117709203
434	61.24246	1.63285405865	0.21294737007
435	61.16292	1.63497753776	0.21351179046
459	59.52754	1.67989473319	0.22528206850
460	59.46036	1.68179283051	0.22577249675
484	57.87050	1.72799616694	0.23754277479
485	57.80519	1.72994861362	0.23803320304
486	57.73011	1.73219836136	0.23859762343
510	56.18652	1.77978647223	0.25036790147
511	56.12310	1.78179743628	0.25085832972
535	54.62248	1.83074816607	0.26262860776
536	54.56083	1.83281671127	0.26311903601
537	54.48997	1.83520023598	0.26368345640
561	53.03301	1.88561808316	0.27545373444
562	52.97315	1.88774862536	0.27594416269
586	51.55675	1.93961011702	0.28771444073
587	51.49857	1.94180166431	0.28820486898
588	51.43168	1.94432692078	0.28876928937
612	50.05649	1.99774276917	0.30053956741
613	50.00000	2.00000000000	0.30102999566

Table 4: The Universal Scale Main Table

Table No 2

"Table locating each degree of all recognized musical scales, in the UNIVERSAL SCALE"

We were not able to find in Sabra's manuscripts a table containing exactly the above description. However, we were able to find in the folder No.11 entitled "Rapports" several ratios of known scales located in their respective order in Sabra's Scale. This folder surely served as a draft in its early development, where no clear descriptions were provided.
This paper, will not include an inventory of the totality of scales considered by Sabra, instead, we will locate some of the most important ones in his Universal Scale. For this purpose, we will draw a table showing the rank of each note in Sabra's scale.

1- 12 Tones equal tempered scale

The 12 tones equal tempered scale is included by definition in Sabra's universal Scale.

N Sabra	Ratio	ET12
1	1	1
52	1.059463094	2
103	1.122462048	3
154	1.189207115	4
205	1.25992105	5
256	1.334839854	6
307	1.414213562	7
358	1.498307077	8
409	1.587401052	9
460	1.681792831	10
511	1.781797436	11
562	1.887748625	12
613	2	13

Table 5: Location of the 12-TET in Sabra's Universal Scale

2- Just intonation

We have dressed the following comparison table showing the nearest rank of the just intonation ratios and expressing the difference with Sabra's scale in cents.

N	Ratios SABRA	Just Ratios		Sabra (Cents)	Just (Cents)	Difference in Cents
1	**1.00000000000**	1	1	**0.00000000000**	**0.00000000000**	0.00000000000
58	**1.06666587797**	16/15	1.066666667	**111.73000519206**	**111.73128526978**	0.00128007772
105	**1.12500000000**	9/8	1.125	**203.91000173069**	**203.91000173078**	0.00000000009
162	**1.19999911272**	6/5	1.2	**315.64000692274**	**315.64128700055**	0.00128007781
198	**1.25000092425**	5/4	1.25	**386.31499394260**	**386.31371386484**	-0.00128007777
255	**1.33333333333**	4/3	1.333333333	**498.04499913466**	**498.04499913461**	-0.00000000004
302	**1.40625103978**	45/32	1.40625	**590.22499567329**	**590.22371559561**	-0.00128007768
359	**1.50000000000**	3/2	1.5	**701.95500086534**	**701.95500086539**	0.00000000004
416	**1.59999881696**	8/5	1.6	**813.68500605740**	**813.68628613517**	0.00128007777
452	**1.66666789900**	5/3	1.666666667	**884.35999307726**	**884.35871299945**	-0.00128007781
495	**1.74989293482**	7/4	1.75	**968.71998615452**	**968.82590646913**	0.10592031461
556	**1.87500138638**	15/8	1.875	**1088.26999480794**	**1088.26871473022**	-0.00128007772
613	**2.00000000000**	2	2	**1200.00000000000**	**1200.00000000000**	0.00000000000

Table 6: Location of the Just Temperament in Sabra's Universal Scale

3- Pythagorean Scale

In the same logic of the previous table.

N	Ratios SABRA	Pythagorean Ratios		Sabra (Cents)	Pythagorean (Cents)	Difference in Cents
1	**1.00000000000**	1	1	**0.00000000000**	**0.00000000000**	0.00000000000
105	**1.12500000000**	9/8	1.125	**203.91000173069**	**203.91000173078**	0.00000000009
209	**1.26562500000**	81/64	1.265625	**407.82000346137**	**407.82000346155**	0.00000000018
255	**1.33333333333**	4/3	1.333333	**498.04499913466**	**498.04499913461**	-0.00000000004
359	**1.50000000000**	3/2	1.5	**701.95500086534**	**701.95500086539**	0.00000000004
463	**1.68750000000**	27/16	1.6875	**905.86500259603**	**905.86500259616**	0.00000000013
567	**1.89843750000**	243/128	1.898438	**1109.77500432671**	**1109.77500432694**	0.00000000022
613	**2.00000000000**	2	2	**1200.00000000000**	**1200.00000000000**	0.00000000000

Table 7: Location of the Pythagorean Scale in Sabra's Universal Scale

4- Other Scales

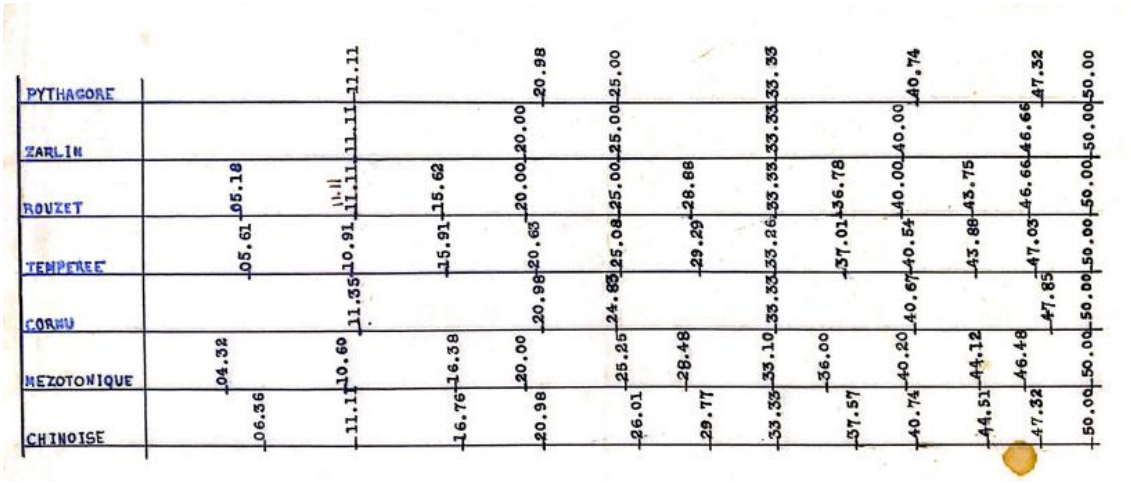

Figure 6: Octave division of various scales, as found in Sabra's Archive

For the remaining scales, Sabra did not reach to the point to develop clearly the emplacement of their relative ranks in his Universal Scale, however, he located some ratios[67] in his system. As this exercise will not be performed in this paper's scope of work, a sample of Sabra's draft will be annexed. (Refer to Appendix B)

[67] The ratios which are mentioned in this specific manuscript, belonged to different divisions of the Arabic Scales

Table No 3

"Table of chords whose regular sequence generates the tonality of the degrees of the UNIVERSAL SCALE"

This table is in fact a series of 12 tables, each based on a Note $n = \{C, C\#, D, D\#, E, F, F\#, G, G\#, A, A\#, B\}$[68]. Theses tables show the rank of each interval of a just harmonic seventh chord, built upon a series of 25 intervals going up by pure fifths starting from the main note n. Below is the first table relative to the note C, as found and calculated by Sabra in his manuscript calculation note No 6, entitled *"Accords"*.

Note C:

Down					Up			
harmonic seventh	lower fifth	lower third	Fundamental		Fundamental	upper third	upper fifth	harmonic seventh
117	**255**	**417**	**613**	**1**	**1**	**197**	**359**	**497**
12.31	25.00	37.57	50.00		0.00	19.91	33.33	42.98
371	**509**	**59**	**355**	**2**	**359**	**555**	**105**	**243**
34.23	43.75	6.35	25.00		33.33	46.61	11.11	23.94
13	**151**	**313**	**509**	**3**	**105**	**301**	**463**	**601**
1.35	15.62	29.77	43.75		11.11	28.81	40.74	49.32
267	**405**	**567**	**151**	**4**	**463**	**47**	**209**	**347**
26.01	36.72	47.32	15.62		40.74	5.08	20.98	32.42
521	**47**	**208**	**405**	**5**	**209**	**405**	**567**	**93**
44.51	5.08	20.98	36.72		20.98	36.72	47.32	9.9
163	**301**	**463**	**47**	**6**	**567**	**151**	**313**	**451**
16.76	28.81	40.74	5.08		47.32	15.62	29.77	39.93
417	**555**	**105**	**301**	**7**	**313**	**509**	**59**	**197**
37.57	46.61	11.11	28.81		29.77	43.75	6.35	17.91
59	**197**	**359**	**555**	**8**	**59**	**255**	**417**	**555**
6.35	19.91	33.33	46.61		6.35	25	37.57	46.61

[68] Sabra considered the notes of 12 Equal tempered scale, thus C# = Db, etc…

313	**451**	**1**	**197**	**9**	**417**	**613**	**163**	**301**
29.77	39.98	0	19.91		37.57	50	16.76	28.81
567	**93**	**255**	**451**	**10**	**163**	**359**	**521**	**47**
47.32	9.9	25	39.93		16.76	33.33	44.51	5.08
209	**347**	**509**	**93**	**11**	**521**	**105**	**267**	**405**
20.98	32.42	43.75	9.9		44.51	11.11	26.01	36.72
463	**601**	**151**	**347**	**12**	**267**	**463**	**13**	**151**
40.74	49.32	15.62	32.42		26.01	40.74	1.35	15.62
105	**243**	**405**	**601**	**13**	**13**	**209**	**371**	**509**
11.11	23.97	36.72	49.32		1.35	20.98	34.23	43.75
359	**497**	**47**	**243**	**14**	**371**	**567**	**117**	**255**
33.33	42.98	5.08	23.94		34.23	47.32	12.31	25
1	**139**	**301**	**497**	**15**	**117**	**313**	**475**	**613**
0	14.47	28.81	42.98		12.31	29.77	41.54	50
255	**393**	**555**	**139**	**16**	**475**	**59**	**221**	**359**
25	35.85	46.61	14.47		41.54	6.35	22.05	33.33
509	**35**	**197**	**393**	**17**	**221**	**417**	**579**	**105**
43.75	3.78	19.91	35.85		22.05	37.57	48.03	11.11
151	**289**	**451**	**35**	**18**	**579**	**163**	**325**	**463**
15.62	27.84	39.93	3.78		48.03	16.76	30.71	40.74
405	**543**	**93**	**289**	**19**	**325**	**521**	**71**	**209**
36.72	45.88	9.9	27.04		30.71	44.51	7.62	20.98
47	**185**	**347**	**543**	**20**	**71**	**267**	**429**	**567**
5.08	18.82	32.42	45.85		7.62	26.01	38.41	47.32
301	**439**	**601**	**185**	**21**	**429**	**13**	**175**	**313**
28.81	39.11	49.32	18.82		38.41	1.35	17.88	29.77
555	**81**	**243**	**439**	**22**	**175**	**371**	**533**	**59**
46.61	8.67	23.94	39.11		17.88	34.23	45.25	6.35
197	**335**	**497**	**81**	**23**	**533**	**117**	**279**	**417**
19.91	31.5	42.98	8.67		45.25	12.31	27.01	37.57
451	**589**	**139**	**335**	**24**	**279**	**475**	**25**	**163**
39.93	48.63	14.47	31.5		27.01	41.54	2.67	16.76
93	**231**	**393**	**589**	**25**	**25**	**221**	**383**	**521**
9.9	22.94	35.85	48.63		2.67	22.05	35.11	44.51

Table 8: Sabra's Table of Chords[69]

[69] Highlighted in green are the ranks of the notes in Sabra's scale
The decimal numbers are the retained string length

Table No 6

"Table of the Commatic Scale reproducing, for the first time, the ratio of each comma with the fundamental."

In the folder No1, from Sabra's manuscripts, entitled "Justesse absolue", we found a table named *"Composition de la véritable Gamme Commatique"* Translated *"The composition of the true Commatic Scale"*. One would link it to the "Commatic Scale" that was presented in the Congress of Cairo, and contained 90 divisions of the octave[70]. However, the number of divisions of this table suggest a more generalized elaboration. Below, we will put a part of it as it was found in the manuscripts.

On the left of the table are the ratios and Logarithms of a series of 26 upward fifths going from the fundamental, while on the right, are the ratios and Logarithms of a series of 26 downward fifths going also from the fundamental. In the center two columns, entitled "Sabra", show the relative rank of each note in Sabra's Universal Scale. The full table, taken from the manuscripts is annexed.[71]

[70] Refer to page 11
[71] Refer to Appendix C

Ref	Ratio	LOG	Sabra	Sabra	LOG	Ratio	Ref
1	1	0.0000000000000000	1	613	0.3010299956639810	2	1
2	1.5	0.1760912590556810	359	255	0.1249387366083000	1.333333333	2
3	1.125	0.0511525224473813	105	509	0.2498774732166000	1.777777778	3
4	1.6875	0.2272437815030630	463	151	0.0737862141609186	1.185185185	4
5	1.265625	0.1023050448947630	209	405	0.1987249507692190	1.580246914	5
6	1.8984375	0.2783963039504440	567	47	0.0226336917135373	1.053497942	6
7	1.423828125	0.1534575673421440	313	301	0.1475724283218370	1.404663923	7
8	1.067871094	0.0285188307338439	59	555	0.2725111649301370	1.872885231	8
9	1.601806641	0.2046100897895250	417	197	0.0964199058744560	1.248590154	9
10	1.20135498	0.0796713531812252	163	451	0.2213586424827560	1.664786872	10
11	1.802032471	0.2557626122369060	521	93	0.0452673834270748	1.109857915	11
12	1.351524353	0.1308238756286060	267	347	0.1702061200353750	1.479810553	12
13	1.013643265	0.0058851390203065	13	601	0.2951448566436750	1.973080737	13
14	1.520464897	0.1819763980759880	371	243	0.1190535975879930	1.315387158	14
15	1.140348673	0.0570376614676878	117	497	0.2439923341962930	1.753849544	15
16	1.710523009	0.2331289205233690	475	139	0.0679010751406122	1.169233029	16
17	1.282892257	0.1081901839150690	221	393	0.1928398117489120	1.558977373	17
18	1.924338385	0.2842814429707500	579	35	0.0167485526932309	1.039318248	18
19	1.443253789	0.1593427063624500	325	289	0.1416872893015310	1.385757664	19
20	1.082440342	0.0344039697541504	71	543	0.2666260259098310	1.847676886	20
21	1.623660513	0.2104952288098320	429	185	0.0905347668541497	1.231784591	21
22	1.217745385	0.0855564922015317	175	439	0.2154735034624500	1.642379454	22
23	1.826618077	0.2616477512572130	533	81	0.0393822444067683	1.094919636	23
24	1.369963558	0.1367090146489130	279	335	0.1643209810150680	1.459892848	24
25	1.027472668	0.0117702780406131	25	589	0.2892597176233680	1.946523798	25
26	1.541209002	0.1878615370962940	383	231	0.1131684585676870	1.297682532	26

Table 9: Sabra's Commatic Scale

The importance of this table resides from the fact that Sabra built his scale upon it. He proceeded by calculating the ratios of 26 fifths going Upward and Downward from each of the 12 Chromatic Notes, then he sorted and organized the results in one table, and found that the difference between the all the steps is approximately 1/12 of the Pythagorean comma, which he later considered as a unit for his scale.

Discussion

As a result of this research, we were able to reveal the parameters that are enough to know by numbers, the essence of Sabra's Universal Scale, and how he tried to divide the Octave. Our findings on this point converge with Sabra's announced parameters. Also, we have dressed a main table, annexed[72]. This table shows all the necessary numbers that are enough to understand Sabra's Scale.

However, Sabra's approach toward "testing and packaging his product", was not successful. From what we have found, Wadia Sabra was not able to complete what he promised to publish. Also, we have observed many discrepancies in the announced numbers of tables, in their enumeration, and in the used methodology and descriptions. On this subject, we did not complete all the announced tables, however, with a certain interest, effort and some calculation exercises, they could all be deducted from our completed main table.

Moreover, as a general view on the evolution of Wadia Sabra's work, some notes shall be considered. We have already mentioned that Sabra started promoting his new theory during the late 1930's. In one of his publications on the subject, "Exposé d'un nouveau Système Perfectionné de Partage des 12 Demi-Tons de l'Octave", dated January 1940, he announced his work with confidence, while citing the advantages of his theory and characteristics. His adopted language suggested the existence of a completed work that was waiting an opportunity to be revealed. However, 8 years later, he was still discussing with H.P. Mullholland, the best possible consideration of a unit, while trying different divisors for the

[72] Refer to Appendix A

octave, and reviewing some mathematical proprieties. In other words, he was reviewing the fundaments of his work. This unconventional succession of steps in creating and announcing a theory, raises many questions about how much Sabra was really confident of his work, and how much was he able to create a balance between his optimism, his confidence about a prospective result and the actual final results. We believe that the lack of equilibrium in this triad was due to a lack of systematic methodology in his work. It is clear that extensive time and efforts were put on the elaboration of the calculations, but we were not able to find any evidence of a methodic description, table format, titled ratios, abbreviations, etc… The elaboration of the tables was still in the draft phase.

Chapter V: Conclusion

Wadia Sabra's optimistic announcements about his theory, has certainly created high expectations in the musicological and "para-musicological" communities then. In addition to the results found in the last chapter, which is that Sabra divided the octave into 612 periodic divisions and used the $1/12^{th}$ of the Pythagorean comma as a unit, we have found that he had never completed the calculation of his tables, thus never completed his announced work. Moreover, we are convinced that he has prematurely announced his work, while he struggled to shape it, and most importantly failed to complete it. His optimistic ambitions led him to explicitly announce some mathematical impossibilities, as stated in our key document by H.P. Mullholland. As it also let him promote the importance of his theory based on replies to his letters written with polite formalities. These replies as shown in chapter 3, ranged from a simple expression of interest to more optimistic assumptions with no concrete ground. We have previously mentioned the term "para-musicological" in order to refer to some writers, culture amateurs and musicians who echoed the most Sabra's optimism and weighted it with their personal unrealistic assumptions.

After nearly a century

With no doubt, Wadia Sabra made an extensive research on different octave divisions in both Oriental and Western music, invested days and years into this subject, and went deep in analyzing and comparing the different divisions of the musical scales. Thus, these efforts are not to be judged nor evaluated. What we are evaluating will go beyond the lack of optimistic results or the possibility of finding them, and arrives to questioning the validity and importance of such work after nearly a century. In order to proceed, we will defer between the title and the content of the work.

As for the content, we have understood Sabra's work as being a comparative representation of different music interval, using a certain degree of approximation. This comparison should've been represented through different tables, as announced by Sabra, which aims to simplify the calculation in the same way of using the logarithmic tables in math and physics then. But, do we still need these tables? Will we still have interest in a unit promoted to be almost always an integer at a cost of a certain acceptable approximation? Absolutely not, a simple computer function can elaborate instantly an exact calculation of any note (existing in a known or unknown scale), and express it in any format with supreme accuracy.

Concerning the quantification of the pitch, it is well known that it has always been performed based on the physical properties of the sound as the series of harmonics, and based on conventional ratios and proportions. Thus, the choice of an octave divider will be affected by how much the real note converge to the nearest quantified point. Sabra's choice of dividing the octave to 612 parts can surely reach a good approximation, but dividing to 1200, as in the cent system, could give it a better approximation. It could even give more exact numbers if unlike Sabra, we consider a decimal expression. Moreover, since the physical properties of sound are continuous, and since the neurological properties of the human ear are not yet discovered to be clearly pitch quantified[73] and differ from one person to another, the aforementioned decimal expression of notes in any system is unavoidable for more accuracy.

Sabra's title contained the word Universal as an adjective to his scale. Since we have excessively talked about the scale, we'll add some thoughts about its adjective. Does a

[73] Recent neurosciences studies suggest a just noticeable difference which differs between cultures and persons (Liu, 2013)

particular division of the octave give the Universality property to a scale? Does the exercise of converting and converging the ratios of the different scales to the nearest point of a certain octave division, makes this division Universal? Didn't we miss that music, scales and divisions are all related to concrete sounds, abstract ideas, collective memories and emotions, which are all by the way considered universal? We believe that universality in music, cannot be exclusive to a particular aspect or work despite how general it is. Universality is adjacent to sounds, ideas and emotions, which are common between all humans. If the real basis of any scale is universal, each existing scale though, will be, by definition, universal.

Prospective ideas

Once the weight of the predefined properties, expectations and ambitions that were put on this scale is unloaded, we will still have a scale based on an undiscovered 612 periodic octave division worth of experimenting. The author himself who was a composer, never used it, nor have we found any evidence in his archive about a possible draft composition on this scale. However, as we have discovered its numbers, and amid the wide expansion of experimental micro-tonal music and the development of its related instruments, trying the sounds and possibilities of this scale, and evaluating its musical qualities, is now easier than ever.

Final Thoughts

Wadia Sabra's work is still considered a part of the first milestone in the Lebanese musicological development despite considering its aspect and development as obsolete nowadays. Although Sabra has failed to conclude the culmination of his work and researches, there are still some particular aspects of his work related to his researches on Oriental music that could be studied later.

Appendix A

Table No1 (As described by Wadia Sabra)

N	String Length	Ratios	Logarithms
1	100.00000	1.00000000000	0.00000000000
2	99.88714	1.00112989063	0.00049042825
3	99.77440	1.00226105791	0.00098085650
4	99.66180	1.00339350328	0.00147128476
5	99.54932	1.00452722820	0.00196171301
6	99.43696	1.00566223410	0.00245214126
7	99.32474	1.00679852243	0.00294256951
8	99.21264	1.00793609465	0.00343299776
9	99.10067	1.00907495219	0.00392342601
10	98.98882	1.01021509652	0.00441385427
11	98.87710	1.01135652909	0.00490428252
12	98.76551	1.01249925136	0.00539471077
13	98.65404	1.01364326477	0.00588513902
14	98.54269	1.01478857079	0.00637556727
15	98.43148	1.01593517089	0.00686599552
16	98.32039	1.01708306652	0.00735642378
17	98.20942	1.01823225914	0.00784685203
18	98.09858	1.01938275023	0.00833728028
19	97.98786	1.02053454124	0.00882770853
20	97.87727	1.02168763366	0.00931813678
21	97.76681	1.02284202894	0.00980856503
22	97.65647	1.02399772856	0.01029899329
23	97.54625	1.02515473400	0.01078942154
24	97.43616	1.02631304672	0.01127984979
25	97.32619	1.02747266821	0.01177027804
26	97.21635	1.02863359995	0.01226070629
27	97.09008	1.02997130797	0.01282512668
28	96.98051	1.03113506289	0.01331555493
29	96.87105	1.03230013274	0.01380598318
30	96.76172	1.03346651898	0.01429641144
31	96.65252	1.03463422311	0.01478683969
32	96.54343	1.03580324663	0.01527726794
33	96.43447	1.03697359101	0.01576769619
34	96.32563	1.03814525775	0.01625812444
35	96.21692	1.03931824834	0.01674855269
36	96.10833	1.04049256429	0.01723898095
37	95.99986	1.04166820709	0.01772940920
38	95.89151	1.04284517823	0.01821983745
39	95.78329	1.04402347923	0.01871026570
40	95.67518	1.04520311157	0.01920069395
41	95.56720	1.04638407677	0.01969112220
42	95.45934	1.04756637633	0.02018155046
43	95.35161	1.04875001176	0.02067197871
44	95.24399	1.04993498457	0.02116240696
45	95.13650	1.05112129627	0.02165283521
46	95.02913	1.05230894837	0.02214326346
47	94.92187	1.05349794239	0.02263369171
48	94.81474	1.05468827984	0.02312411997
49	94.70774	1.05587996224	0.02361454822
50	94.60085	1.05707299111	0.02410497647
51	94.49408	1.05826736798	0.02459540472
52	94.38743	1.05946309436	0.02508583297
53	94.28090	1.06066017178	0.02557626122
54	94.17450	1.06185860177	0.02606668948
55	94.06821	1.06305838585	0.02655711773
56	93.96204	1.06425952556	0.02704754598
57	93.85600	1.06546202242	0.02753797423
58	93.75007	1.06666587797	0.02802840248
59	93.64426	1.06787109375	0.02851883073
60	93.53857	1.06907767129	0.02900925899
61	93.43300	1.07028561213	0.02949968724
62	93.32755	1.07149491781	0.02999011549
63	93.22222	1.07270558988	0.03048054374
64	93.11701	1.07391762987	0.03097097199
65	93.01192	1.07513103933	0.03146140024
66	92.90694	1.07634581982	0.03195182850
67	92.80209	1.07756197287	0.03244225675
68	92.69735	1.07877950005	0.03293268500
69	92.59273	1.07999840289	0.03342311325
70	92.48823	1.08121868297	0.03391354150
71	92.38384	1.08244034182	0.03440396975
72	92.27958	1.08366338102	0.03489439801
73	92.17543	1.08488780212	0.03538482626
74	92.07140	1.08611360668	0.03587525451
75	91.96749	1.08734079626	0.03636568276
76	91.86369	1.08856937244	0.03685611101
77	91.76001	1.08979933677	0.03734653926
78	91.64084	1.09121658904	0.03791095965

79	91.53741	1.09244954443	0.03840138790	125	86.90383	1.15069728256	0.06096108748
80	91.43410	1.09368389294	0.03889181616	126	86.80575	1.15199744463	0.06145151573
81	91.33090	1.09491963612	0.03938224441	127	86.70778	1.15329907575	0.06194194398
82	91.22783	1.09615677555	0.03987267266	128	86.60992	1.15460217756	0.06243237224
83	91.12487	1.09739531282	0.04036310091	129	86.49743	1.15610370404	0.06299679262
84	91.02202	1.09863524950	0.04085352916	130	86.39981	1.15740997478	0.06348722088
85	90.91929	1.09987658717	0.04134395741	131	86.30230	1.15871772146	0.06397764913
86	90.81668	1.10111932741	0.04183438567	132	86.20489	1.16002694575	0.06446807738
87	90.71418	1.10236347182	0.04232481392	133	86.10760	1.16133764933	0.06495850563
88	90.61180	1.10360902198	0.04281524217	134	86.01042	1.16264983385	0.06544893388
89	90.50953	1.10485597947	0.04330567042	135	85.91335	1.16396350100	0.06593936213
90	90.40738	1.10610434588	0.04379609867	136	85.81638	1.16527865245	0.06642979039
91	90.30535	1.10735412282	0.04428652692	137	85.71953	1.16659528988	0.06692021864
92	90.20343	1.10860531186	0.04477695518	138	85.62279	1.16791341497	0.06741064689
93	90.10162	1.10985791461	0.04526738343	139	85.52615	1.16923302939	0.06790107514
94	89.99993	1.11111193267	0.04575781168	140	85.42962	1.17055413483	0.06839150339
95	89.89836	1.11236736763	0.04624823993	141	85.33321	1.17187673297	0.06888193164
96	89.79690	1.11362422109	0.04673866818	142	85.23690	1.17320082551	0.06937235990
97	89.69555	1.11488249466	0.04722909643	143	85.14070	1.17452641413	0.06986278815
98	89.59432	1.11614218994	0.04771952469	144	85.04461	1.17585350052	0.07035321640
99	89.49320	1.11740330854	0.04820995294	145	84.94863	1.17718208637	0.07084364465
100	89.39220	1.11866585207	0.04870038119	146	84.85275	1.17851217337	0.07133407290
101	89.29131	1.11992982213	0.04919080944	147	84.75698	1.17984376323	0.07182450115
102	89.19053	1.12119522034	0.04968123769	148	84.66133	1.18117685764	0.07231492941
103	89.08987	1.12246204831	0.05017166594	149	84.56578	1.18251145830	0.07280535766
104	88.98932	1.12373030766	0.05066209420	150	84.47033	1.18384756692	0.07329578591
105	88.88889	1.12500000000	0.05115252245	151	84.37500	1.18518518519	0.07378621416
106	88.78857	1.12627112696	0.05164295070	152	84.27977	1.18652431482	0.07427664241
107	88.68836	1.12754369015	0.05213337895	153	84.18465	1.18786495752	0.07476707066
108	88.58826	1.12881769119	0.05262380720	154	84.08964	1.18920711500	0.07525749892
109	88.48828	1.13009313172	0.05311423545	155	83.99474	1.19055078898	0.07574792717
110	88.38841	1.13137001336	0.05360466371	156	83.89994	1.19189598115	0.07623835542
111	88.28866	1.13264833774	0.05409509196	157	83.80525	1.19324269325	0.07672878367
112	88.18901	1.13392810648	0.05458552021	158	83.71066	1.19459092699	0.07721921192
113	88.08948	1.13520932122	0.05507594846	159	83.61619	1.19594068408	0.07770964017
114	87.99006	1.13649198359	0.05556637671	160	83.52182	1.19729196625	0.07820006843
115	87.89075	1.13777609523	0.05605680496	161	83.42755	1.19864477522	0.07869049668
116	87.79156	1.13906165778	0.05654723322	162	83.33339	1.19999911272	0.07918092493
117	87.69248	1.14034867287	0.05703766147	163	83.23934	1.20135498047	0.07967135318
118	87.59351	1.14163714214	0.05752808972	164	83.14540	1.20271238020	0.08016178143
119	87.49465	1.14292706725	0.05801851797	165	83.05156	1.20407131365	0.08065220968
120	87.39590	1.14421844983	0.05850894622	166	82.95783	1.20543178254	0.08114263794
121	87.29726	1.14551129153	0.05899937447	167	82.86420	1.20679378861	0.08163306619
122	87.19874	1.14680559401	0.05948980273	168	82.77068	1.20815733360	0.08212349444
123	87.10032	1.14810135890	0.05998023098	169	82.67726	1.20952241925	0.08261392269
124	87.00202	1.14939858786	0.06047065923	170	82.58395	1.21088904730	0.08310435094

171	82.49074	1.21225721948	0.08359477919	217	78.30176	1.27711048637	0.10622847091
172	82.39764	1.21362693755	0.08408520745	218	78.21339	1.27855348154	0.10671889916
173	82.30465	1.21499820325	0.08457563570	219	78.12512	1.27999810713	0.10720932741
174	82.21176	1.21637101834	0.08506606395	220	78.03694	1.28144436500	0.10769975566
175	82.11897	1.21774538455	0.08555649220	221	77.94887	1.28289225697	0.10819018391
176	82.02629	1.21912130365	0.08604692045	222	77.86089	1.28434178491	0.10868061217
177	81.93372	1.22049877738	0.08653734870	223	77.77302	1.28579295066	0.10917104042
178	81.84124	1.22187780751	0.08702777696	224	77.68524	1.28724575606	0.10966146867
179	81.74888	1.22325839579	0.08751820521	225	77.59757	1.28870020298	0.11015189692
180	81.64270	1.22484920768	0.08808262560	226	77.50999	1.29015629326	0.11064232517
181	81.55056	1.22623315332	0.08857305385	227	77.42251	1.29161402876	0.11113275342
182	81.45852	1.22761866267	0.08906348210	228	77.33513	1.29307341135	0.11162318168
183	81.36659	1.22900573749	0.08955391035	229	77.24785	1.29453444287	0.11211360993
184	81.27475	1.23039437955	0.09004433860	230	77.16066	1.29599712521	0.11260403818
185	81.18303	1.23178459063	0.09053476685	231	77.06045	1.29768253169	0.11316845857
186	81.09140	1.23317637249	0.09102519511	232	76.97348	1.29914877102	0.11365888682
187	80.99988	1.23456972692	0.09151562336	233	76.88661	1.30061666704	0.11414931507
188	80.90846	1.23596465568	0.09200605161	234	76.79983	1.30208622163	0.11463974332
189	80.81715	1.23736116056	0.09249647986	235	76.71315	1.30355743664	0.11513017157
190	80.72594	1.23875924334	0.09298690811	236	76.62657	1.30503031397	0.11562059983
191	80.63483	1.24015890580	0.09347733636	237	76.54009	1.30650485549	0.11611102808
192	80.54382	1.24156014972	0.09396776462	238	76.45371	1.30798106308	0.11660145633
193	80.45292	1.24296297690	0.09445819287	239	76.36742	1.30945893863	0.11709188458
194	80.36212	1.24436738912	0.09494862112	240	76.28123	1.31093848401	0.11758231283
195	80.27142	1.24577338817	0.09543904937	241	76.19514	1.31241970112	0.11807274108
196	80.18083	1.24718097584	0.09592947762	242	76.10914	1.31390259184	0.11856316934
197	80.09033	1.24859015394	0.09641990587	243	76.02324	1.31538715806	0.11905359759
198	79.99994	1.25000092425	0.09691033413	244	75.93744	1.31687340168	0.11954402584
199	79.90965	1.25141328858	0.09740076238	245	75.85174	1.31836132460	0.12003445409
200	79.81946	1.25282724873	0.09789119063	246	75.76613	1.31985092870	0.12052488234
201	79.72938	1.25424280649	0.09838161888	247	75.68062	1.32134221589	0.12101531059
202	79.63940	1.25565996369	0.09887204713	248	75.59521	1.32283518808	0.12150573885
203	79.54951	1.25707872211	0.09936247538	249	75.50989	1.32432984716	0.12199616710
204	79.45973	1.25849908358	0.09985290364	250	75.42467	1.32582619504	0.12248659535
205	79.37005	1.25992104989	0.10034333189	251	75.33954	1.32732423363	0.12297702360
206	79.28047	1.26134462288	0.10083376014	252	75.25451	1.32882396485	0.12346745185
207	79.19100	1.26276980435	0.10132418839	253	75.16958	1.33032539059	0.12395788010
208	79.10162	1.26419659611	0.10181461664	254	75.08474	1.33182851278	0.12444830836
209	79.01235	1.26562500000	0.10230504489	255	75.00000	1.33333333333	0.12493873661
210	78.92317	1.26705501783	0.10279547315	256	74.91535	1.33483985417	0.12542916486
211	78.83410	1.26848665141	0.10328590140	257	74.83080	1.33634807721	0.12591959311
212	78.74512	1.26991990259	0.10377632965	258	74.74635	1.33785800438	0.12641002136
213	78.65625	1.27135477319	0.10426675790	259	74.66199	1.33936963760	0.12690044962
214	78.56748	1.27279126503	0.10475718615	260	74.57772	1.34088297880	0.12739087787
215	78.47881	1.27422937995	0.10524761440	261	74.49355	1.34239802991	0.12788130612
216	78.39023	1.27566911979	0.10573804266	262	74.40948	1.34391479286	0.12837173437

263	74.32550	1.34543326959	0.12886216262	309	70.55116	1.41741118113	0.15149585434
264	74.24161	1.34695346203	0.12935259087	310	70.47153	1.41901270074	0.15198628259
265	74.15782	1.34847537212	0.12984301913	311	70.39200	1.42061602989	0.15247671084
266	74.07413	1.34999900181	0.13033344738	312	70.31255	1.42222117063	0.15296713909
267	73.99053	1.35152435303	0.13082387563	313	70.23320	1.42382812500	0.15345756734
268	73.90702	1.35305142773	0.13131430388	314	70.15393	1.42543689505	0.15394799559
269	73.82361	1.35458022785	0.13180473213	315	70.07475	1.42704748284	0.15443842385
270	73.74029	1.35611075536	0.13229516038	316	69.99567	1.42865989042	0.15492885210
271	73.65707	1.35764301219	0.13278558864	317	69.91667	1.43027411984	0.15541928035
272	73.57393	1.35917700030	0.13327601689	318	69.83776	1.43189017316	0.15590970860
273	73.49090	1.36071272166	0.13376644514	319	69.75894	1.43350805245	0.15640013685
274	73.40796	1.36225017821	0.13425687339	320	69.68021	1.43512775976	0.15689056510
275	73.32511	1.36378937192	0.13474730164	321	69.60157	1.43674929716	0.15738099336
276	73.24235	1.36533030475	0.13523772989	322	69.52301	1.43837266673	0.15787142161
277	73.15969	1.36687297866	0.13572815815	323	69.44455	1.43999787052	0.15836184986
278	73.07712	1.36841739563	0.13621858640	324	69.36617	1.44162491062	0.15885227811
279	72.99464	1.36996355762	0.13670901465	325	69.28788	1.44325378910	0.15934270636
280	72.91226	1.37151146660	0.13719944290	326	69.20968	1.44488450803	0.15983313461
281	72.82997	1.37306112455	0.13768987115	327	69.13157	1.44651706949	0.16032356287
282	72.73538	1.37484675052	0.13825429154	328	69.05355	1.44815147557	0.16081399112
283	72.65329	1.37640017698	0.13874471979	329	68.97561	1.44978772835	0.16130441937
284	72.57129	1.37795535864	0.13923514804	330	68.89777	1.45142582991	0.16179484762
285	72.48939	1.37951229749	0.13972557630	331	68.82001	1.45306578236	0.16228527587
286	72.40757	1.38107099550	0.14021600455	332	68.74234	1.45470758776	0.16277570412
287	72.32585	1.38263145467	0.14070643280	333	68.65306	1.45659939258	0.16334012451
288	72.24423	1.38419367700	0.14119686105	334	68.57557	1.45824519058	0.16383055276
289	72.16269	1.38575766446	0.14168728930	335	68.49818	1.45989284815	0.16432098102
290	72.08125	1.38732341906	0.14217771755	336	68.42087	1.46154236740	0.16481140927
291	71.99989	1.38889094278	0.14266814581	337	68.34365	1.46319375042	0.16530183752
292	71.91863	1.39046023764	0.14315857406	338	68.26652	1.46484699933	0.16579226577
293	71.83746	1.39203130563	0.14364900231	339	68.18947	1.46650211622	0.16628269402
294	71.75639	1.39360414876	0.14413943056	340	68.11251	1.46815910322	0.16677312227
295	71.67540	1.39517876903	0.14462985881	341	68.03564	1.46981796243	0.16726355053
296	71.59451	1.39675516844	0.14512028706	342	67.95885	1.47147869597	0.16775397878
297	71.51371	1.39833334901	0.14561071532	343	67.88215	1.47314130596	0.16824440703
298	71.43299	1.39991331276	0.14610114357	344	67.80554	1.47480579451	0.16873483528
299	71.35237	1.40149506169	0.14659157182	345	67.72901	1.47647216376	0.16922526353
300	71.27184	1.40307859783	0.14708200007	346	67.65257	1.47814041582	0.16971569178
301	71.19141	1.40466392318	0.14757242832	347	67.57622	1.47981055282	0.17020612004
302	71.11106	1.40625103978	0.14806285657	348	67.49995	1.48148257689	0.17069654829
303	71.03080	1.40783994965	0.14855328483	349	67.42377	1.48315649017	0.17118697654
304	70.95064	1.40943065482	0.14904371308	350	67.34767	1.48483229479	0.17167740479
305	70.87056	1.41102315731	0.14953414133	351	67.27166	1.48650999288	0.17216783304
306	70.79057	1.41261745915	0.15002456958	352	67.19574	1.48818958659	0.17265826129
307	70.71068	1.41421356237	0.15051499783	353	67.11990	1.48987107806	0.17314868955
308	70.63087	1.41581146902	0.15100542608	354	67.04415	1.49155446942	0.17363911780

355	66.96848	1.49323976284	0.17412954605	401	63.56774	1.57312501764	0.19676323776
356	66.89290	1.49492696045	0.17461997430	402	63.49600	1.57490247685	0.19725366601
357	66.81740	1.49661606441	0.17511040255	403	63.42433	1.57668194440	0.19774409427
358	66.74199	1.49830707688	0.17560083080	404	63.35275	1.57846342255	0.19823452252
359	66.66667	1.50000000000	0.17609125906	405	63.28125	1.58024691358	0.19872495077
360	66.59143	1.50169483594	0.17658168731	406	63.20983	1.58203241976	0.19921537902
361	66.51627	1.50339158686	0.17707211556	407	63.13849	1.58381994336	0.19970580727
362	66.44120	1.50509025493	0.17756254381	408	63.06723	1.58560948667	0.20019623552
363	66.36621	1.50679084230	0.17805297206	409	62.99605	1.58740105197	0.20068666378
364	66.29131	1.50849335115	0.17854340031	410	62.92495	1.58919464154	0.20117709203
365	66.21649	1.51019778365	0.17903382857	411	62.85394	1.59099025767	0.20166752028
366	66.14176	1.51190414197	0.17952425682	412	62.78300	1.59278790265	0.20215794853
367	66.06711	1.51361242829	0.18001468507	413	62.71214	1.59458757877	0.20264837678
368	65.99255	1.51532264478	0.18050511332	414	62.64136	1.59638928833	0.20313880503
369	65.91807	1.51703479364	0.18099554157	415	62.57066	1.59819303363	0.20362923329
370	65.84367	1.51874887703	0.18148596982	416	62.50005	1.59999881696	0.20411966154
371	65.76936	1.52046489716	0.18197639808	417	62.42951	1.60180664062	0.20461008979
372	65.69513	1.52218285619	0.18246682633	418	62.35905	1.60361650693	0.20510051804
373	65.62099	1.52390275633	0.18295725458	419	62.28867	1.60542841820	0.20559094629
374	65.54692	1.52562459978	0.18344768283	420	62.21837	1.60724237672	0.20608137454
375	65.47295	1.52734838871	0.18393811108	421	62.14815	1.60905838482	0.20657180280
376	65.39905	1.52907412534	0.18442853933	422	62.07801	1.61087644480	0.20706223105
377	65.32524	1.53080181186	0.18491896759	423	62.00795	1.61269655900	0.20755265930
378	65.25152	1.53253145048	0.18540939584	424	61.93796	1.61451872973	0.20804308755
379	65.17787	1.53426304341	0.18589982409	425	61.86806	1.61634295931	0.20853351580
380	65.10431	1.53599659284	0.18639025234	426	61.79823	1.61816925007	0.20902394405
381	65.03083	1.53773210099	0.18688068059	427	61.72849	1.61999760434	0.20951437231
382	64.95744	1.53946957008	0.18737110884	428	61.65882	1.62182802445	0.21000480056
383	64.88413	1.54120900232	0.18786153710	429	61.58923	1.62366051273	0.21049522881
384	64.79986	1.54321329970	0.18842595748	430	61.51972	1.62549507153	0.21098565706
385	64.72672	1.54495696195	0.18891638574	431	61.45029	1.62733170318	0.21147608531
386	64.65367	1.54670259434	0.18940681399	432	61.38093	1.62917041001	0.21196651356
387	64.58070	1.54845019910	0.18989724224	433	61.31166	1.63101119439	0.21245694182
388	64.50781	1.55019977847	0.19038767049	434	61.24246	1.63285405865	0.21294737007
389	64.43501	1.55195133467	0.19087809874	435	61.16292	1.63497753776	0.21351179046
390	64.36229	1.55370486994	0.19136852699	436	61.09389	1.63682488356	0.21400221871
391	64.28965	1.55546038651	0.19185895525	437	61.02494	1.63867431665	0.21449264696
392	64.21709	1.55721788662	0.19234938350	438	60.95607	1.64052583940	0.21498307521
393	64.14461	1.55897737252	0.19283981175	439	60.88727	1.64237945417	0.21547350346
394	64.07222	1.56073884644	0.19333024000	440	60.81855	1.64423516333	0.21596393171
395	63.99991	1.56250231063	0.19382066825	441	60.74991	1.64609296923	0.21645435997
396	63.92767	1.56426776735	0.19431109650	442	60.68135	1.64795287424	0.21694478822
397	63.85552	1.56603521884	0.19480152476	443	60.61286	1.64981488075	0.21743521647
398	63.78346	1.56780466735	0.19529195301	444	60.54445	1.65167899112	0.21792564472
399	63.71147	1.56957611515	0.19578238126	445	60.47612	1.65354520773	0.21841607297
400	63.63956	1.57134956450	0.19627280951	446	60.40787	1.65541353297	0.21890650122

447	60.33969	1.65728396920	0.21939692948	493	57.27556	1.74594525151	0.24203062119
448	60.27159	1.65915651883	0.21988735773	494	57.21092	1.74791797868	0.24252104944
449	60.20357	1.66103118423	0.22037778598	495	57.14635	1.74989293482	0.24301147769
450	60.13562	1.66290796779	0.22086821423	496	57.08186	1.75187012245	0.24350190594
451	60.06775	1.66478687192	0.22135864248	497	57.01743	1.75384954408	0.24399233420
452	59.99996	1.66666789900	0.22184907073	498	56.95308	1.75583120224	0.24448276245
453	59.93224	1.66855105144	0.22233949899	499	56.88880	1.75781509946	0.24497319070
454	59.86460	1.67043633164	0.22282992724	500	56.82460	1.75980123827	0.24546361895
455	59.79703	1.67232374199	0.22332035549	501	56.76047	1.76178962119	0.24595404720
456	59.72955	1.67421328491	0.22381078374	502	56.69641	1.76378025077	0.24644447545
457	59.66213	1.67610496281	0.22430121199	503	56.63242	1.76577312955	0.24693490371
458	59.59480	1.67799877810	0.22479164024	504	56.56850	1.76776826006	0.24742533196
459	59.52754	1.67989473319	0.22528206850	505	56.50466	1.76976564485	0.24791576021
460	59.46036	1.68179283051	0.22577249675	506	56.44088	1.77176528646	0.24840618846
461	59.39325	1.68369307246	0.22626292500	507	56.37718	1.77376718745	0.24889661671
462	59.32622	1.68559546149	0.22675335325	508	56.31356	1.77577135037	0.24938704496
463	59.25926	1.68750000000	0.22724378150	509	56.25000	1.77777777778	0.24987747322
464	59.19238	1.68940669043	0.22773420975	510	56.18652	1.77978647223	0.25036790147
465	59.12557	1.69131553522	0.22822463801	511	56.12310	1.78179743628	0.25085832972
466	59.05884	1.69322653679	0.22871506626	512	56.05976	1.78381067250	0.25134875797
467	58.99219	1.69513969758	0.22920549451	513	55.99649	1.78582618346	0.25183918622
468	58.92561	1.69705502004	0.22969592276	514	55.93329	1.78784397173	0.25232961448
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470	58.79268	1.70089215971	0.23067677926	516	55.80711	1.79188639048	0.25331047098
471	58.72632	1.70281398182	0.23116720752	517	55.74412	1.79391102612	0.25380089923
472	58.66004	1.70473797538	0.23165763577	518	55.68121	1.79593794937	0.25429132748
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476	58.39567	1.71245571322	0.23361934877	522	55.43027	1.80406857030	0.25625304049
477	58.32976	1.71439060088	0.23410977703	523	55.36771	1.80610697047	0.25674346874
478	58.26393	1.71632767475	0.23460020528	524	55.30522	1.80814767381	0.25723389699
479	58.19818	1.71826693730	0.23509063353	525	55.24280	1.81019068292	0.25772432524
480	58.13249	1.72020839101	0.23558106178	526	55.18045	1.81223600041	0.25821475350
481	58.06688	1.72215203835	0.23607149003	527	55.11817	1.81428362888	0.25870518175
482	58.00135	1.72409788180	0.23656191828	528	55.05597	1.81633357094	0.25919561000
483	57.93589	1.72604592383	0.23705234654	529	54.99383	1.81838582922	0.25968603825
484	57.87050	1.72799616694	0.23754277479	530	54.93176	1.82044040633	0.26017646650
485	57.80519	1.72994861362	0.23803320304	531	54.86977	1.82249730488	0.26066689475
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489	57.53486	1.73807658219	0.24006890818	535	54.62248	1.83074816607	0.26262860776
490	57.46993	1.74004041863	0.24055933643	536	54.56083	1.83281671127	0.26311903601
491	57.40507	1.74200647399	0.24104976469	537	54.48997	1.83520023598	0.26368345640
492	57.34028	1.74397475078	0.24154019294	538	54.42847	1.83727381152	0.26417388465

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541	54.24439	1.84350860623	0.26564516941
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545	53.99992	1.85185459038	0.26760688241
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547	53.87810	1.85604174085	0.26858773892
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555	53.39355	1.87288523091	0.27251116493
556	53.33329	1.87500138638	0.27300159318
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558	53.21298	1.87924087309	0.27398244969
559	53.15292	1.88136420974	0.27447287794
560	53.09293	1.88348994553	0.27496330619
561	53.03301	1.88561808316	0.27545373444
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563	52.91337	1.88988157484	0.27643459094
564	52.85365	1.89201693432	0.27692501920
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566	52.73441	1.89629489417	0.27790587570
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568	52.61545	1.90058252674	0.27888673220
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571	52.43750	1.90703215978	0.28035801696
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574	52.26016	1.91350367968	0.28182930171
575	52.20117	1.91566572955	0.28231972996
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589	51.37363	1.94652379754	0.28925971762
590	51.31565	1.94872315653	0.28975014588
591	51.25774	1.95092500056	0.29024057413
592	51.19989	1.95312933244	0.29073100238
593	51.14210	1.95533615496	0.29122143063
594	51.08438	1.95754547096	0.29171185888
595	51.02673	1.95975728324	0.29220228713
596	50.96914	1.96197159463	0.29269271539
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601	50.68216	1.97308073709	0.29514485664
602	50.62496	1.97531010252	0.29563528490
603	50.56783	1.97754198689	0.29612571315
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607	50.33993	1.98649477074	0.29808742615
608	50.28311	1.98873929256	0.29857785441
609	50.22636	1.99098635045	0.29906828266
610	50.16968	1.99323594727	0.29955871091
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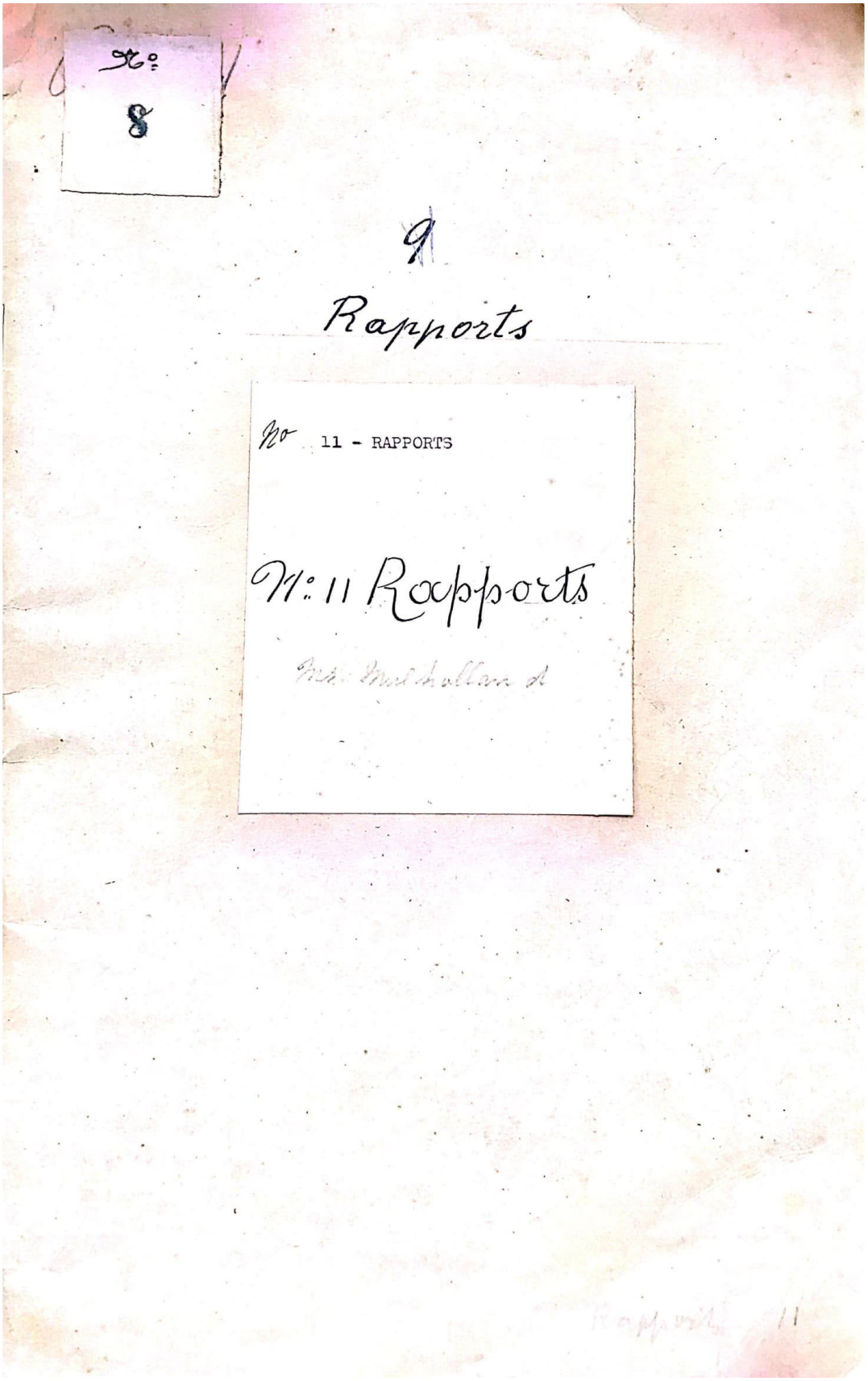
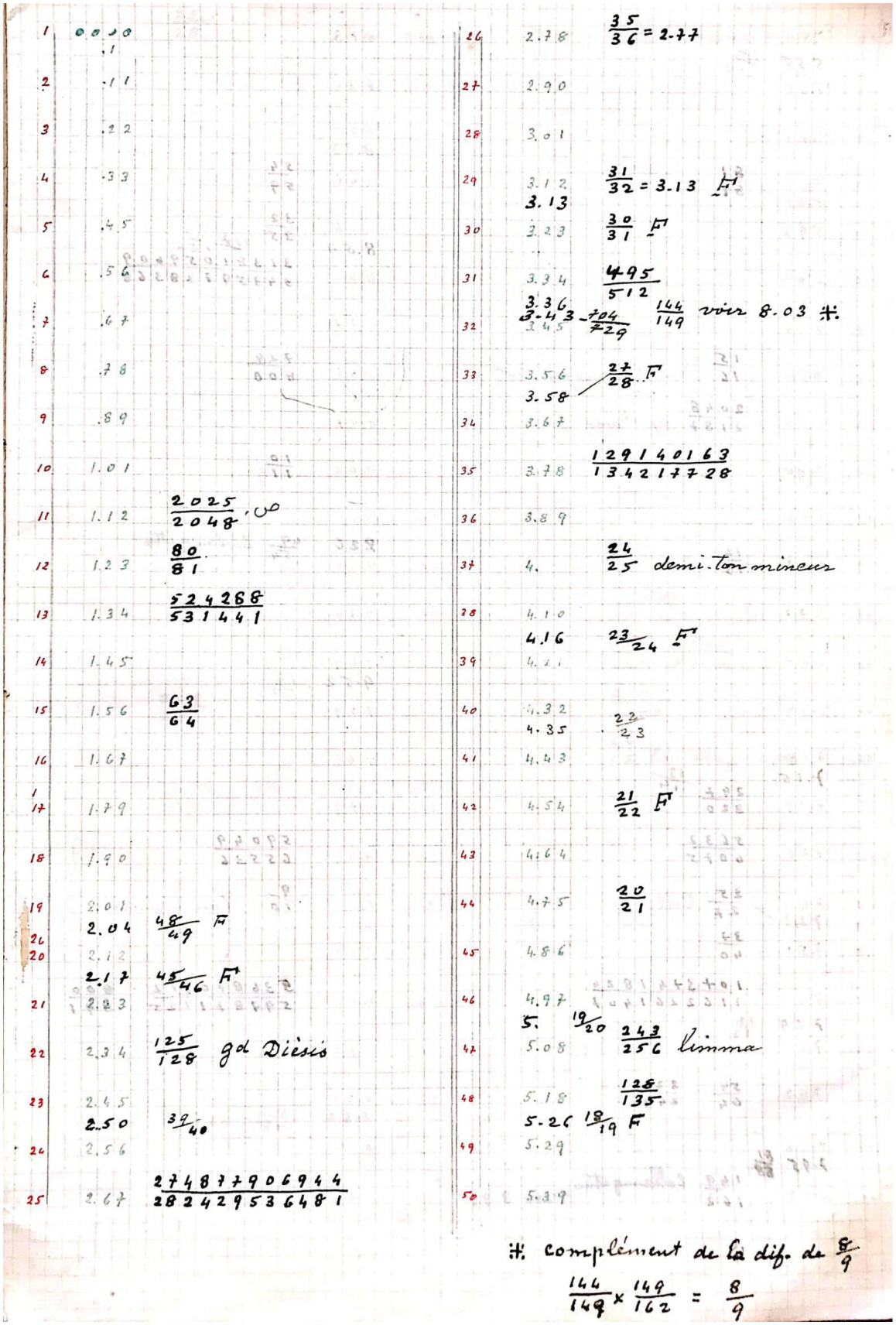
Appendix B

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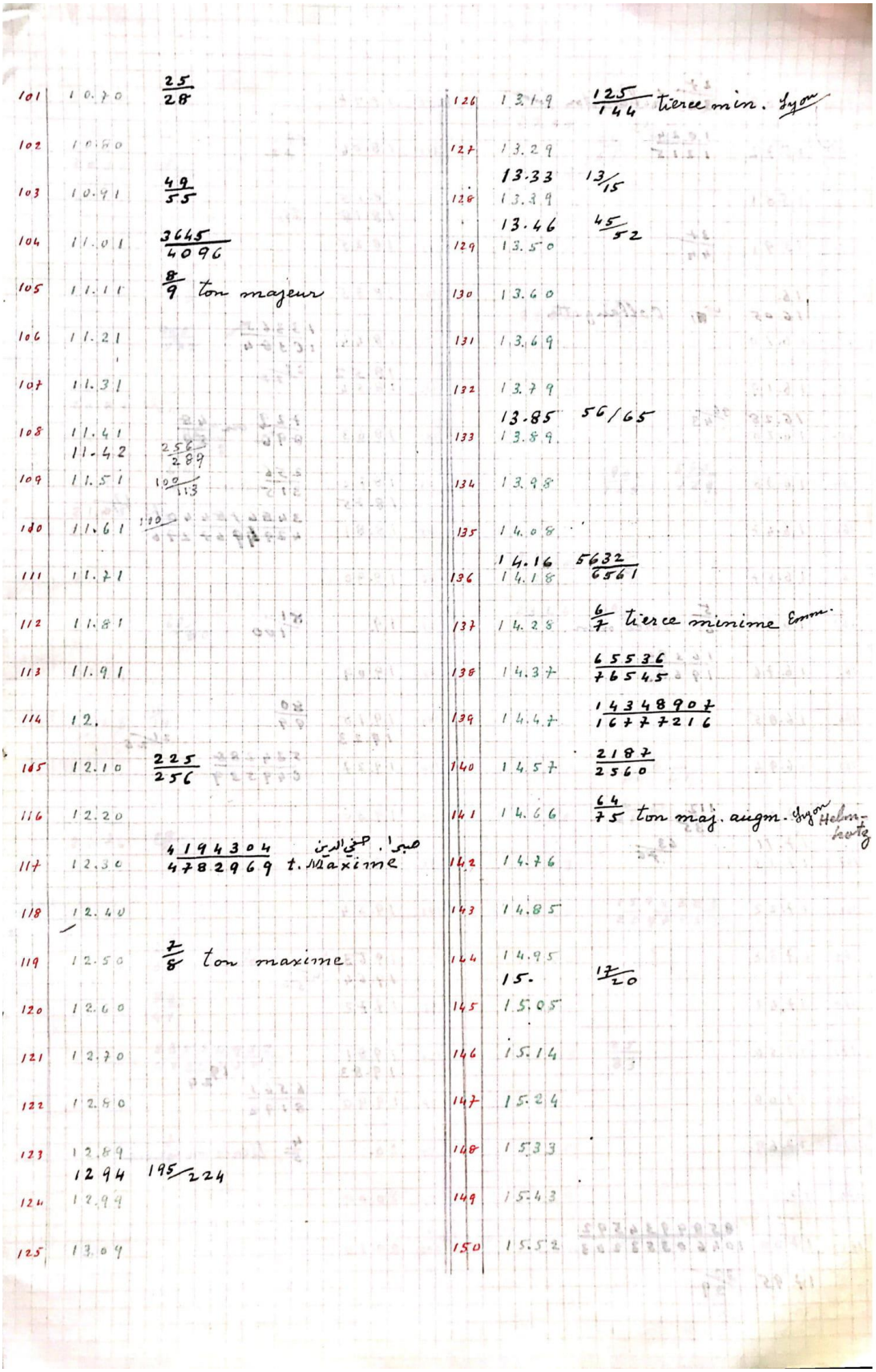
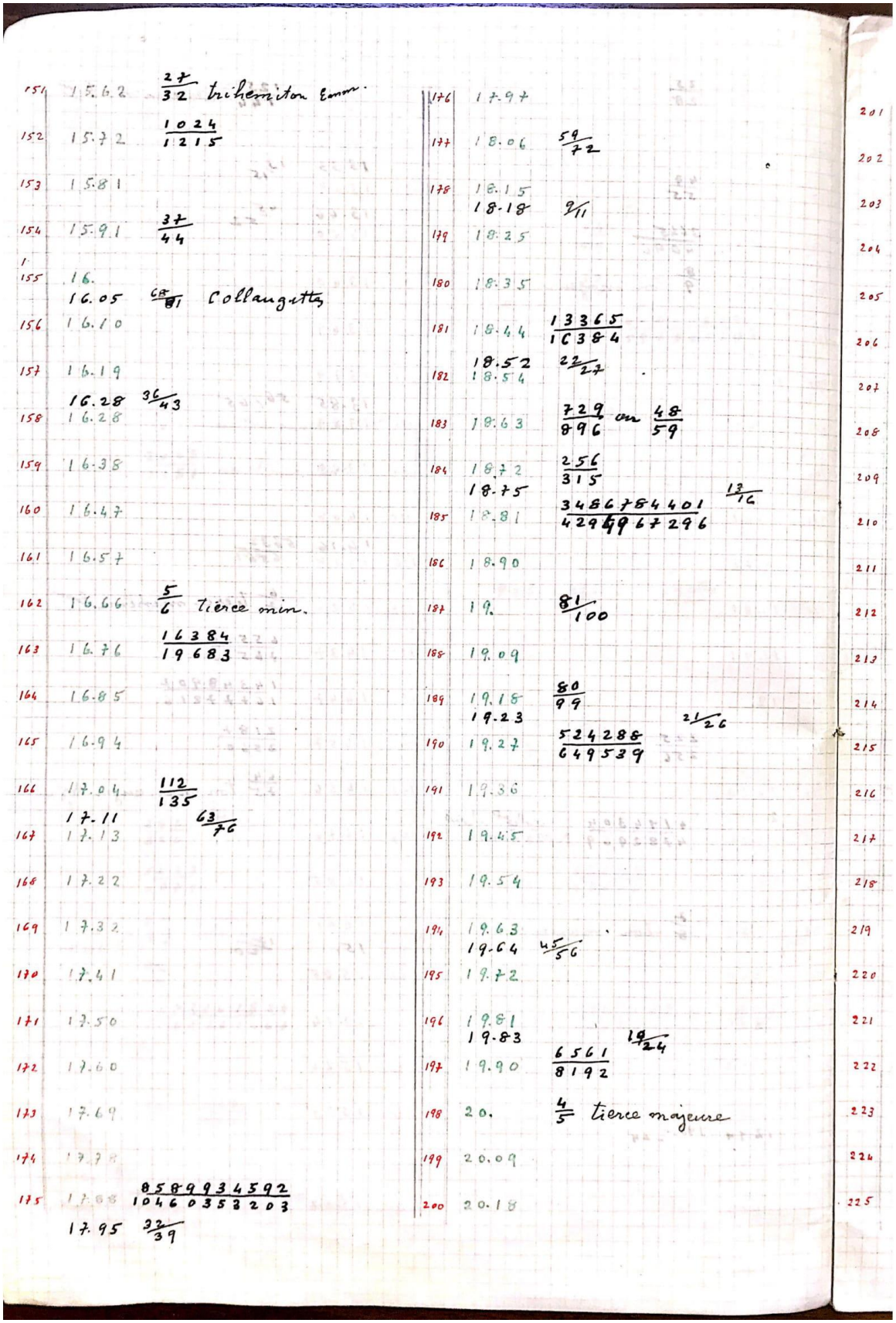
2 .11
3 .22
4 .33
5 .45
6 .56
7 .67
8 .78
9 .89
10 1.01
11 1.12 2025/2048 ; 80/81
12 1.23
13 1.34 524288/531441
14 1.45
15 1.56 63/64
16 1.67
17 1.79
18 1.90
19 2.01
20 2.04 48/49 F
2.17 45/46 F
21 2.23
22 2.34 125/128 gd Diesis
23 2.45
2.50 39/40
24 2.56
25 2.67 2748779069444/2824295364811
26 2.78 35/36 = 2.77
27 2.90
28 3.01
29 3.12 31/32 = 3.13 F
3.13
30 3.23 30/31 F
31 3.34 495/512
3.36
32 3.45 3.43 704/729 144/149 voir 8.03 ∺
33 3.56 27/28 F
3.58
34 3.67
35 3.78 1291401632/1342177728
36 3.89
37 4. 24/25 demi-ton mineur
38 4.10
4.16 23/24 F
39 4.21
40 4.32
4.35 23/23
41 4.43
42 4.54 21/22 F
43 4.64
44 4.75 20/21
45 4.86
46 4.97
5. 19/20
42 5.08 243/256 limma
48 5.18 125/135
5.26 18/19 F
49 5.29
50 5.39
∺ complément de la dif. de 8/9
144/149 × 149/162 = 8/9


101 10.70 25/28
102 10.80
103 10.91 49/55
104 11.01 3645/4096
105 11.11 8/9 ton majeur
106 11.21
107 11.31
108 11.41
11.42 256/289
109 11.51 100/113
110 11.61 100/
111 11.71
112 11.81
113 11.91
114 12.
115 12.10 225/256
116 12.20
117 12.30 4194304/4782969 t. Maxime صبا . صفي الدين
118 12.40
119 12.50 7/8 ton maxime
120 12.60
121 12.70
122 12.80
123 12.89
1294 195/224
124 12.99
125 13.09
126 13.19 125/144 tierce min. Lyon
127 13.29
13.33 13/15
128 13.39
13.46 45/52
129 13.50
130 13.60
131 13.69
132 13.79
13.85 56/65
133 13.89
134 13.98
135 14.08
14.16 5632/6561
136 14.18
137 14.28 6/7 tierce minime Emm.
138 14.37 65536/76545
139 14.47 14348907/16777216
140 14.57 2187/2560
141 14.66 64/75 ton maj. augm. Lyon Helmholtz
142 14.76
143 14.85
144 14.95
15. 17/20
145 15.05
146 15.14
147 15.24
148 15.33
149 15.43
150 15.52


151 15.62 27/32 trihemiton comm.
152 15.72 1024/1215
153 15.81
154 15.91 37/44
155 16.
16.05 Collaugetta
156 16.10
157 16.19
16.28 36/43
158 16.28
159 16.38
160 16.47
161 16.57
162 16.66 5/6 tierce min.
163 16.76 16384/19683
164 16.85
165 16.94
166 17.04 112/135
17.11 63/76
167 17.13
168 17.22
169 17.32
170 17.41
171 17.50
172 17.60
173 17.69
174 17.78
175 17.88 8589934592/10460353203
17.95 32/39
176 17.97
177 18.06 59/72
178 18.15
18.18 9/11
179 18.25
180 18.35
181 18.44 13365/16384
182 18.52 22/27
18.54
183 18.63 729/896 ou 48/59
184 18.72 256/315
18.75
185 18.81 3486784401/4294967296 13/16
186 18.90
187 19. 81/100
188 19.09
189 19.18 80/99
19.23 21/26
190 19.27 524288/649539
191 19.36
192 19.45
193 19.54
194 19.63
19.64 45/56
195 19.72
196 19.81
19.83 19/24
197 19.90 6561/8192
198 20. 4/5 tierce majeure
199 20.09
200 20.18
201
202
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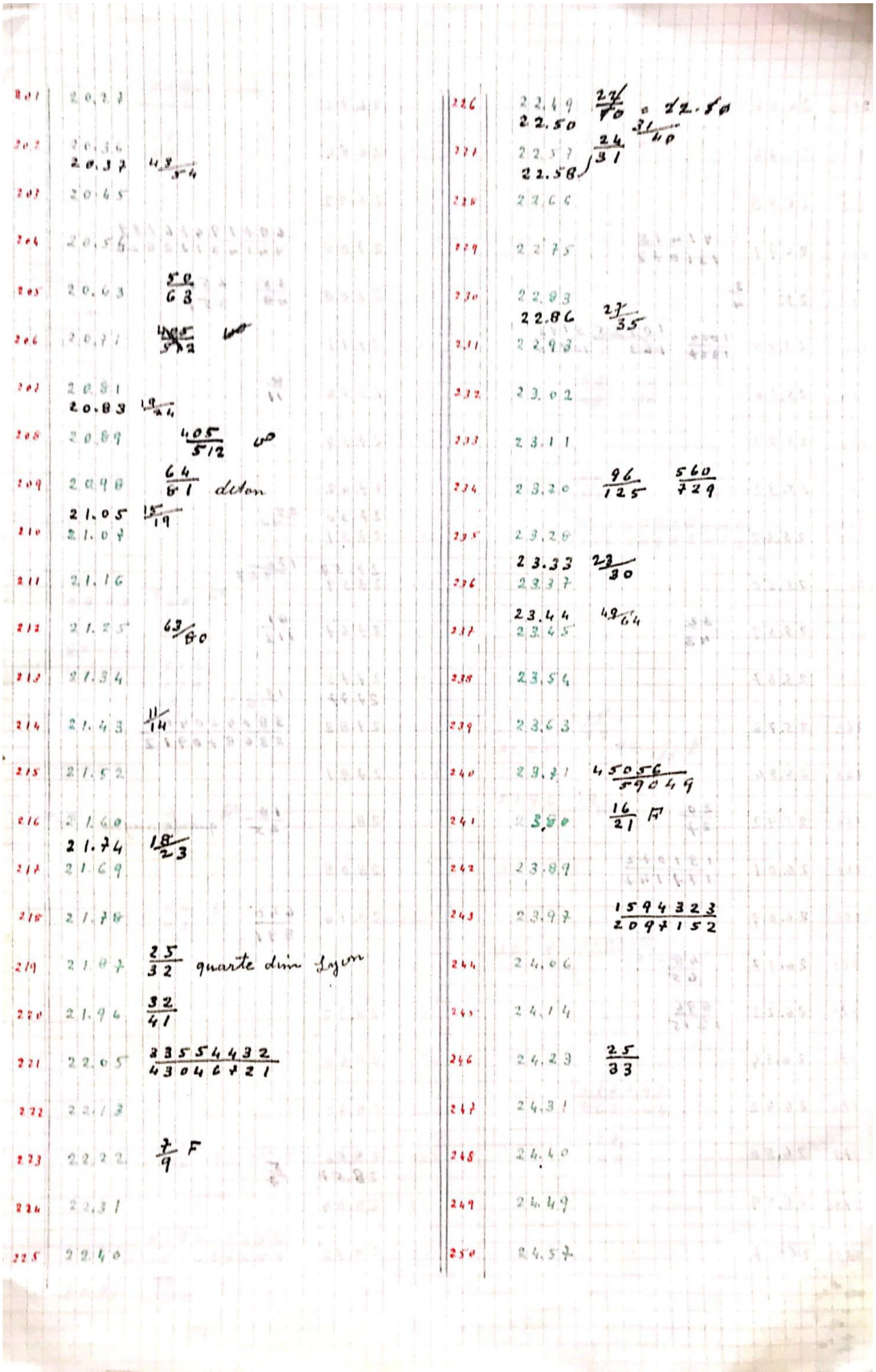
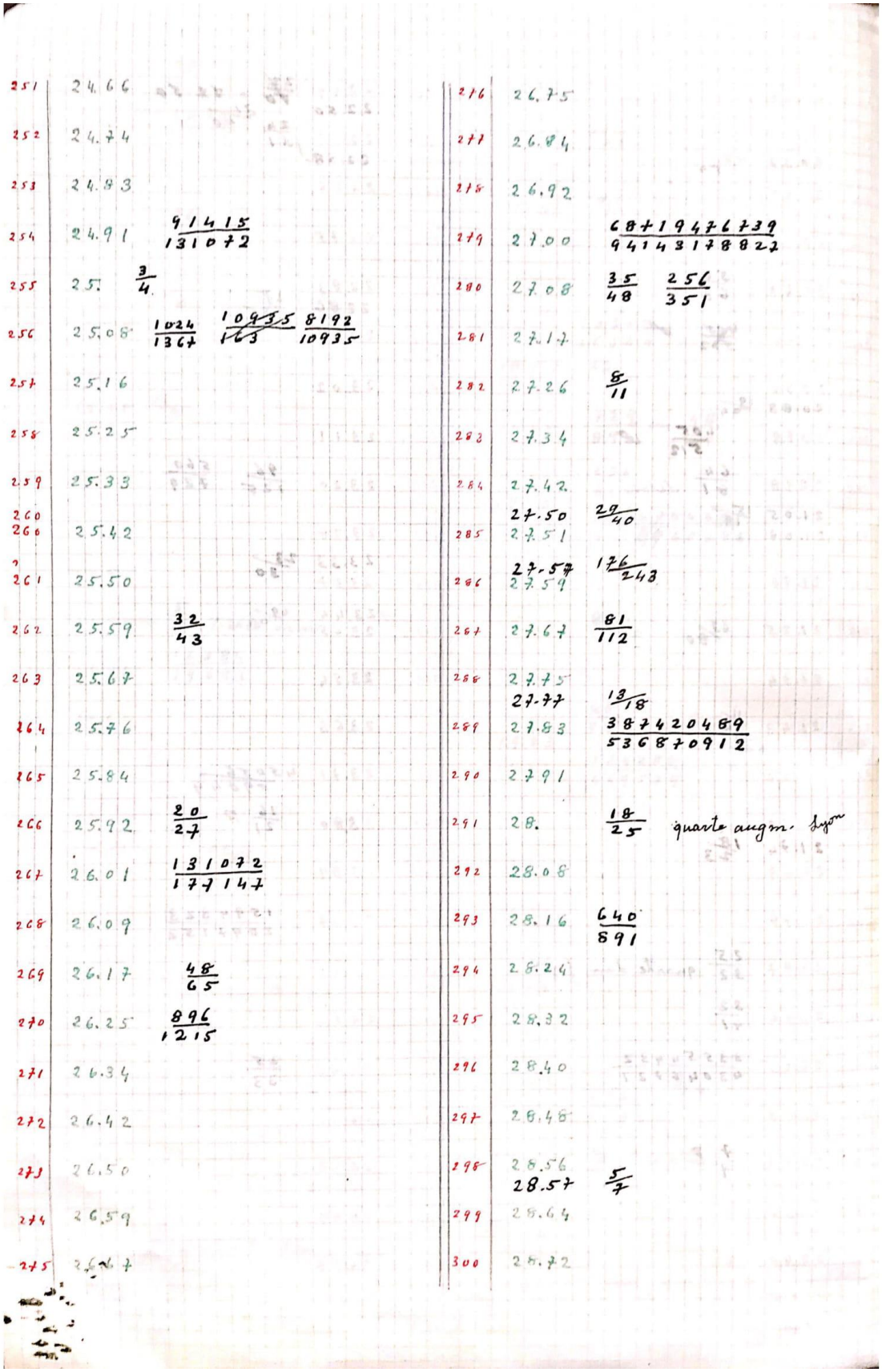

201 20.22
202 20.36
20.37 43/54
203 20.45
204 20.54
205 20.63 50/63
206 20.71 405/512 ∞
207 20.81
20.83 18/24
208 20.89 405/512 ∞
209 20.98 64/81 dodon
21.05 15/19
210 21.07
211 21.16
212 21.25 63/80
213 21.34
214 21.43 11/14
215 21.52
216 21.60
21.74 18/23
217 21.79
218 21.88
219 21.97 25/32 quarte dim Lyon
220 22.04 32/41
221 22.05 23554432/43046721
222 22.13
223 22.22 7/9 F
224 22.31
225 22.40
226 22.49 27/40 = 22.50
22.50
227 22.57 24/31
22.58 31/40
228 22.66
229 22.75
230 22.83
22.86 27/35
231 22.93
232 23.02
233 23.11
234 23.20 96/125 560/729
235 23.28
23.33 23/30
236 23.37
23.44 49/64
237 23.45
238 23.54
239 23.63
240 23.71 45056/59049
241 23.80 16/21 F
242 23.89
243 23.97 15943232/20971521
244 24.06
245 24.14
246 24.23 25/33
247 24.31
248 24.40
249 24.49
250 24.57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257 25.16
258 25.25
259 25.33
260
266 25.42
261 25.50
262 25.59 32/43
263 25.67
264 25.76
265 25.84
266 25.92 20/27
267 26.01 131072/177147
268 26.09
269 26.17 48/65
270 26.25 896/1215
271 26.34
272 26.42
273 26.50
274 26.59
275 26.67
276 26.75
277 26.84
278 26.92
279 27.00 6871947673 9414317882 2
280 27.08 35/48 256/351
281 27.17
282 27.26 5/11
283 27.34
284 27.42
27.50 29/40
285 27.51
286 27.57 176/243
27.59
287 27.67 81/112
288 27.75
27.77 13/18
289 27.83 3874204889/5368709122
290 27.91
291 28. 18/25 quarta augm. Lyon
292 28.08
293 28.16 640/891
294 28.24
295 28.32
296 28.40
297 28.48
298 28.56
28.57 5/7
299 28.64
300 28.72

301	28.81	729/1024
302	28.88	32/45
303	28.96	
304	29.04	
305	29.13	
	29.16	17/18
306	29.20	
307	29.29	
308	29.36	
309	29.44	
310	29.52	
311	29.60	
	29.63	19/27 F
312	29.68	45/64
	29.69	
313	29.76	512/729
	29.83	40/57 F
314	29.84	
315	29.92	
316	30.	7/10 F
317	30.08	
318	30.16	44/63 F
319	30.24	
320	30.31	
321	30.39	
322	30.47	
323	30.55	25/36 quinte dim Lyon
324	30.63	
325	30.71	2684354561/3874204889
	30.77	~~36/52~~ Av. 9/13
326	30.79	9/13
327	30.86	56/81 F
328	30.94	
329	31.02	
330	31.10	
331	31.17	64/93
332	31.25	33/48 11/16
333	31.34	
334	31.42	24/35
335	31.50	941431788227/1374389534472
336	31.57	
337	31.65	
338	31.73	
339	31.81	15/22
	31.85	92/135 F
340	31.85	
	31.94	49/72 F.
341	31.96	
342	32.04	
343	32.11	
	32.18	340448/581441
344	32.19	
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346	32.34	
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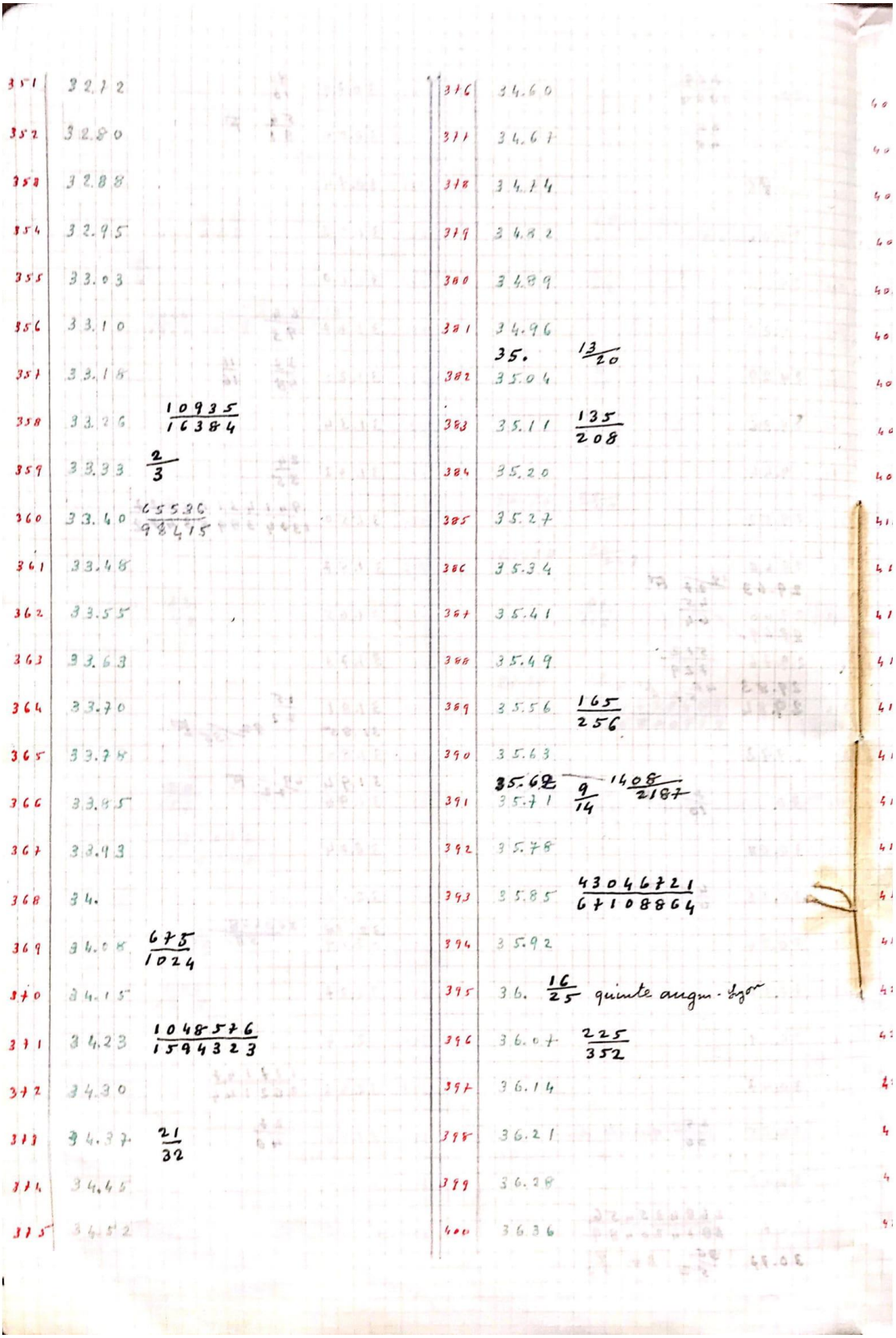
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357 33.18
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359 33.33 2/3
360 33.40 65536/98415
361 33.48
362 33.55
363 33.63
364 33.70
365 33.78
366 33.85
367 33.93
368 34.
369 34.08 675/1024
370 34.15
371 34.23 1048576/1594323
372 34.30
373 34.37 21/32
374 34.45
375 34.52
376 34.60
377 34.67
378 34.74
379 34.82
380 34.89
381 34.96
35. 13/20
382 35.04
383 35.11 135/208
384 35.20
385 35.27
386 35.34
387 35.41
388 35.49
389 35.56 165/256
390 35.63
35.62 1408/2187
391 35.71 9/14
392 35.78
393 35.85 43046721/67108864
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395 36. 16/25 quinte augm. byzar
396 36.04 225/352
397 36.14
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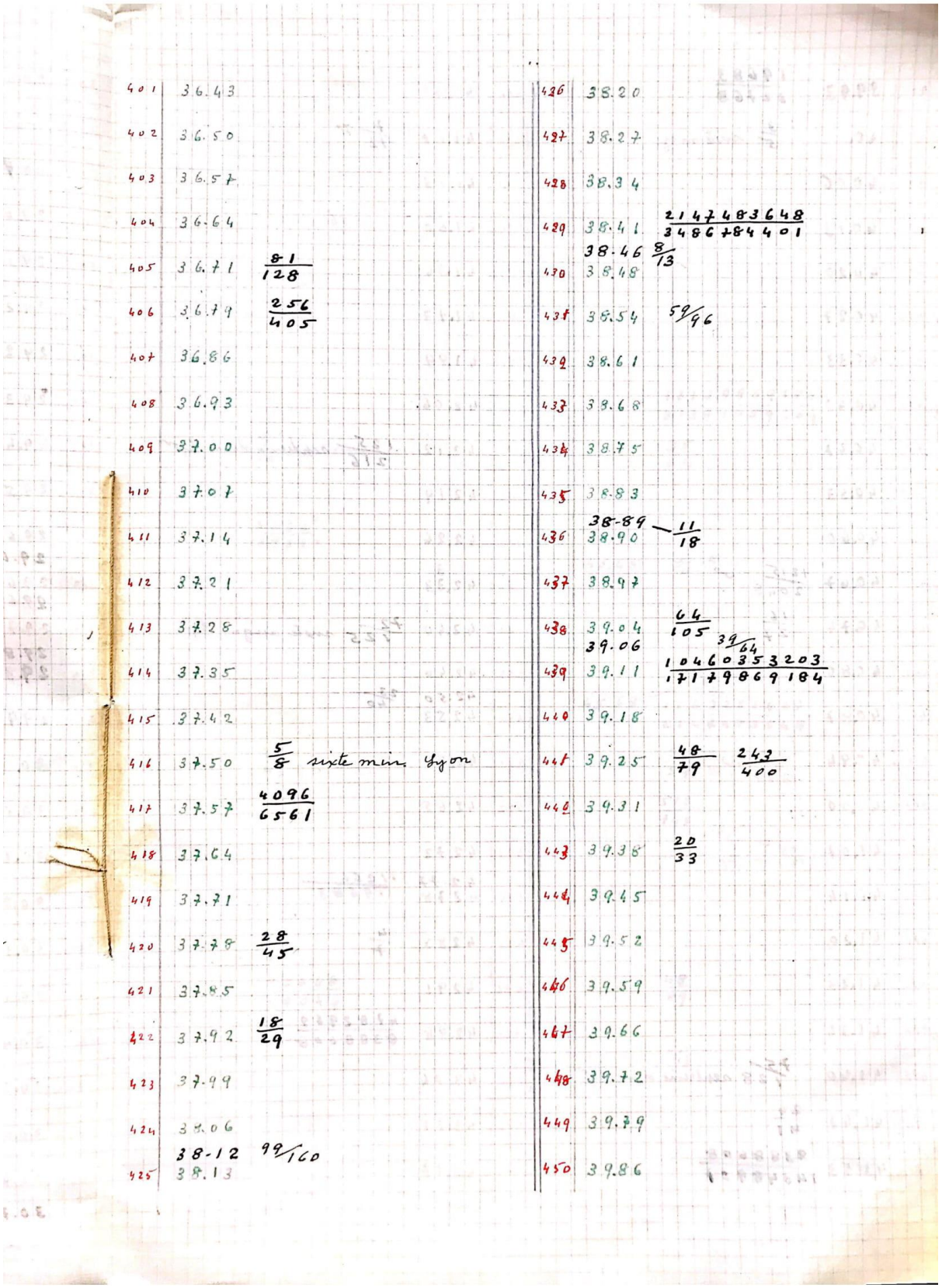
399 36.28
400 36.36



401 36.43
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404 36.64
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406 36.79 256/405
407 36.86
408 36.93
409 37.00
410 37.07
411 37.14
412 37.21
413 37.28
414 37.35
415 37.42
416 37.50 5/8 sixte mineur hypon
417 37.57 4096/6561
418 37.64
419 37.71
420 37.78 28/45
421 37.85
422 37.92 18/29
423 37.99
424 38.06
38-12 99/160
425 38.13

426 38.20
427 38.27
428 38.34
429 38.41 2147483648/3486784401
38.46 8/13
430 38.48
431 38.54 59/96
432 38.61
433 38.68
434 38.75
435 38.83
38-89 — 11/18
436 38.90
437 38.97
438 39.04 64/105
39.06 39/64
439 39.11 1046035203/1717986918 4
440 39.18
441 39.25 48/79 243/400
442 39.31
443 39.38 20/33
444 39.45
445 39.52
446 39.59
447 39.66
448 39.72
449 39.79
450 39.86

451	39.93	$\frac{19683}{32768}$		476	41.40	
452	40.	$\frac{3}{5}$ sixte maj.		477	41.67	$\frac{7}{12}$ Fa
453	40.06			478	41.73	
454	40.13			479	41.78 / 41.80	$\frac{33}{56}$
455	40.20			480	41.86	
456	40.27			481	41.93	
457	40.33			482	41.99	
458	40.40			483	42.06.	
459	40.47			484	42.12	$\frac{125}{216}$ septième dim. Lyon
460	40.53			485	42.19	
461	40.60			486	42.26	
462	40.67	$\frac{1215}{2048}$ Do		487	42.33	
463	40.74	$\frac{16}{27}$		488	42.40	$\frac{72}{125}$ sixte augm. Lyon
464	40.80			489	42.46	
465	40.87				42.50	$\frac{23}{40}$
466	40.94	$\frac{512}{867}$ Do		490	42.53	
467	41.00			491	42.59	
468	41.07	$\left(\frac{11}{16} \times \frac{6}{7}\right) - \frac{33}{56}$ Yekta		492	42.65	
469	41.14			493	42.72	
470	41.20				42.77	$\frac{11264}{19683}$
471	41.27			494	42.78	
472	41.34			495	42.85	$\frac{4}{7}$
473	41.40	$\frac{75}{128}$ septième dim. Lyon		496	42.91	
474	41.47	$\frac{24}{41}$		497	42.98	$\frac{47829 69}{83886 08}$
475	41.53	$\frac{8388608}{14348907}$		498	43.04	
				499	43.11	
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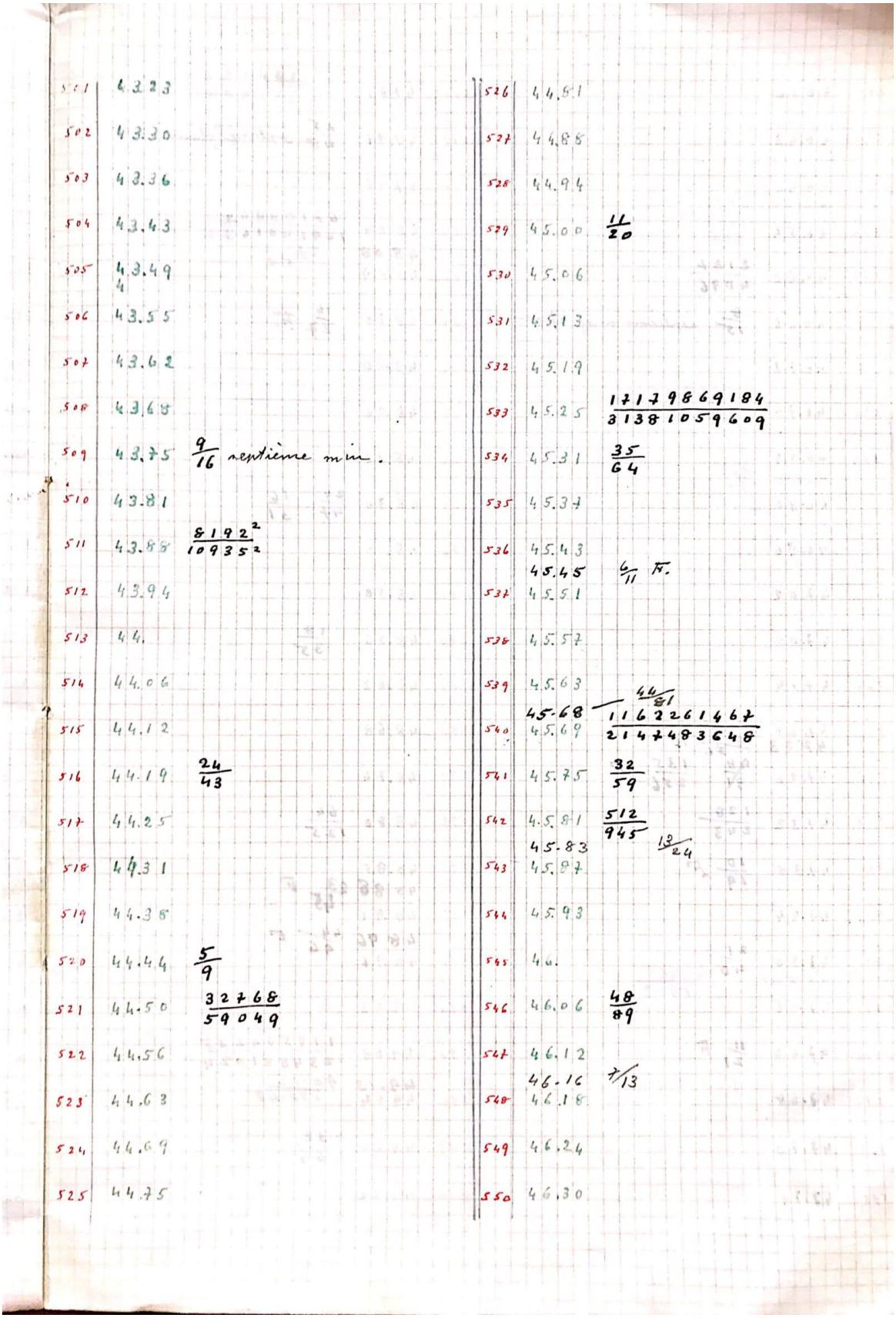
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502 43.30
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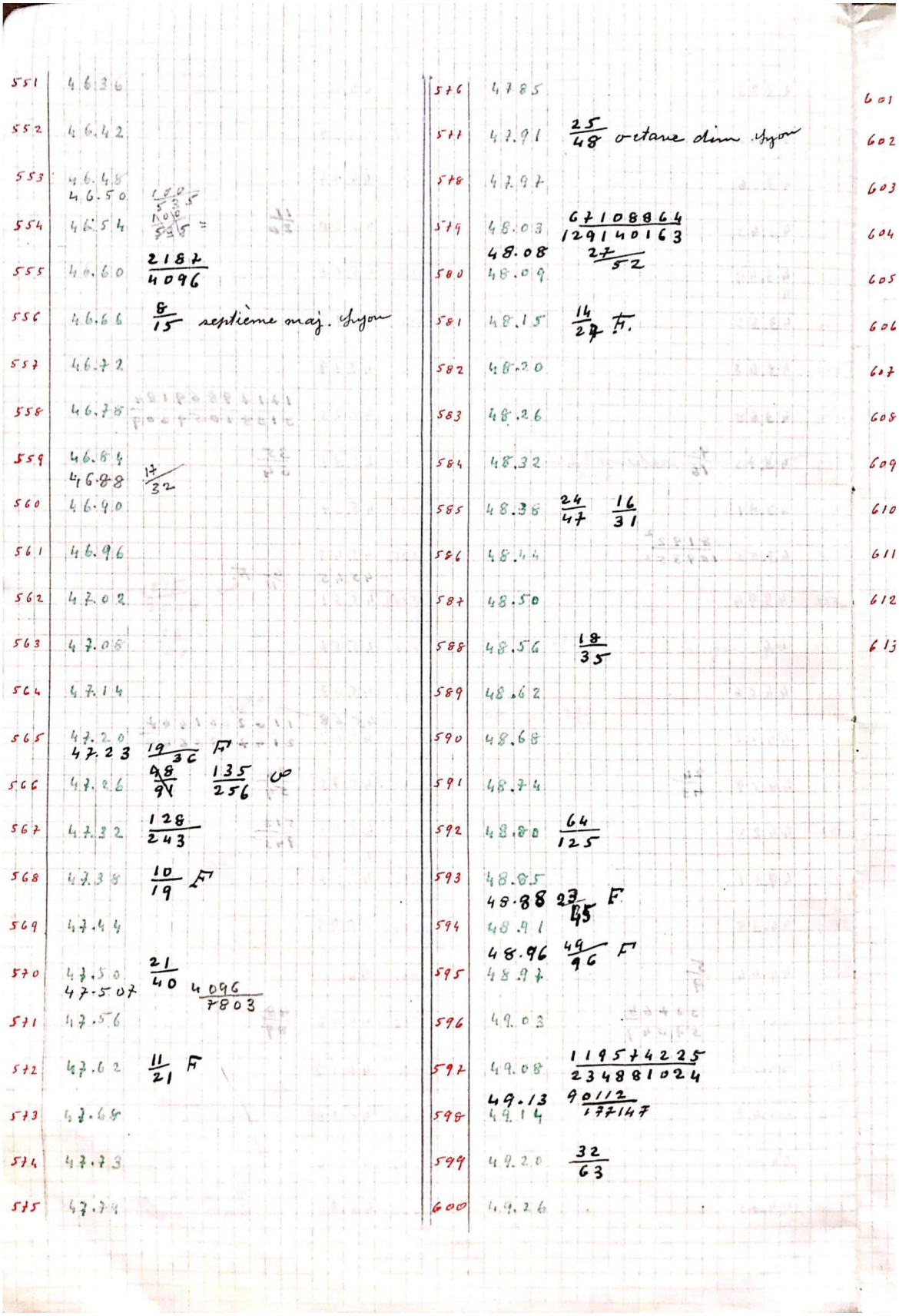
504 43.43
505 43.49 4
506 43.55
507 43.62
508 43.68
509 43.75 9/16 neuvième min.
510 43.81
511 43.88 8192²/10935²
512 43.94
513 44.
514 44.06
515 44.12
516 44.19 24/43
517 44.25
518 44.31
519 44.38
520 44.44 5/9
521 44.50 32768/59049
522 44.56
523 44.63
524 44.69
525 44.75
526 44.81
527 44.88
528 44.94
529 45.00 11/20
530 45.06
531 45.13
532 45.19
533 45.25 1717986918 4/3138105960 9
534 45.31 35/64
535 45.37
536 45.43
45.45 6/11 M.
537 45.51
538 45.57
539 45.63
45.68 44/81
540 45.69 1162261467/2147483648
541 45.75 32/59
542 45.81 512/945
45.83 13/24
543 45.87
544 45.93
545 46.
546 46.06 48/89
547 46.12
46.16 7/13
548 46.18
549 46.24
550 46.30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46.50
554 46.54 100/535 =
555 46.60 2182/4096
556 46.66 8/15 septième maj. Lyon
557 46.72
558 46.78
559 46.84
46.88 17/32
560 46.90
561 46.96
562 47.02
563 47.08
564 47.14
565 47.20
47.23 19/36 F
566 47.26 48/91 135/256 U
567 47.32 128/243
568 47.38 10/19 F
569 47.44
570 47.50 21/40
47.507 4096/7803
571 47.56
572 47.62 11/21 F
573 47.68
574 47.73
575 47.74
576 47.85
577 47.91 25/48 octave dim. Lyon
578 47.97
579 48.03 67108864/129140163
48.08 27/52
580 48.09
581 48.15 14/27 H.
582 48.20
583 48.26
584 48.32
585 48.38 24/47 16/31
586 48.44
587 48.50
588 48.56 18/35
589 48.62
590 48.68
591 48.74
592 48.80 64/125
593 48.85
48.88 23/45 F
594 48.91
48.96 49/96 F
595 48.97
596 49.03
597 49.08 1195744225/2348810241
49.13 90112/177147
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600 49.26
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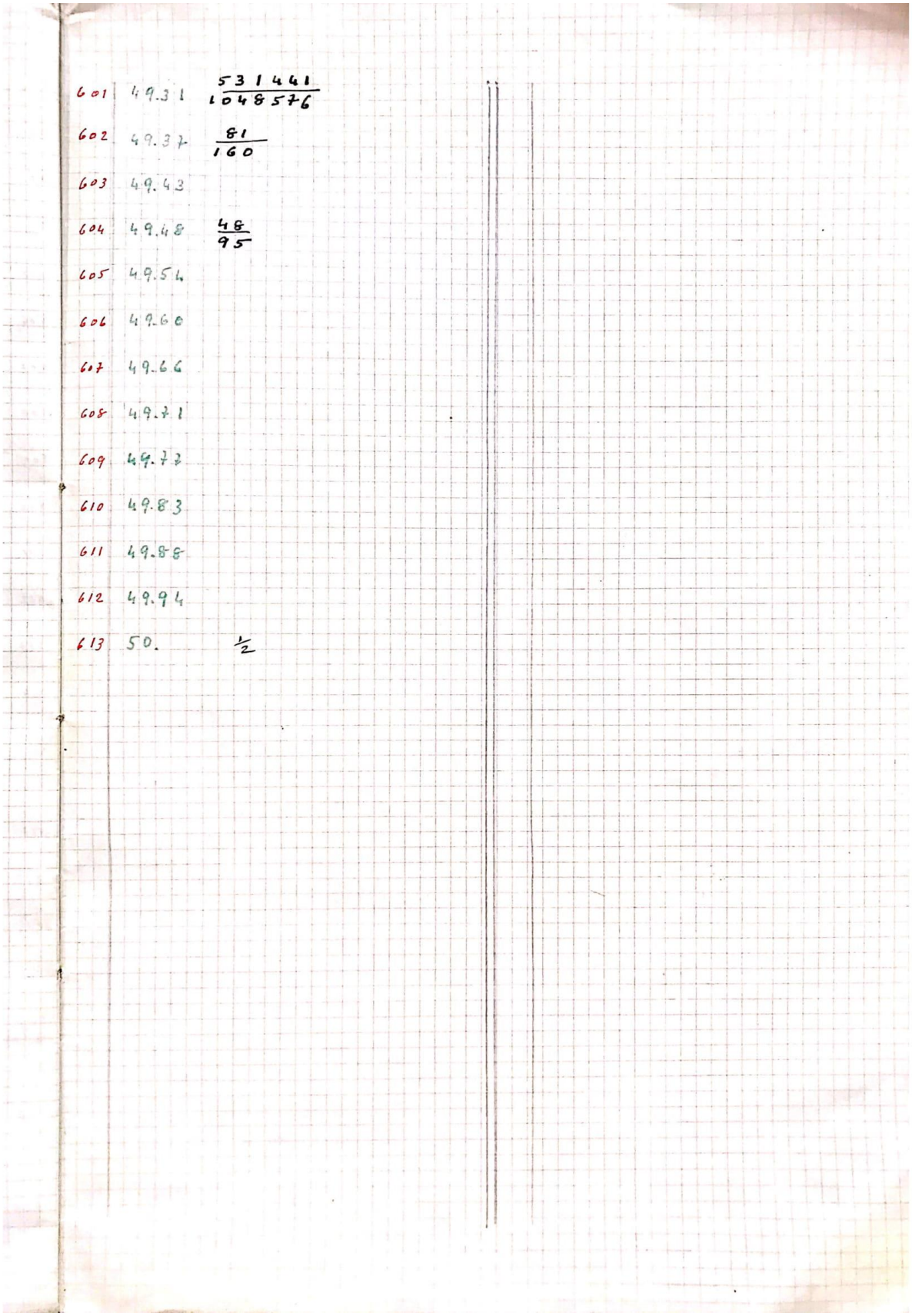
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602 49.37 81/160
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610 49.83
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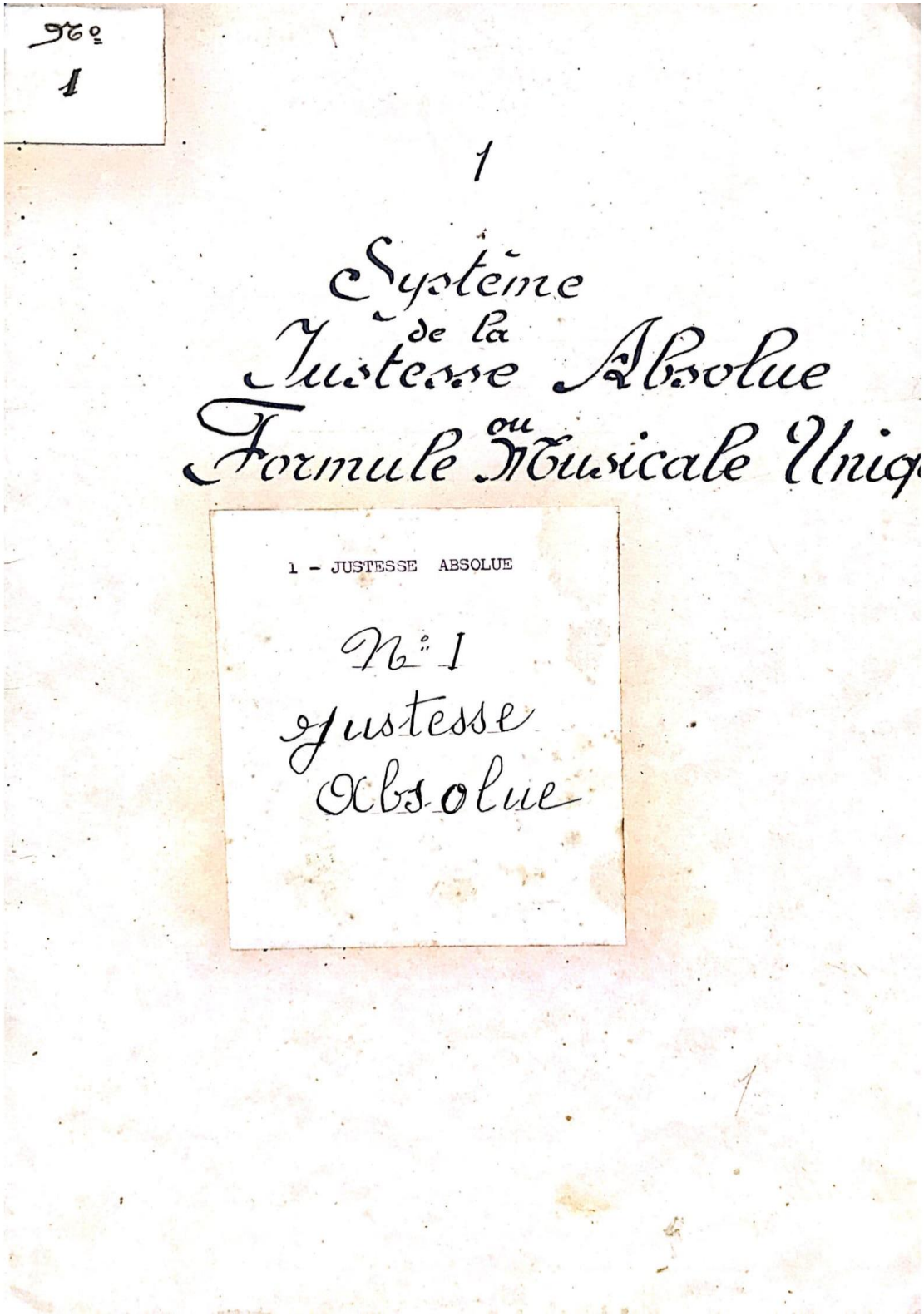
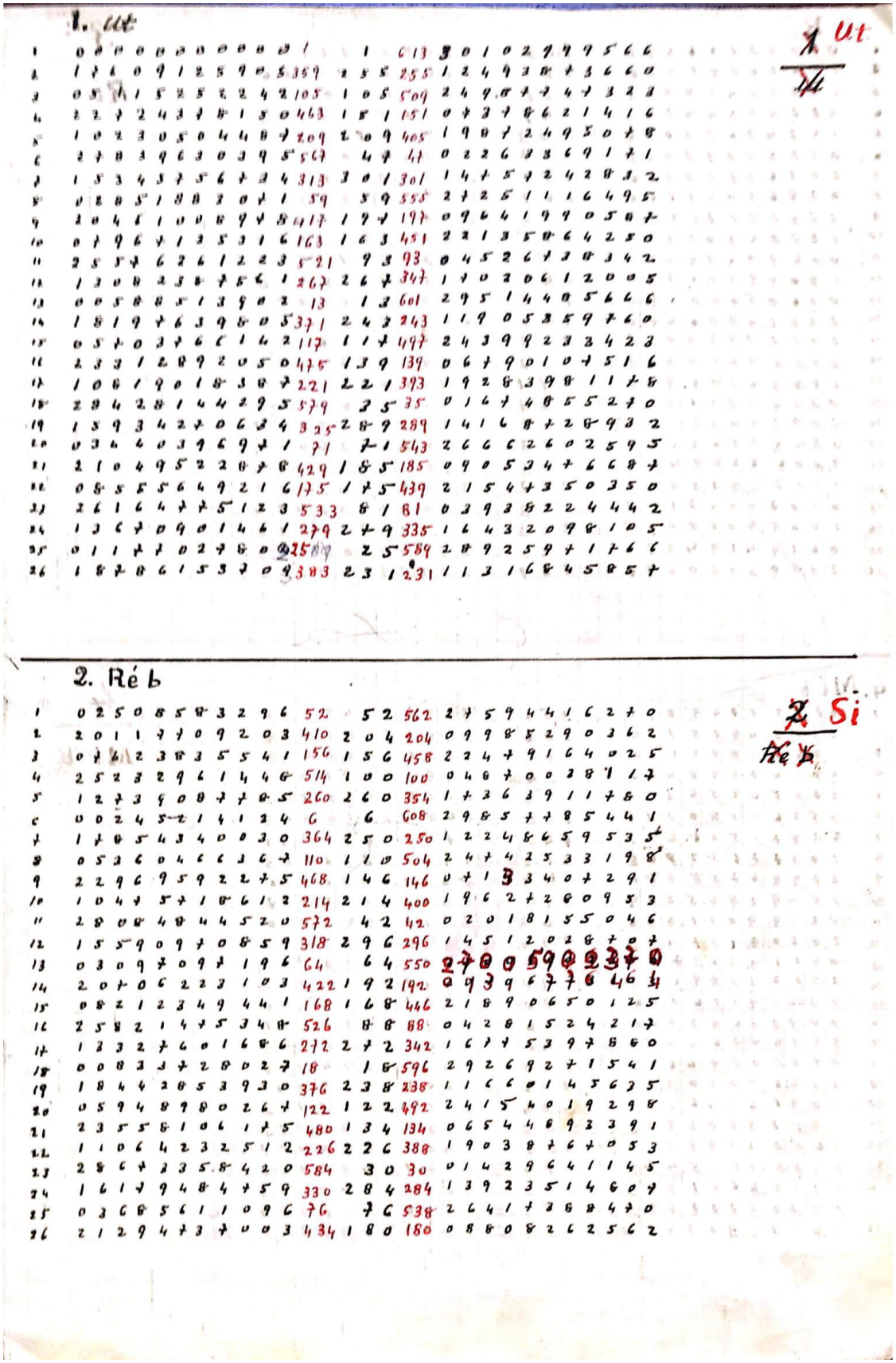
612 49.94
613 50. 1/2

Appendix C


1
Système
de la
Justesse Absolue
ou
Formule Musicale Uniq
1 - JUSTESSE ABSOLUE
N° 1
Justesse
Absolue


1. ut
1 ut
2. Ré b
2 Si
Ré b

3. Ré

~~3~~ Si b

~~Ré~~

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3	1 0 1 3 2 4 1 8 8 3 7	207	2 0 7	407	1 9 9 7 0 5 8 0 7 2 8
4	2 7 7 4 1 5 4 4 7 4 5	565	4 9	49	0 2 3 6 1 4 5 4 8 2 1
5	1 5 2 4 7 6 7 1 0 8 4	311	3 0 3	303	1 4 8 5 5 7 2 8 4 8 2
6	0 2 7 5 3 7 9 7 4 2 1	57	5 7	557	2 7 3 4 9 2 0 2 1 4 5
7	2 0 3 6 2 9 2 3 3 2 6	415	1 9 9	199	0 9 7 4 0 0 7 6 2 3 7
8	0 7 8 6 9 0 4 9 6 6 6	161	1 6 1	453	2 2 2 3 3 9 4 9 9 0 0
9	2 5 4 7 8 1 7 5 5 7 3	519	9 5	95	0 4 6 2 4 8 2 3 9 9 2
10	1 2 9 8 4 3 0 1 9 1 1	265	2 6 5	349	1 7 1 1 8 6 9 7 6 3 5
11	0 0 4 9 0 4 2 8 2 5 0	11	1 1	603	2 9 6 1 2 5 7 1 3 1 6
12	1 8 0 9 9 5 5 4 1 5 5	369	2 4 5	245	1 2 0 0 3 4 4 5 4 1 0
13	0 5 6 0 5 6 8 0 4 9 2	115	1 1 5	499	2 4 4 9 7 3 1 9 0 7 3
14	2 3 2 1 4 8 0 6 4 0 0	473	1 4 1	141	0 6 8 8 8 1 9 3 1 6 6
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16	2 8 3 3 0 0 5 8 6 4 5	577	3 7	37	0 1 7 7 2 9 4 0 9 2 0
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18	0 3 3 4 2 3 1 1 3 2 1	69	6 9	545	2 6 7 6 0 6 8 8 2 4 5
19	2 0 9 5 1 4 3 7 2 2 8	427	1 8 7	187	0 9 1 5 1 5 6 2 3 3 7
20	0 8 4 5 7 5 6 3 5 6 6	173	1 7 3	441	2 1 6 4 5 4 3 6 0 0 0
21	2 6 0 6 6 6 8 9 4 7 3	531	8 3	83	0 4 0 3 6 3 1 0 0 9 2
22	1 3 5 7 2 8 1 5 8 1 1	277	2 7 7	337	1 6 5 3 0 1 8 3 7 5 5
23	0 1 0 7 8 9 4 2 1 5 2	23	2 3	591	2 9 0 2 4 0 5 7 4 1 6
24	1 8 6 8 8 0 6 8 0 5 5	361	2 3 3	233	1 1 4 1 4 9 3 1 5 1 0
25	0 6 1 9 4 1 9 4 3 9 2	127	1 2 7	487	2 3 9 0 8 8 0 5 1 7 3
26	2 3 8 0 3 3 2 0 2 9 7	485	1 2 9	129	0 6 2 9 9 6 7 9 2 6 6

4. Mi b

~~4~~ La

~~Mi b~~

1	0 7 5 2 5 7 4 9 8 9 1	154	1 5 4	460	2 2 5 7 7 2 4 9 6 7 5	
2	2 5 1 3 4 8 7 5 7 9 8	512	1 0 2	102	0 4 9 6 8 1 2 3 7 6 7	
3	1 2 6 4 1 0 0 2 1 3 5	258	2 5 8	356	1 7 4 6 1 9 9 7 4 3 0	
4	0 0 1 4 7 1 2 8 4 7 5	4	4	610	2 9 9 5 5 8 7 1 0 9 1	
5	1 7 7 5 6 2 5 4 3 8 0	362	2 5 2	252	1 2 3 4 6 7 4 5 1 8 5	
6	0 5 2 6 2 3 8 0 7 1 7	108	1 0 8	506	2 4 8 4 0 6 1 8 8 4 8	
7	2 2 8 7 1 5 0 6 6 2 5	466	1 4 8	148	0 7 2 3 1 4 9 2 9 4 1	
8	1 0 3 7 7 6 3 7 9 6 2	212	2 1 2	402	1 9 7 2 5 3 6 6 6 0 3	
9	2 7 9 8 6 7 5 8 8 7 0	570	4 4	44	0 2 1 1 6 2 4 0 6 9 6	
10	1 5 4 9 2 8 8 5 2 0 9	316	2 9 8	298	1 4 6 1 0 1 1 4 3 5 7	
11	0 2 9 9 9 0 1 1 5 4 6	62	6 2	552	2 7 1 0 3 9 8 8 0 2 0	
12	2 0 6 0 8 1 3 7 4 5 3	420	1 9 4	194	0 9 4 9 4 8 6 2 1 1 2	
13	0 8 1 1 4 2 6 3 7 9 1	166	1 6 6	448	2 1 9 8 8 7 3 5 7 7 5	
14	2 5 7 2 3 3 8 9 6 9 8	524	9 0	90	0 4 3 7 9 6 0 9 8 6 7	
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17	1 8 3 4 4 7 6 8 2 8 0	374	2 4 0	240	1 1 7 5 8 2 3 1 2 8 5	
18	0 5 8 5 0 8 9 4 6 1 7	120	1 2 0	494	2 4 2 5 2 1 0 4 9 4 8	
19	2 3 4 6 0 0 2 0 5 2 5	478	1 3 6	136	0 6 6 4 2 9 7 9 0 4 1	
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22	1 6 0 8 1 3 9 9 1 0 9	328	2 8 6	~~328~~	1 4 0 2 1 6 0 0 4 5 7	286
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25	0 8 7 0 2 7 7 7 6 9 2	178	1 7 8	~~178~~	2 1 4 0 0 2 2 1 8 7 4	436
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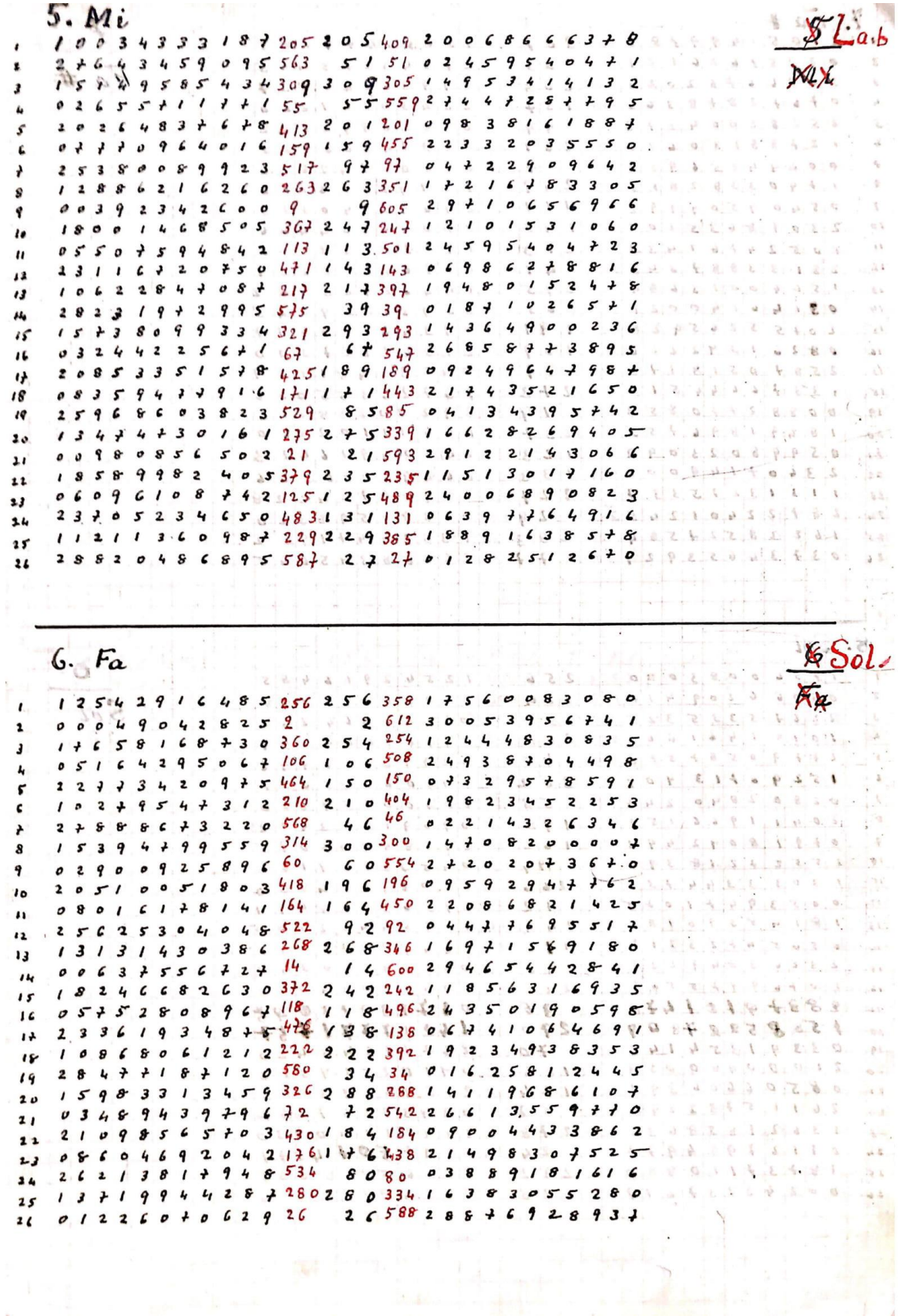
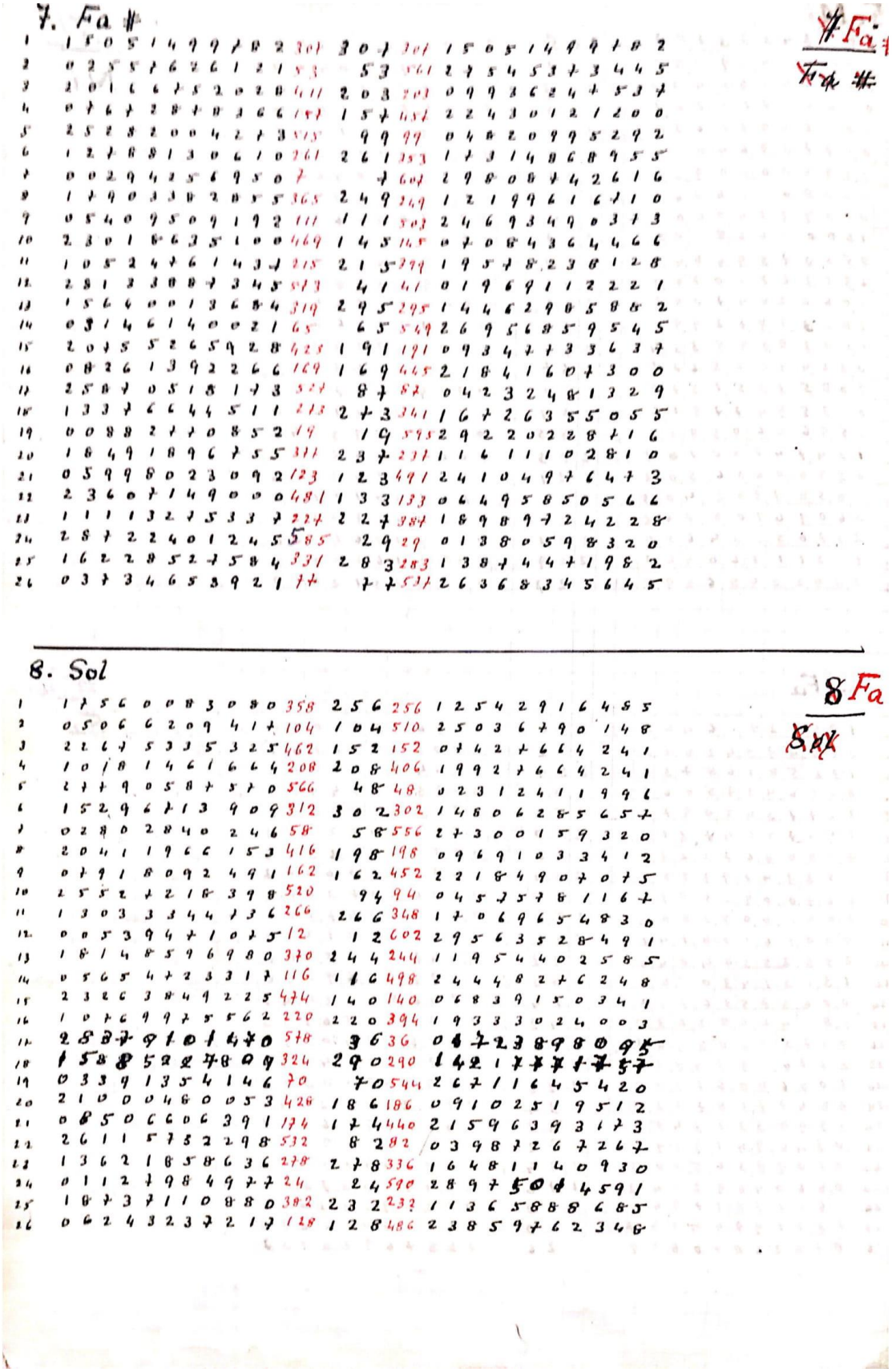

5. Mi
5 Lab
Mix
6. Fa
6 Sol
Fa


7. Fa #
8. Sol
8 Fa

9. Lab

~~9~~ Mi

~~La b~~

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3	2 5 1 8 3 9 1 8 6 2 3 513 1 0 1 101 0 4 9 1 9 0 8 0 9 4 2
4	1 2 6 9 0 0 4 4 9 6 0 259 2 5 9 355 1 7 4 1 2 9 5 4 6 0 5
5	0 0 1 9 6 1 7 1 3 0 0 5 5 609 2 9 9 0 6 8 2 8 2 6 6
6	1 7 8 0 5 2 9 7 2 0 5 363 2 5 1 251 1 2 2 9 7 7 0 2 3 6 0
7	0 5 3 1 1 4 2 3 5 4 2 109 1 0 9 505 2 4 7 9 1 5 7 6 0 2 3
8	2 2 9 2 0 5 4 9 4 5 0 467 1 4 7 147 0 7 1 8 2 4 5 0 1 1 6
9	1 0 4 2 6 6 7 5 7 8 7 213 2 1 3 401 1 9 6 7 6 3 2 3 7 7 8
10	2 8 0 3 5 8 0 1 6 9 5 571 4 3 43 0 2 0 6 7 1 9 7 8 7 1
11	1 5 5 4 1 9 2 8 0 3 4 317 2 9 7 297 1 4 5 6 1 0 7 1 5 3 2
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13	2 0 6 5 7 1 8 0 2 7 8 421 1 9 3 193 0 9 4 4 5 8 1 9 2 8 7
14	0 8 1 6 3 3 0 6 6 1 6 167 1 6 7 447 2 1 9 3 9 6 9 2 9 5 0
15	2 5 7 7 2 4 3 2 5 2 3 525 8 9 89 0 4 3 3 0 5 6 7 0 4 2
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24	0 3 6 3 6 5 6 8 2 7 175 7 5 539 2 6 4 6 6 4 3 1 2 9 5
25	2 1 2 4 5 6 9 4 1 7 8 433 1 8 1 181 0 8 8 5 7 3 0 5 3 8 7
26	0 8 7 5 1 8 2 0 5 1 7 179 1 7 9 435 2 1 3 5 1 1 7 9 0 4 9

10. La

~~10~~ Mib

~~La~~

1	2 2 5 7 7 2 4 9 6 7 5 460 1 5 4 154 0 7 5 2 5 7 4 9 8 9 1
2	1 0 0 8 3 3 7 6 0 1 2 206 2 0 6 408 2 0 0 1 9 6 2 3 5 5 3
3	2 7 6 9 2 5 0 1 9 2 0 564 5 0 50 0 2 4 1 0 4 9 7 6 4 6
4	1 5 1 9 8 6 2 8 2 5 9 310 3 0 4 304 1 4 9 0 4 3 7 1 3 0 7
5	0 2 7 0 4 7 5 4 5 9 6 56 5 6 558 2 7 3 9 8 2 4 4 9 7 0
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7	0 7 8 2 0 0 0 6 8 4 1 160 1 6 0 454 2 2 2 8 2 9 9 2 7 2 5
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13	2 3 1 6 5 7 6 3 5 7 5 472 1 4 2 142 0 6 9 3 7 2 3 5 9 9 1
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15	2 8 2 8 1 0 1 5 8 2 0 576 3 8 38 0 1 8 2 1 9 8 3 7 4 6
16	1 5 7 8 7 1 4 2 1 5 9 322 2 9 2 292 1 4 3 1 5 8 5 7 4 0 7
17	0 3 2 9 3 2 6 8 4 9 6 68 6 8 546 2 6 8 0 9 7 3 1 0 7 0
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19	0 8 4 0 8 5 2 0 7 4 1 172 1 7 2 442 2 1 6 9 4 4 7 8 8 2 5
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23	1 8 6 3 9 0 2 5 2 3 0 380 2 3 4 234 1 1 4 6 3 9 7 4 3 3 5
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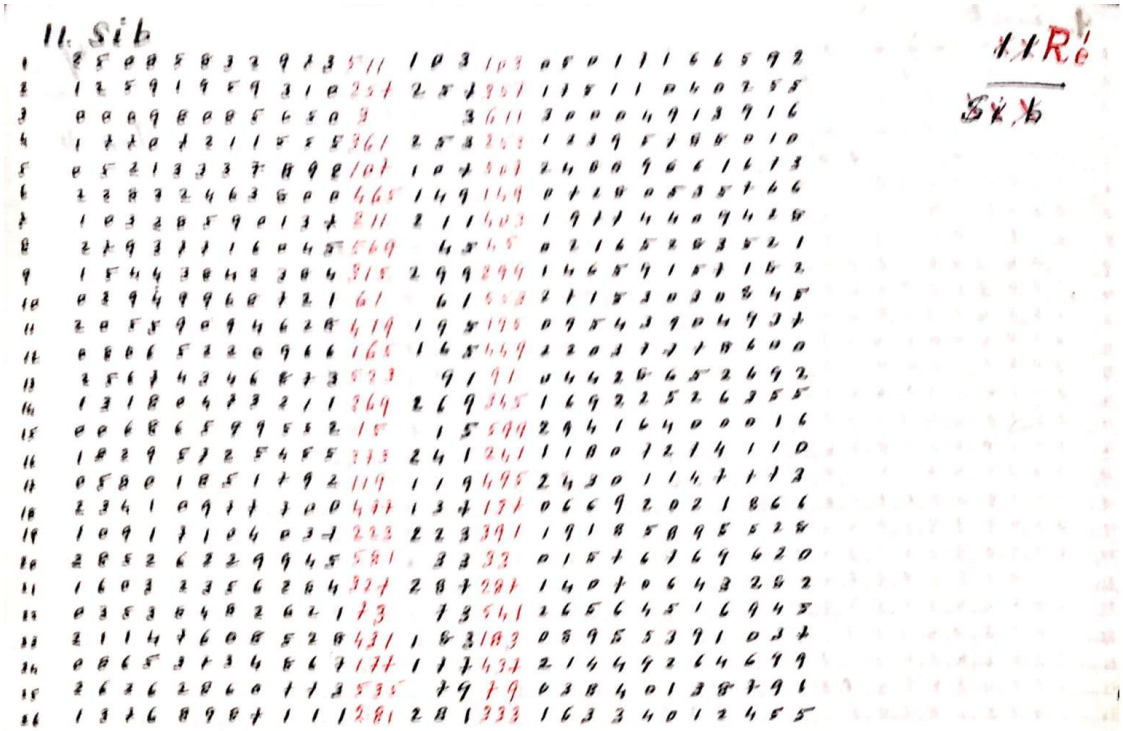
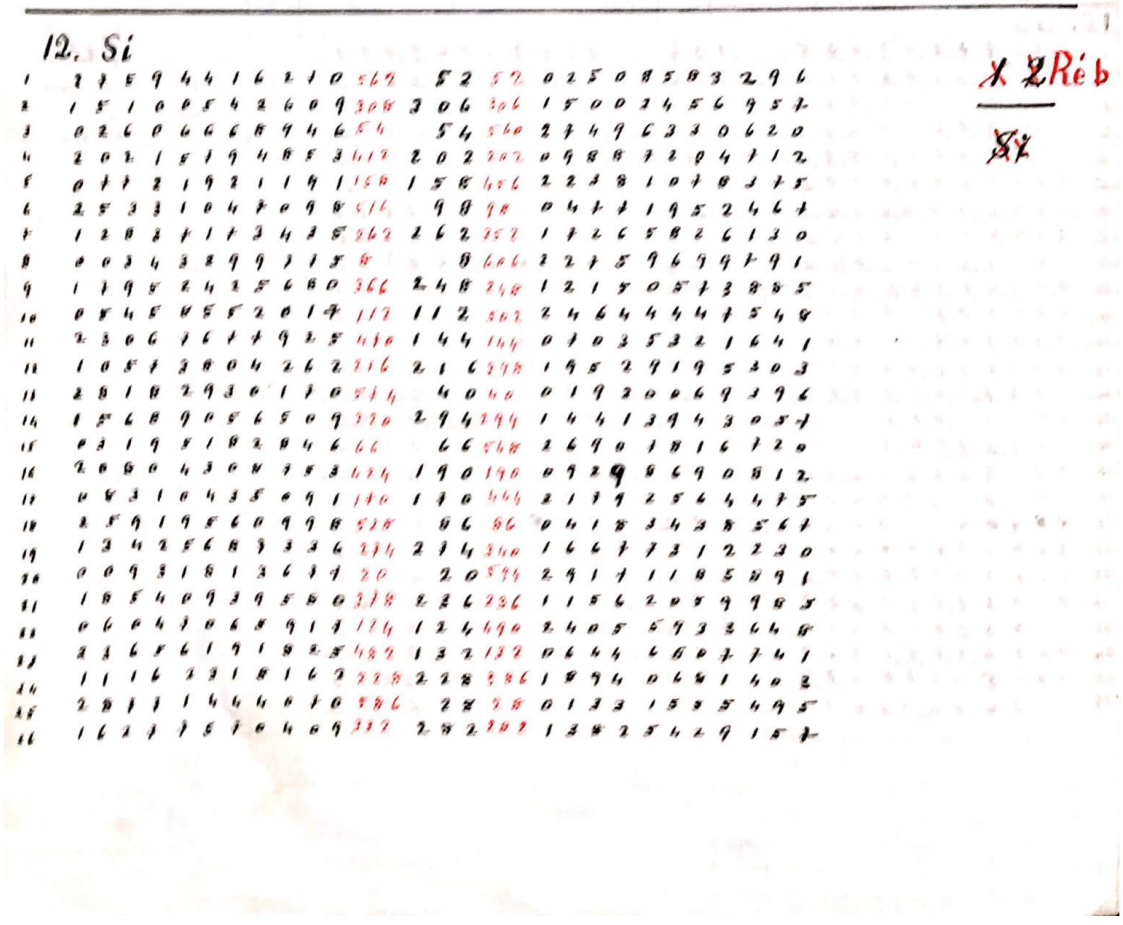

11. Sib
1 1 Ré
Si b


12. Si
1 2 Réb
Si

Composition de la Véritable Gamme Commatique

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3	2498	7747	3216	5999	0626	43.75	11.11	0511	3
4	0737	8621	4160	9186	6419	15.62	40.74	2272	4
5	1981	2495	0769	2186	1731	36.72	20.98	1023	5
6	0226	3369	1713	5373	7523	5.08	47.32	2783	6
7	1475	7242	8321	8373	2836	28.81	29.77	1534	7
8	2725	1116	4930	1372	8149	46.61	6.36	0285	8
9	0964	1990	5874	4560	3941	19.91	37.51	2046	9
10	2213	5864	2482	7559	9254	39.94	16.77	0796	10
11	0452	6738	3427	0747	5044	9.91	44.51	2557	11
12	1702	0612	0035	3747	0359	32.42	26.01	1308	12
13	2951	4485	6643	6746	5672	49.32	1.35	0058	13
14	1190	5359	7587	9934	1464	23.98	34.23	1819	14
15	2439	9233	4196	2933	6747	42.98	12.31	0570	15
16	0679	0107	5140	6121	2569	14.48	41.54	2331	16
17	1928	3981	1748	9120	7882	35.86	22.05	1081	17
18	0163	4855	2693	2308	3674	3.78	48.03	2842	18
19	1416	8728	9301	5307	8987	27.84	30.71	1592	19
20	2666	2602	5909	8307	4300	45.88	7.62	0346	20
21	0905	3476	6954	1495	0092	18.82	38.41	2104	21
22	2154	7350	3462	4494	5404	39.12	17.89	0855	22
23	0393	8224	4406	7682	1193	8.66	45.26	2616	23
24	1643	2098	1015	0681	6510	31.51	27.01	1364	24
25	2892	5971	7623	3661	1823	48.63	2.67	0113	25
26	1131	6845	8567	6868	7615	22.94	35.11	1878	26
27	2381	0719	5175	9866	2928	42.21	13.48	0629	27

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