ALGEBRAS WITH PSEUDO-RIEMANNIAN BILINEAR FORMS

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ABSTRACT

The purpose of this dissertation is to study pseudo-Riemannian algebras, which are algebras with pseudo-Riemannian non-degenerate symmetric bilinear forms. The paper([1]), the authors *Zhiqi Chen, Ke Liang, and Fuhai Zhu* find that pseudo-Riemannian algebras whose left centers are isotropic play a key role and show that the decomposition of pseudo-Riemannian algebras whose left centers are isotropic into indecomposable non-degenerate ideals is unique up to a special automorphism. Furthermore, if the left center equals the center, the orthogonal decomposition of any pseudo-Riemannian algebra into indecomposable non-degenerate ideals is unique up to an isometry.

Chapter 1

Introduction

In this chapter, many fundamental definitions and examples needed throughout the dissertation are sumed up.

1.1 Algebras

Definition 1.1.1. *Let K be a field. An algebra over K (K - algebra) is a vector space A endowed with a bilinear operation $x, y \in A \longmapsto x.y \in A$.*
Recall that bilinearity means that for each $x \in A$ left and right multiplications by x are linear tranformations of vector spaces (i.e preserve sum and multiplication by a scalar).

Throughout this dissertation, Algebras are assumed to be of finite dimension over the complex number field.

1.2 Pseudo-Riemannian algebra

Let A be an algebra with a bilinear product $A \times A \longmapsto A$ denoted by $(a, b) \longmapsto ab$. The purpose of the dissertation is to study the pairs (A, f) where f denotes a non-degenerate symmetric bilinear form on A satisfying:

$$f(xy, z) + f(y, xz) = 0, for\ all\ x, y, z \in A.$$

In abuse of notation we will use the term pseudo-Riemannian algebra for denoting such a pair.

Definition 1.2.1. *A bilinear form f on A is called pseudo-Riemannian if*

$$f(xy, z) + f(y, xz) = 0, for\ all\ x, y, z \in A.$$

Definition 1.2.2. *The pair (A, f) is called a pseudo-Riemannian algebra if f is a pseudo-Riemannian non-degenerate symmetric bilinear form on A.*

1.3 Example of Pseudo-Riemannian algebra: Lie algebras

In this section, we will introduce an example about pseudo-Remannian algebra called Lie algebra.To completely understand this example, some basic definitions are collected .

The simplest example of a pseudo-Riemannian algebra A is when A is a Lie algebra, endowed with the product given by the Lie bracket $[-,-]$, and symmetric bilinear form given by the Killing form, for those Lie algebras whose Killing form are non-degenerate.

To completely understand this example, it is necessary to introduce some notations and definitions of Lie algebra and Killing form ([2]) which help us understand pseudo-Riemannian algebras.

1.3.1 Lie Algebra

A Lie algebra is a vector space $\mathfrak{g}$ over some field F together with a binary operation $[.,.] : \mathfrak{g} \times \mathfrak{g} \longmapsto \mathfrak{g}$ called the Lie bracket, which satisfies the following axioms:

1. Bilinearity

   $[ax + by, z] = a[x, z] + b[y, z]$,

   $[z, ax + by] = a[z, x] + b[z, y]$, for all scalars $a, b \in F$ and all elements $x, y, z \in \mathfrak{g}$.

2. Alternating on $\mathfrak{g}$

   $[x, x] = 0$ for all $x \in \mathfrak{g}$.

3. The Jacobi Identity

   $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$, for all $x, y, z \in \mathfrak{g}$. (1.1)

As an example of Lie bracket, we have the commutator $[A, B] = AB - BA$ of two n by n matrices.

Definition 1.3.1. *A non abelian Lie algebra* $\mathfrak{g}$ *is called simple if it has no non-trivial ideals.*

Definition 1.3.2. *We define a Lie algebra* $\mathfrak{g}$ *to be semi-simple if it is the finite direct sum of simple Lie algebras* $\mathfrak{g}_i$:

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \ldots \oplus \mathfrak{g}_n.$$

1.3.2 Adjoint Representation of a Lie Algebra

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{C}$. Then the linear mapping

$$ad : \mathfrak{g} \longrightarrow End(\mathfrak{g})$$

given by

$$x \longmapsto ad_x$$

is a representation of a Lie algebra and is called the adjoint representation of the algebra.

Within $End(\mathfrak{g})$, the lie bracket is, by definition, given by the commutator of the two operators:

$$[ad_x, ad_y] = ad_x \circ ad_y - ad_y \circ ad_x$$

where $\circ$ denotes composition of linear map.

Moreover, by using the above definition of the Lie bracket, we notice that there is an equation equivalent to the Jacobi identity presented in the following lemma.

Lemma 1.3.3. *We have the the Jacobi identity,(1.1) takes the form*

$$([ad_x, ad_y])(z) = \left(ad_{[x,y]}\right)(z) \tag{1.2}$$

where x, y, z are arbitrary elements of $\mathfrak{g}$.

Proof. By taking the first side of equation (1.2), we get:

$([ad_x, ad_y])(z) = (ad_x \circ ad_y - ad_y \circ ad_x)(z) = ad_x \circ ad_y(z) - ad_y \circ ad_x(z)$

$= [x, ad_y(z)] - [y, ad_x(z)] = [x, [y, z]] - [y, [x, z]]$.

Now we take the second part of the equation (1.2), we have $\left(ad_{[x,y]}\right)(z) = [[x, y], z]$.

As a result of (1.2), we have: $[x, [y, z]] - [y, [x, z]] = [[x, y], z]$.

Then $[x, [y, z]] + [y, [z, x]] - [[x, y], z] = 0$. Therefore $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ which is the Jacobi identity (1.1).

□

1.3.3 Killing form

We represent the definition of trace of a linear operator and some of its properties that will be used throughout this subsection:

Given some linear map $f : V \longrightarrow V$ ($V$is a finite-dimensional vector space) generally, we can define the trace of this map by considering the trace of matrix representation of f, that is, choosing a basis for V and describing f as a matrix relative to this basis, and taking the trace of this square matrix.The result will not depend on the basis chosen, since different bases will give rise to similar matrices, allowing for the possibility of a basis-independent definition for the trace of a linear map.

As an illustration, We declare the matrix of f with respect to the basis $\{v_1, v_2, ..., v_n\}$, also called the matrix representation of f, to be

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

an $n \times n$ matrix. Also known as an $n-$ square matrix or a square matrix of order n.

Then the trace of the square matrix A is defined to be the sum of the elements on the main diagonal (the diagonal from the upper left to the lower right) of A, i.e.,

$$tr(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}$$

Here we list some properties of a trace :

1. The trace is a linear mapping. That is,

   $tr(A + B) = tr(A) + tr(B)$,

   $tr(cA) = c\,\mathrm{tr}(A)$.

   for all square matrices A and B, and all scalars c.

2. A matrix and its transpose have the same trace:

   $\mathrm{tr}(A) = \mathrm{tr}(A^{\mathrm{T}})$.

   This follows immediately from the fact that transposing a square matrix does not affect elements along the main diagonal.

3. The matrices in a trace of a product can be switched:

   $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

Now we define the Killing form by the following:

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field $\mathbb{C}$. The **Killing form** κ is the bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \longmapsto \mathbb{C}$ defined by $\kappa(x, y) = Tr(ad_x \circ ad_y)$ for all $x, y, z \in \mathfrak{g}$.

It has the following properties:

1. It is bilinear.

2. It is symmetric.

3. It is ad invariant:

$$\kappa([y, x], z) + \kappa(x, [y, z]) = 0, \ for\ all\ x, y, z \in \mathfrak{g}.$$

Definition 1.3.4. *The Killing form is said to be non-degenerate if for all* $y = 0, \kappa(x, y) = 0$ *implies* $x = 0$.

Theorem 1.3.5 (Cartan criterion). *A Lie algebra* $\mathfrak{g}$ *over* $\mathbb{C}$ *is semi-simple if and only if the Killing form is non-degenerate.*

All Lie algebras mentioned in this dissertation are finite dimensional Lie algebras over $\mathbb{C}$.

Chapter 2

Preliminaries

In this chapter, we will list some definitions :

Definition 2.0.6. *$f|_{A\times A}$ is non-degenerate if $x \in A$ and $f(x,y) = 0$, for all $y \in A$, then $x = 0$.*

Definition 2.0.7. *Let H be a subspace of A. If $AH \subseteq H$, then H is called a left ideal of A. If $HA \subseteq H$, then H is called a right ideal of A. If H is both a left ideal and a right ideal, then H is an ideal. The algebra A is called abelian if $A \neq 0$ and $xy = 0$, for all $x, y \in A$.*

A symmetric positive definite bilinear form on a real finite-dimensional vector space is said to be a Riemannian inner product, while in general, a non-degenerate symmetric bilinear form (which is allowed to be indefinite) is called a pseudo-Riemannian inner product. This justifies the terminology in the following definition.

Definition 2.0.8. *Let (A, f) be a pseudo-Riemannian algebra and H be a subspace of A. If $f(x,y) = 0$, for any $x, y \in H$, then H is called isotropic. If $f|_{H\times H}$ is non-*

degenerate, then H is called non-degenerate.

Definition 2.0.9. *Let (A, f) be a pseudo-Riemannian algebra. If there exist non-trivial and non-degenerate ideals A_1 and A_2 such that $A = A_1 \oplus A_2$, then (A, f) is called decomposable, otherwise indecomposable. Furthermore, if $f(A_1, A_2) = 0$, then the decomposition $A = A_1 \oplus A_2$ is called an orthogonal decomposition.*

Definition 2.0.10. *The pair (A, f) is called irreducible if it has no non-trivial non-degenerate ideal.*

Definition 2.0.11. *Let (A, f) be a pseudo-Riemannian algebra. An automorphism π of A is called an isometry if π preserves the bilinear form i.e.,*

$$f(\pi(x), \pi(y)) = f(x, y), for\ all\ x, y \in A.$$

The following notation will be used in this dissertation.

Let $H^{\perp}$ denote the subspace of A orthogonal to H with respect to f, i.e.,

$$H^{\perp} = \{x \in A | f(x, y) = 0, \forall y \in H\}.$$

Let $LC(A)$ denote the left center of A, i.e.,

$$LC(A) = \{x \in A | yx = 0, \forall y \in A\}.$$

Proposition 2.0.12. *$LC\ (A)$ is a left ideal.*

Proof. Let $x \in LC(A)$ and Let $Ax \in ALC(A)$. We have, $Ax = 0$. This implies, $ALC(A) = 0$. It follows that $ALC(A) = 0 \subseteq LC(A)$. Therefore $LC(A)$ is a left ideal.

□

Let $Z(A)$ denote the center of A, i.e.,

$$Z(A) = \{x \in A | xy = yx = 0, \forall y \in A\}.$$

Proposition 2.0.13. *$Z\ (A)$ is an ideal.*

Proof. Let $x \in Z(A)$. We have $Ax = 0$ also $xA = 0$.

Therefore $Z(A)$ is an ideal.

□

Chapter 3

Now we will introduce some important propositions that will be used throughout the dissertation :

Proposition 3.0.14. *Let (A, f) be a pseudo-Riemannian algebra. Then $LC(A) = (AA)^{\perp}$. As a consequence, dim LC(A)+dim AA= dim A.*

Proof. Let $x \in (AA)^{\perp}$ such that $(AA)^{\perp} = \{x \in A | f(x, yz) = 0, \forall y, z \in A\}$. Since (A, f) is a pseudo-Riemannian algebra. We have, $f(yx, z) + f(x, yz) = 0$, for all $x, y, z \in A$. This implies $f(yx, z) = 0$, for all $y, z \in A$. But f is non-degenerate on A. Then $yx = 0$, for all $y \in A$. It follows that $x \in LC(A)$. So $(AA)^{\perp} \subseteq LC(A)$.

Conversely, we need to show that $LC(A) \subseteq (AA)^{\perp}$. Let $x \in LC(A)$ i.e., $yx = 0$, for all $y \in A$. Then $f(yx, z)$, for all $y, z \in A$. Since f on A is pseudo-Riemannian, we know that $f(yx, z) + f(x, yz) = 0$, for all $x, y, z \in A$. Then $f(x, yz) = 0$, for all $y, z \in A$. We have, $x \in (AA)^{\perp}$. Therefore, $LC(A) = (AA)^{\perp}$.
Because *dim* $(\mathrm{AA})^{\perp}$ = dim A - dim AA. *We have,* dim $(AA)^{\perp}$ + dim AA = dim A. *Consequently, dim LC(A) +dim AA = dim A .*

□

Proposition 3.0.15. *Let (A, f) be a pseudo-Riemannian algebra and H an ideal of A. Then $H^{\perp}$ is a left ideal and $HH^{\perp} = 0$.*

Proof. Let H an ideal of A, then $AH \subseteq H$ and $HA \subseteq H$.

we want to show that $f(H, AH^{\perp}) = 0$.

Since f is pseudo-riemannian, we have $f(H, AH^{\perp}) = -f(AH, H^{\perp}) = 0$.

This implies $AH^{\perp} \subseteq H^{\perp}$. Therefore, $H^{\perp}$ is left ideal.

Moreover, we have $f(A, HH^{\perp}) = -f(HA, H^{\perp}) = 0$.

Because f is non-degenerate on A. It follows that $HH^{\perp} = 0$.

□

Proposition 3.0.16. *Let (A, f) be a pseudo-Riemannian algebra. Then there exists a decomposition $A = \bigoplus_{i=1}^{l} A_i$ of A into indecomposable non-degenerate ideals.*

Proof. If A is indecomposable $(A = A_1)$.

f is non-degenerate on A_1 because $A = A_1$ and f is is non-degenerate on A.

Clearly A_1 is indecomposable since A is indecomposable.

If A is decomposable, then there exist a non-trivial and non-degenerate ideals A_1 and A_2 such that

$$A = A_1 \oplus A_2. \tag{3.1}$$

(3.1) holds for *dim* A $= 2$.

dim A$=$ *dim* A_1 + dim A_2 - dim $A_1 \cap A_2$, *where* $\mathrm{A}_1 \cap A_2 = 0$.

Then $2 =$ *dim* A_1 + dim A_2. *So* dim $A_1 =$ dim $A_2 = 1$.

This implies that A_1 and A_2 are indecomposable.

Suppose it is true for A of dimension up to $n-1$. Prove it is true for *dim* $A = n$. We already have $A = A_1 \oplus A_2$ which are non-trivial and non-degenerate ideals. Both A_1 and A_2 can be decomposed into indecomposable non-degenerate ideals since the dimension of each is less than n and the conclusion is true for dimension up to $n-1$ (By induction).Therefore $A = A_1 \oplus A_2$ is decomposed into indecomposable non-degenerate ideals.

Then there exists a decomposition $A = \bigoplus_{i=1}^{l} A_i$ of A into indecomposable non-degenerate ideals.

□

Chapter 4

Pseudo-Riemannian algebras whose left centers are not isotropic

In this chapter we focus on pseudo-Riemannian algebras whose left center is not isotropic.

Proposition 4.0.17. *Let A be an abelian algebra. If f is a non-degenerate symmetric bilinear form on A, then (A, f) is a pseudo-Riemannian algebra. Furthermore, there exists an orthogonal decomposition $A = A_1 \oplus A_2 \oplus ..... \oplus A_n$ into indecomposable non-degenerate ideals such that $dim A_i = 1$.*

Proof. We have $f(xy, z) + f(y, xz) = 0$, for all $x, y, z \in A$ since A is an abelian algebra. Therefore f is a pseudo-Riemannian bilinear form.
Moreover, f is a non-degenerate symmetric bilinear form on A. Then the pair (A, f) is a pseudo-Riemannian algebra.
Since A is abelian then any subspace H is an ideal because $AH \subseteq H$ and $HA \subseteq H$, since $AH = 0$ and $HA = 0$.
We let A abelian, let $x \neq 0$, $x \in A$ then there exists $y \neq 0$ such that $f(x, y) \neq 0$ since f is non-degenerate on A. We claim that there exists $e_1 \in A$ such that $f(e_1, e_1) \neq 0$.

If $f(x,x) \neq 0$, choose $e_1 = x$.

On the other hand, if $f(x,x) = 0$, and $f(y,y) \neq 0$, so choose $e_1 = y$.

Otherwise, if $f(x,x) = 0$, and $f(y,y) = 0$, we choose $e_1 = x + y$.

Since $f(x+y, x+y) = f(x,x) + f(x,y) + f(y,x) + f(y,y) = 0 + f(x,y) + f(y,x) + 0$.

But f is symmetric then $f(x+y, x+y) = 2f(x,y) \neq 0$.

Then there exist at least one $e_1 \in A$, such that $f(e_1, e_1) \neq 0$.

Let A_1 be the subspace of A generated by e_1 so A_1 is an ideal and *dim* $\mathrm{A}_1 = 1$ *then* A_1 is indecomposable.

Let $x \in A_1$ such that $f(x,y) = 0$, for all $y \in A_1$.

Let $x = \lambda_1 e_1$ and $y = \lambda_2 e_1$, where $\lambda_1, \lambda_2 \in \mathbb{C}$. So $f(\lambda_1 e_1, \lambda_2 e_1) = 0$, for all $\lambda_2 \in \mathbb{C}$ since f is bilinear, we get: $\lambda_1 \lambda_2 f(e_1, e_1) = 0$, for all λ_2 but $f(e_1, e_1) \neq 0$. So $\lambda_1 = 0$ then $x = 0$. Consequently, f is non-degenerate on A_1.

Let $A_2 = A_1^{\perp}$; A_2 is an ideal. Since A_1 is non-degenerate then A can be written as $A = A_1 \oplus A_1^{\perp}$. Obviously, $f(A_1, A_1^{\perp}) = 0$.

Let $x \in A_1^{\perp}$ such that $f(x, A_1^{\perp}) = 0$ but $x \in A_1^{\perp}$, so $f(x, A_1) = 0$.

However $A = A_1 \oplus A_1^{\perp}$. Hence $f(x, A) = 0$ and $x \in A$ because f is non-degenerate on A, therefore $x = 0$. Then f is non-degenerate on $A_1^{\perp}$.

Therefore, the decomposition $A = A_1 \oplus A_1^{\perp}$ is an orthogonal decomposition.

$A = A_1 \oplus A_1^{\perp}$ holds for *dim* A = 2. Then *dim* $\mathrm{A}_1^{\perp}$ = dim A - dim A_1 = 2 - 1 = 1. *Accordingly,* $\mathrm{A}_1^{\perp}$ is indecomposable.

Suppose it is true for A of dimension $n-1$. We already have $A = A_1 \oplus A_1^{\perp}$ where A_1 and $A_1^{\perp}$ are non-trivial and non-degenerate ideals with *dim* $\mathrm{A}_1 = 1$ *and* f $(\mathrm{A}_1, A_1^{\perp}) = 0$ then *dim* $\mathrm{A}_1^{\perp}$ = n-1.

Hence by induction there exists an n-1 *indecomposable, non-degenerate ideals* $A_2, ..., A_n$ such that $A_1^{\perp} = A_2 \oplus .... \oplus A_n$ with $f(A_i, A_j) = 0$, for $i \neq j$ and *dim* $A_i = 1,\ 2 \leq i \leq n$. Moreover, since $f(A_1, A_1^{\perp}) = 0$ and $A = A_1 \oplus A_2 \oplus ..... \oplus A_n$. Therefore $f(A_1, A_i) = 0$, $2 \leq i \leq n$.

□

Here we present an example and we show that this (A, f) is pseudo-Riemannian algebra whose left center is not isotropic:

Example:

Let (H, f_H) be an abelian pseudo-Riemannian algebra and (I, f_I) a pseudo-Riemannian algebra with the product $\circ$. Let

$$so(\mathbf{I}) = \{A \in End\ I \,|\, f_I(A(x), y) + f_I(x, A(y)) = 0\}.$$

Given a linear mapping $L : H \longmapsto so(\mathbf{I})$ denoted by $x \longmapsto L_x$, define a product $*$ on vector space $A = H + I$ (direct sum as subspaces) by

$$x * y = 0, \forall x, y \in H,$$

$$x * y = 0, \forall x \in I, y \in H,$$

$$x * y = x \circ y, \forall x, y \in I,$$

$$x * y = L_x(y), \forall x \in H, y \in I,$$

and define a symmetric bilinear form f on A by

$$f(x,y) = f_H(x,y), \forall x,y \in H,$$

$$f(x,y) = f_I(x,y), \forall x,y \in I,$$

$$f(x,y) = 0, \forall x \in H, y \in I.$$

We need to show that (A,f) is a pseudo-Riemannian and the left center is not isotropic:

Let $x = h_1 + l_1$, $y = h_2 + l_2$, and $z = h_3 + l_3$ where $h_1, h_2, h_3 \in H$ and $l_1, l_2, l_3 \in I$. We have, $f(x * y, z) + f(y, x * z)$

$= f((h_1 + l_1) * (h_2 + l_2), h_3 + l_3) + f(h_2 + l_2, (h_1 + l_1) * (h_3 + l_3))$

$$= f(\overbrace{h_1 * h_2}^{=0}, h_3 + l_3) + f(h_1 * l_2, h_3 + l_3) + f(\overbrace{l_1 * h_2}^{=0}, h_3 + l_3) + f(l_1 * l_2, h_3 + l_3) + f(h_2 + l_2, \overbrace{h_1 * h_3}^{=0}) + f(h_2 + l_2, h_1 * l_3) + f(h_2 + l_2, \overbrace{l_1 * h_3}^{=0}) + f(h_2 + l_2, l_1 * l_3)$$

$= f(h_1 * l_2, h_3) + f(h_1 * l_2, l_3) + f(l_1 * l_2, h_3) + f(l_1 * l_2, l_3) + f(h_2, h_1 * l_3) + f(l_2, h_1 * l_3) + f(h_2, l_1 * l_3) + f(l_2, l_1 * l_3)$

$$= \overbrace{f(L_{h_1}(l_2), h_3)}^{=0} + f(L_{h_1}(l_2), l_3) + \overbrace{f(l_1 \circ l_2, h_3)}^{=0} + f(l_1 \circ l_2, l_3) + \underbrace{f(h_2, L_{h_1}(l_3))}_{=0} + f(l_2, L_{h_1}(l_3)) + \underbrace{f(h_2, l_1 \circ l_3)}_{=0} + f(l_2, l_1 \circ l_3)$$

$$= f_I(L_{h_1}(l_2), l_3) + f_I(l_1 \circ l_2, l_3) + f_I(l_2, L_{h_1}(l_3)) + f_I(l_2, l_1 \circ l_3) = 0.$$

Since $L_{h_1} \in so(\mathbf{I})$ and f_I is a pseudo Riemannian algebra.

Therefore (A, f) is pseudo-Riemannian.

Now we want to show that f is non-degenerate on A.

Let $f(x, A) = 0$ and $x \in A$. However $A = H + I$ then $f(x, H) = 0$ and $f(x, I) = 0$.

Let $x = h_1 + l_1$, where $h_1 \in H$ and $l_1 \in I$.

Since $f(x, H) = 0$ this implies that $f(h_1, H) + \overbrace{f(l_1, H)}^{=0} = 0$.

So $f_H(h_1, H) = 0$ and $h_1 \in H$. But f_H is non-degenerate on H then $h_1 = 0$. Similarly, $l_1 = 0$. Finally, $x = 0$.

Hence (A, f) is a pseudo-Riemannian algebra.

One can see that the left center of $LC(A)$ is not isotropic:

Since $A * H = 0$ then $H \subset LC(A)$ and $H \neq 0$.

There exists an $x \neq 0$ and $x \in H$ But since f_H is non-degenerate.

Therefore there exist $y \neq 0$ where $y \in H$, such that $f(x, y) = f_H(x, y) \neq 0$. Then there exist $x, y \in LC(A)$ such that $f(x, y) \neq 0$.

Consequently, $LC(A)$ is not isotropic.

We will state Zorn's lemma below and use it later to prove some propositions in this dissertation:

Theorem 4.0.18 (Zorn's Lemma.). *Suppose a non-empty partially ordered set P has the property that every non-empty chain has an upper bound in P. Then the set P contains at least one maximal element.*

Moreover, Zorn's Lemma is equivalent to the well-ordering theorem and the axiom of choice.

Theorem 4.0.19. *Let (A, f) be a pseudo-Riemannian algebra whose left center is not isotropic. Then there exist a sequence of non-degenerate subalgebras of A such that*

$$A = A_0 \supset A_1 \supset \ldots \supset A_n,$$

where A_i is an ideal of A_{i-1}, the quotient algebra A_{i-1}/A_i is an abelian for each $i \in \{1, 2, \ldots, n\}$, and the left center of A_n is isotropic.

Proof. Let $A_0 = A$. Since $LC(A)$ is not isotropic.

There exists $x, y \in LC(A)$ such that $f(x, y) \neq 0$.

We claim that there exists an $e_1 \in LC(A)$ such that $f(e_1, e_1) \neq 0$.

If $f(x, x) \neq 0$, choose $e_1 = x$.

However, if $f(x, x) = 0$ and $f(y, y) \neq 0$, we choose $e_1 = y$.

Also if $f(x, x) = 0$ and $f(y, y) = 0$, so choose $e_1 = x + y$.

Since $f(x+y, x+y) = f(x, x) + f(x, y) + f(y, x) + f(y, y) = 0 + f(x, y) + f(y, x) + 0$.

But f is symmetric then $f(x + y, x + y) = 2f(x, y) \neq 0$.

Then there exist $e_1 \in LC(A)$ such that $f(e_1, e_1) \neq 0$.

Let H_1 be the subspace of $LC(A)$ generated by e_1.

We have to show that H_1 is non-degenerate. Let $x \in H_1$ such that $f(x, y) = 0$, for all $y \in H_1$.

Let $x = \lambda_1 e_1$ and $y = \lambda_2 e_1$, where $\lambda_1, \lambda_2 \in \mathbb{C}$. So $f(\lambda_1 e_1, \lambda_2 e_1) = 0$ for all $\lambda_2 \in \mathbb{C}$ since f is bilinear, we get: $\lambda_1 \lambda_2 f(e_1, e_1) = 0$, for all λ_2 but $f(e_1, e_1) \neq 0$. So $\lambda_1 = 0$ then $x = 0$.

Consequently, $f|_{H_1 \times H_1}$ is non-degenerate.

By Zorn's Lemma (4.0.18), we can pick H_1 to be a maximal non-degenerate subspace of $LC(A)$.

Let $A_1 = H_1^{\perp}$. Prove that A_1 is an ideal of $A = A_0$.

$f(H_1, AA_1) = -f(\overbrace{AH_1}^{=0}, A_1) = 0$ because f is a pseudo-Riemannian and $H_1 \subset LC(A)$.

So $AA_1 \subseteq A_1$.

And $f(A_1 A, H_1) = -f(A, \overbrace{A_1 H_1}^{=0}) = 0$. Then $A_1 A \subseteq A_1$.

Therefore A_1 is an ideal.

We want to prove that A_1 is non-degenerate:

Now let $x \in H_1^{\perp}$ such that $f(x, H_1^{\perp}) = 0$. Otherwise, $x \in H_1^{\perp}$ this implies that $f(x, H_1) = 0$.

Since H_1 is non-degenerate then A can be written as $A = H_1 \oplus H_1^{\perp}$. Therefore $f(x, A) = 0$ and $x \in A$ but f is non-degenerate on A so $x = 0$. Then f is non-degenerate on A_1.

Prove that the quotient is abelian: $A_0 | A_1 = \{x + A_1 | x \in A\}$.

Let $x = c + d$ and $y = e + f$ where $c, e \in A_1$ and $d, f \in A_1^{\perp}$ then $(xy) + A_1 =$

$[(c+d)(e+f)] + A_1 = ce + cf + de + df + A_1$.

We have, $df = 0$ since $d, f \in A_1^{\perp}$ and $A_1^{\perp} = H_1 \subset LC(A)$.

And we have $cf, de \in AA_1^{\perp} \subset A \cap A_1^{\perp} = 0$. Hence $(xy) + A_1 = A_1$. Then $A|A_1$ is abelian.

Now if $LC(A_1)$ is isotropic, we're done.

If $LC(A_1)$ is not isotropic, we continue this process by induction till we reach a certain A_n such that $LC(A_n)$ is isotropic so we stop because the algebra A is of a finite dimension.

□

Chapter 5

Pseudo-Riemannian algebras whose left centers are isotropic

We notice that pseudo-Riemannian algebras whose left centers are isotropic play a crucial role.

Proposition 5.0.20. *Let (A, f) be a pseudo-Riemannian algebra whose left center is isotropic. Then (A, f) is decomposable if and only if there exist non-trivial ideals A_1 and A_2 of A such that $A = A_1 \oplus A_2$.*

Proof. ($\Rightarrow$)*It is obvious. Since (A, f) is decomposable.*

($\Leftarrow$)*If there exist non-trivial ideals* A_1 and A_2 of A such that $A = A_1 \oplus A_2$. We need to show that f is non-degenerate on A_1 and A_2. Assume that f is degenerate on A_1 then there exists a non-zero element $(x \neq 0)$, $x \in A_1$ such that $f(x, A_1) = 0$.

If $x \in A_1A_1$ *then* $f(x, A) = 0$.

Because we have $f(x,A_2) \subseteq f(A_1A_1, A_2) = -f(A_1, A_1A_2)$. *But since* A_1 *is an ideal* and $A_2 \subseteq A$ then $A_1A_2 \subseteq A_1$ and since A_2 is an ideal also $A_1 \subseteq A$ so $A_1A_2 \subseteq A_2$. Therefore, $A_1A_2 \subseteq A_1 \cap A_2$.

However, $A = A_1 \oplus A_2$ then $A_1A_2 \subseteq A_1 \cap A_2 = 0$. Hence $f(x, A_2) \subseteq f(A_1A_1, A_2) = -f(A_1, \overbrace{A_1A_2}^{0}) = 0$. Now we have $f(x, A_1) = 0$ and $f(x, A_2) = 0$ then $f\ (x,\ A) = 0$ however f is non-degenerate on A so $x = 0$. It is a contradiction, then $x \notin A_1A_1$.

Because $LC(A)$ is isotropic. We have $LC(A) \subseteq (LC(A))^{\perp} = ((AA)^{\perp})^{\perp} = AA$ by proposition *(3.0.14)*. Accordingly, $LC(A) \subseteq AA$. Moreover, we have $A = A_1 + A_2$ so $AA = A_1A_1 + A_1A_2 + A_2A_1 + A_2A_2 = A_1A_1 + A_2A_2$.

If $x \in AA$, hence $x = s + t$ this implies $t \in A_1 \cap A_2 = 0$ so $t = 0$.

Consequently, $x = s \in A_1A_1$.It is a contradiction, $x \notin AA$ therefore $x \notin LC(A)$.

Thus there exists $w \in A$ such that $wx \neq 0$ since $A = A_1 + A_2$ then $w = y_1 + y_2$, where $y_1 \in A_1$ and $y_2 \in A_2$. So $y_1x + y_2x \neq 0$. Note that $y_2x = 0$ because $y_2 \in A_2$, $x \in A_1$ and $A_2A_1 = 0$.

Thus $y_1x \neq 0$ then $\exists\ y \in A_1$ such that $yx \neq 0$. Therefore there exists $z \in A$ such that $f(yx, z) \neq 0$ since f is non-degenerate on A.

Since A_1 is an ideal of A and $y \in A_1$ we have $yz \in A_1$, which contradicts the choice of x.

Namely $f|A_1 \times A_1$ is non-degenerate. Similarly, $f|A_2 \times A_2$ is non-degenerate.

□

The following is to show that the decomposition of any pseudo-Riemannian algebra whose left center is isotropic into non-degenerate indecomposable ideals is unique up to an automorphism.

Let (A, f) be a pseudo-Riemannian algebra whose left center is isotropic and let

$$A = A_1 \oplus A_2 \oplus ..... \oplus A_n,$$

$$A = A'_1 \oplus A'_2 \oplus ..... \oplus A'_m$$

be decompositions of A. Here A_i, A'_j, $1 \leq i \leq n$, $1 \leq j \leq m$, are indecomposable non-degenerate ideals of A.

It is easy to check that $A_1A_1 \neq 0$. In fact, we assume that $A_1A_1 = 0$. Thus $A_1 \subseteq LC(A)$ so $AA_1 = 0$. But $A = A_1 \oplus A_2 \oplus ..... \oplus A_n$ this implies $(A_1 \oplus A_2 \oplus ..... \oplus A_n)A_1 = 0$ we notice that $A_iA_1 = 0$, for all $i = 1, 2, ..., n$ and $A_i \cap A_1 = 0$. However, $LC(A)$ is isotropic and A_1 is non-degenerate. It is a contradiction. Because $A_1A_1 = \bigoplus_{j=1}^{m} A_1A'_j$, we have $A_1A'_j \neq 0$ for some j. Without loss of generality, assume that $A_1A'_1 \neq 0$. Let $H_1 = \bigoplus_{j=2}^{n} A_j$ and $H'_1 = \bigoplus_{j=2}^{m} A'_j$, which are non-degenerate ideals of A by proposition (5.0.20).

Then

$$A = A_1 \oplus H_1,$$

$$A = A'_1 \oplus H'_1.$$

We introduce some propositions that we use throughout this chapter:

Proposition 5.0.21. *$A_1A'_1 \subseteq A_1 \cap A'_1$ and $A_1H'_1 \subseteq A_1 \cap H'_1$*

Proof. Since A_1 is an ideal ($A_1A \subseteq A_1$) and $A_1' \subset A$ then $A_1A_1' \subseteq A_1$. Moreover, A_1' is an ideal ($AA_1' \subseteq A_1'$) and $A_1 \subset A$ so $A_1A_1' \subseteq A_1'$.

Consequently, $AA_1' \subseteq A \cap A_1'$. Likewise, $A_1H_1' \subseteq A_1 \cap H_1'$.

□

Proposition 5.0.22. *Intersection of two ideals is an ideal.*

Proof. Let A_1 and A_2 be ideals. We have, $A(A_1 \cap A_2) \subseteq AA_1 \cap AA_2 \subseteq A_1 \cap A_2$. Also, $(A_1 \cap A_2)A \subseteq A_1A \cap A_2A \subseteq A_1 \cap A_2$. Thus $A_1 \cap A_2$ is an ideal.

□

Proposition 5.0.23. *Sum of two ideals is an ideal.*

Proof. Let A_1 and A_2 be ideals. We have, $A(A_1 + A_2) \subseteq AA_1 + AA_2 \subseteq A_1 + A_2$. Similarly $(A_1 + A_2)A \subseteq A_1A + A_2A \subseteq A_1 + A_2$.

□

Proposition 5.0.24. $LC(A_1) = LC(A) \cap A_1$.

Proof. Let $x \in LC(A_1)$ we have $x \in A_1$ such that $A_1x = 0$. Clearly $x \in A$. Since $A = A_1 + H_1$. As a result, $Ax = A_1x + H_1x = 0$ because $x \in A_1$ and $H_1A_1 \subseteq H_1 \cap A_1 = \{0\}$. Therefore, $x \in A$; $Ax = 0$. So $x \in LC(A)$. Consequently, $LC(A_1) \subseteq LC(A) \cap A_1$. Conversly, let $x \in LC(A) \cap A_1$ so $x \in LC(A)$ which means $x \in A$ such that $Ax = 0$. But $A_1 \subset A$ then $A_1x = 0$ and we already have $x \in A_1$. Therefore, $x \in LC(A_1)$. As a consequence, $LC(A) \cap A_1 \subseteq LC(A_1)$.

□

Lemma 5.0.25. $A_1 \cap H'_1 = 0$ *and* $A'_1 \cap H_1 = 0$.

Proof. $B_1 = A_1 \cap A'_1$ and $B_2 = A_1 \cap H'_1$.

We notice that $B_1 \cap B_2 = 0$ since $(A_1 \cap A'_1) \cap (A_1 \cap H'_1) = A_1 \cap A'_1 \cap H'_1 = 0$.

And $LC(A_1) \subseteq LC(A) \cap A_1$. We have $LC(A)$ is isotropic, then $LC(A_1)$ is isotropic.

Clearly, $A_1A_1 = A_1A = A_1(A'_1 \oplus A'_2 \oplus ..... \oplus A'_m) = A_1(A'_1 \oplus H'_1) = A_1A'_1 \oplus A_1H'_1 \subseteq B_1 \oplus B_2$.

(1) If $A_1 = B_1 \oplus B_2$.

A_1 is indecomposable and $B_1 \neq 0$ since $A_1A'_1 \subseteq A_1 \cap A'_1 \neq 0$. Consequently, by proposition (5.0.20) $B_2 = 0$.

(2) If $A_1 \neq B_1 \oplus B_2$.

Since $B_1 = A_1 \cap A'_1 \subseteq A_1$ and $B_2 = A_1 \cap H'_1 \subseteq A_1$ then $B_1 \oplus B_2 \subset A_1$.

So there exist $x \in A_1$ such that $x \notin B_1 \oplus B_2$. Then $x = x_1 + x_2$, where $x_1 \in A'_1$ and $x_2 \in H'_1$. Using the other decomposition,

$$x_1 = x_1^1 + x_1^2 \textit{ and } x_2 = x_2^1 + x_2^2,$$

where $x_1^1, x_2^1 \in A_1$, $x_1^2, x_2^2 \in H_1$. So $x = x_1^1 + x_2^1 + x_1^2 + x_2^2 = x_1^1 + x_2^1$ and $x_1^2 + x_2^2 = 0$ since $x_1^2, x_2^2 \in H_1$ and $x \in A_1$ then $x_1^2 + x_2^2 \in A_1 \cap H_1 = \{0\}$.

One can easily check that $A_1x_1^1 \subseteq A_1A'_1$. *Since* $x_1 = x_1^1 + x_1^2$ *then* $x_1^1 = x_1 - x_1^2$. *So* $A_1x_1^1 \subseteq A_1x_1 - A_1x_1^2$ *and we know that* $A_1x_1^2 \subseteq A_1H_1 \subseteq A_1 \cap H_1 = 0$. *This implies* $A_1x_1^1 \subseteq A_1x_1 \subseteq A_1A'_1$.

Similarly,

$$x_1^1 A_1 \subseteq A_1' A_1,\ A_1 x_2^1 \subseteq A_1 H_1', \text{and}\ x_2^1 A_1 \subseteq H_1' A_1.$$

If $x_1^1 \notin B_1 \oplus B_2$, *let*

$$B_1^{(1)} = B_1 + \mathbb{C}x_1^1,$$

$$B_2^{(1)} = B_2.$$

If $x_1^1 \in B_1 \oplus B_2$, *then* $x_2^1 \notin B_1 \oplus B_2$. *Let*

$$B_1^{(1)} = B_1,$$

$$B_2^{(1)} = B_2 + \mathbb{C}x_2^1.$$

It is clear that B_1 *and* B_2 *are ideals of* A_1. *Also,* $\mathbb{C}x_1^1$ *and* $\mathbb{C}x_2^1$ *are ideals of* A_1 *by above. Therefore,* $B_1^{(1)}$ *and* $B_2^{(1)}$ *are ideals of* A_1 *and*

$$B_1^{(1)} \cap B_2^{(1)} = 0.$$

Thus by taking $B_1^{(1)}$ *and* $B_2^{(1)}$ *according to the first case,* $B_1^{(1)} = B_1 + \mathbb{C}x^1{}_1, B_2^{(1)} = B_2$. *Let* $y \in B_1^{(1)} \cap B_2^{(1)}$. *Then* $y \in B_1^{(1)}$ *so* $y = a + nx_1^1$, *where* $a \in B_1$, $n \in \mathbb{C}$. *And* $y \in B_2^{(1)}$ *so* $y = b$, *where* $b \in B_2$. *However,* $y = y$ *so* $b = a + nx_1^1$. *Then* $x_1^1 = \frac{1}{n}(b - a)$ *if* $n \neq 0$. *It is a contradiction. So* $n = 0$. *Therefore,* $y = a = b \in B_1 \cap B_2 = \{0\}$. *As a result,* $B_1^{(1)} \cap B_2^{(1)} = \{0\}$. *Likewise for* $B_1^{(1)}$ *and* $B_2^{(1)}$ *in the second case.*

If $A_1 = B_1^{(1)} \oplus B_2^{(1)}$ *using similar argument as in* (1), $B_2^{(1)} = 0$. *In particular,* $A_1 \cap H_1' = 0$.

If $A_1 \neq B_1^{(1)} \oplus B_2^{(1)}$, since $dim A_1 < \infty$, repeating the discussion in (2), we may choose $B_1^{(k)}$ and $B_2^{(k)}$ such that $A_1 = B_1^{(k)} \oplus B^{(k)}{}_2$ where $B_1^{(k)}$ and $B_2^{(k)}$ are ideals of A_1.

Using similar argument as in (1), $B_2^{(k)} = 0$. In particular, $A_1 \cap H_1' = 0$.

Similarly, $A_1' \cap H_1 = 0$.

□

Lemma 5.0.26. *The projection $\pi_1 : A_1 \longmapsto A_1'$ is an isomorphism and preserves the bilinear form.*

Proof. We let $x \in A_1 \subseteq A = A_1' + H_1'$ so $x = x_1 + x_2$, where $x_1 \in A_1'$ and $x_2 \in H_1'$. Since $\pi_1 : A_1 \longmapsto A_1'$, then $\pi_1(x) = x_1$.

Now we let $x, y \in A_1$ such that $x = x_1 + x_2$ and $y = y_1 + y_2$, where $x_1, y_1 \in A_1'$ and $x_2, y_2 \in H_1'$. We notice that $\pi_1(x) = x_1$ and $\pi_1(y) = y_1$.

The map $\pi_1 : A_1 \longmapsto A_1'$ is a linear map. Because it preserves the following operations:

(1) Addition : It is clear that $\pi_1(x + y) = \pi_1(x) + \pi_1(y)$. Because $x + y = x_1 + x_2 + y_1 + y_2$ then $\pi_1(x + y) = \pi_1(x_1 + x_2 + y_1 + y_2) = x_1 + y_1 = \pi_1(x) + \pi_1(y)$.

(2) Scalar multiplication : we have, $\pi_1(kx) = \pi_1(kx_1 + kx_2) = kx_1 = k\pi_1(x)$, where $k \in \mathbb{C}$.

. (3) Multiplication: $xy = (x_1+x_2)(y_1+y_2) = x_1y_1+x_1y_2+x_2y_1+x_2y_2 = x_1y_1+x_2y_2$. Because $x_1y_2 \subseteq A'_1H'_1 \subseteq A'_1 \cap H'_1 = 0$. Similarly, $x_2y_1 = 0$.
Then $\pi_1(xy) = x_1y_1 = \pi_1(x)\pi_1(y)$.

Let $x \in ker\ \pi_1$, then $x \in A_1$ and $\pi_1(x) = \{0\} = x_1$ so $x = x_2 \in H'_1$. Therefore, $x \in A_1 \cap H'_1 = 0$ by Lemma (5.0.25). It follows that $ker\pi_1 \subseteq A_1 \cap H'_1 = 0$, then $ker\pi_1 = 0$. As a result, π_1 is injective. Thus $dimA_1 = dim(Im(\pi_1)) \leq dimA'_1$. *By defining the projection* π'_1 similarly as π_1 we have, $dimA'_1 \leq dimA_1$. *This implies that* $dimA_1 = dimA'_1$. Therefore, π_1 is an isomorphism from A_1 to A'_1.
It is clear that $A'_1x_2 = 0$ since $A'_1x_2 \subseteq A'_1H'_1 \subseteq A'_1 \cap H'_1 = 0$. Also, $H'_1x_2 = 0$ because $x_2 = x - x_1$ so $H'_1x_2 \subseteq H'_1x - H'_1x_1$ then $H'_1x_2 \subseteq H'_1x \subseteq H'_1A_1 \subseteq H'_1 \cap A_1 = 0$. Thus $Ax_2 = 0$. Consequently, $x_2 \in LC(A)$.
Therefore, $f(x,x) = f(x_1 + x_2, x_1 + x_2) = f(x_1,x_1) + 2f(x_1,x_2)$ since $LC(A)$ is isotropic.
Now because H'_1 is non-degenerate , Let $x_1 = h_1 + h_2$, where $h_1 \in H'_1$ and $h_2 \in H'^{\perp}_1$. So $h_1 = x_1 - h_2$, then $H'_1h_1 \subseteq H'_1(x_1 - h_2) = 0$ so $h_1 \in LC(H'_1)$. Moreover, $A'_1h_1 = 0$ then $h_1 \in LC(A'_1)$. Accordingly, $h_1 \in LC(A)$.
It follows that

$$f(x,x) = f(x_1,x_1) + 2f(x_1,x_2) = f(x_1,x_1) + 2f(h_1,x_2) + 2f(h_2,x_2) = f(x_1,x_1) + 0$$

As a result,

$$f(x,x) = f(\pi_1(x), \pi_1(x)).$$

Therefore, π_1 preserves the bilinear form.

□

Furthermore, we have

$$A_1H'_1 = H'_1A_1 = A'_1H_1 = H_1A'_1 = 0.$$

Indeed $A_1H'_1$, $H'_1A_1 \in A_1 \cap H'_1 = 0$. Likewise, $A'_1H_1 = H_1A'_1 \subseteq H_1 \cap A'_1 = 0$.

Moreover, (1) $AA_1 = (A_1 + H_1)A_1 = A_1A_1$ and $AA_1 = (A'_1 + H'_1)A_1 = A'_1A_1$ then $A_1A_1 = A'_1A_1$.

Also, (2) $A'_1A = A'_1(A_1 + H_1) = A'_1A_1$ and $A'_1A = A'_1(A'_1 + H'_1) = A'_1A'_1$ then $A'_1A = A'_1A_1 = A'_1A'_1$.

By (1) and (2) we have $A_1A_1 = A'_1A_1 = A'_1A'_1$.

We also notice that (3) $A_1A = A_1(A_1+H_1) = A_1A_1$, and $A_1A = A_1(A'_1+H'_1) = A_1A'_1$.

Therefore, $A_1A = A_1A_1 = A_1A'_1$.

By (1), (2) and (3) we have,

$$A_1A_1 = A_1A'_1 = A'_1A_1 = A'_1A'_1.$$

Repeating the above discussion for $j = 2, 3, ..., n$, we have :

Theorem 5.0.27. *Let (A, f) be a pseudo-Riemannian algebra whose left center is isotropic and let*

$$A = A_1 \oplus A_2 \oplus ..... \oplus A_n,$$

$$A = A'_1 \oplus A'_2 \oplus ... \oplus A'_m$$

be decompositions of A. Here A_i, A'_j, $1 \le i \le n$, $1 \le j \le m$, *are indecomposable ideals of A. Then we have:*

1. $n = m$.

2. *Changing the subscripts if necessary, we can get*

$$dim A_j = dim A'_j,$$

$$A_j A_j = A_j A'_j = A'_j A_j = A'_j A'_j,$$

$$A_j A'_k = A'_j A_k = 0, j \neq k.$$

3. *The projections* $\pi_i : A_i \longmapsto A'_i$ *are isomorphism and preserves the bilinear form, so* $\pi = (\pi_1, ..., \pi_n)$ *is an automorphism of A.*

Proof. We already have (2) and (3) where they are clear.

Now, we assume $n < m$. According to (2) we have,

$$\begin{array}{ccc} dim A_1 & = & dim A'_1 \quad , \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ dim A_n & = & dim A'_n \quad , \end{array}$$

By (3) the projections are isomorphism. Then

$$dimA = dimA_1 + ... + dimA_n,$$

$$dimA = dimA'_1 + ... + dimA'_n.$$

Therefore, $dimA'_{n+1} = ... = dimA'_m = 0.$

Then $A'_{n+1} = A'_{n+2} = ... = A'_m = 0.$

Consequently, $n = m$.

□

Chapter 6

Pseudo-Riemannian algebras whose left centers equal the centers

In this chapter we focus on pseudo-Riemannian algebras whose left centers equal the centers.

Theorem 6.0.28. *Let (A, f) be a pseudo-Riemannian algebra whose left center equals the center. If the left center is not isotropic, then there exist non-degenerate ideals A_1 and A_2 such that $A = A_1 \oplus A_2$, where $f(A_1, A_2) = 0$, $A_1A_1 = 0$ and the left center of A_2 is isotropic.*

Proof. Since the left center of A is not isotropic.Then there exist $x, y \in LC(A)$ such that $f(x, y) \neq 0$.

Now we claim that there exists $e_1 \in LC(A)$ such that $f(e_1, e_1) \neq 0$.

If $f(x, x) \neq 0$, choose $e_1 = x$.

On the otherhand, if $f(x, x) = 0$ and $f(y, y) \neq 0$, we choose $e_1 = y$.

Otherwise, if $f(x, x) = 0$ and $f(y, y) = 0$, choose $e_1 = x + y$ since $f(x + y, x + y) = f(x, x) + f(x, y) + f(y, x) + f(y, y) = 0 + f(x, y) + f(y, x) + 0$. But f is symmetric then $f(x+y, x+y) = 2f(x,y) \neq 0$.

As a result, there exist $e_1 \in LC(A)$ such that $f(e_1, e_1) \neq 0$.

Let H be the subspace of $LC(A)$ generated by e_1.

We have to show that H is non-degenerate. Let $x \in H$ such that $f(x,y) = 0$, for all $y \in H$. Let $x = \lambda_1 e_1$ and $y = \lambda_2 e_1$, where $\lambda_1, \lambda_2 \in \mathbb{C}$. So $f(\lambda_1 e_1, \lambda_2 e_1) = 0$, for all $\lambda_2 \in \mathbb{C}$ since f is bilinear, we get: $\lambda_1 \lambda_2 f(e_1, e_1) = 0$, for all λ_2 but $f(e_1, e_1) \neq 0$. So $\lambda_1 = 0$ then $x = 0$.

Consequently, f is non-degenerate on H.

By Zorn's Lemma (4.0.18), we can pick H_1 to be a maximal non-degenerate subspace of $LC(A)$.

Let $B_1 = H_1^{\perp}$. So $B_1^{\perp} = H_1$.

Now we want to prove that B_1 is non-degenerate ideal.

We have f is a pseudo- Riemannian. Then $f(H_1, AB_1) = -f(AH_1, B_1) = 0$, where $AH_1 = 0$ thus $H_1 \subseteq LC(A)$.

Moreover, $f(B_1 A, H_1) = -f(A, B_1 H_1) = 0$, where $B_1 H_1 = 0$ because $H_1 \subseteq LC(A)$ and $B_1 \subset A$. Consequently, B_1 is an ideal of A.

Now, let $x \in H_1^{\perp}$ such that $f(x, H_1^{\perp}) = 0$ but $x \in H_1^{\perp}$ so by its definition, we have $f(x, H_1) = 0$. Since H_1 is non-degenerate then A can be written as $A = H_1 \oplus H_1^{\perp}$. Therefore, $f(x, A) = 0$ and $x \in A$. Thus $x = 0$. So f is non-degenerate on B_1.

Now because B_1 is non-degenerate then A can be written as $A = B_1 \oplus {B_1}^{\perp}$. Clearly $B_1 \subset A$ and $f(B_1, {B_1}^{\perp}) = 0$.

Assume that $LC(B_1^{\perp})$ is isotropic. Now we need to prove that $B_1^{\perp}$ is non-degenerate ideal.

We let $x \in B_1^{\perp}$ such that $f(x, B_1^{\perp}) = 0$. However, $x \in B_1^{\perp}$ this implies that $f(x, B_1) = 0$ and we have $A = B_1 \oplus B_1^{\perp}$. Then $f(x, A) = 0$ and $x \in A$. Because f is non-degenerate on A so $x = 0$.

As a consequence, we have f is non-degenerate on $B_1^{\perp}$.

In fact, $f(AB_1^{\perp}, B_1) = -f(B_1^{\perp}, AB_1) = 0$ since $AB_1 \subseteq B_1$ and $f(B_1^{\perp}, B_1) = 0$. Moreover, $f(B_1^{\perp}A, B_1) = -f(A, B_1^{\perp}B_1) = 0$ because $B_1^{\perp}B_1 \subseteq B_1 \cap B_1^{\perp} = 0$. Hence $B_1^{\perp}$ is an ideal.

To completely prove the theorem we want to show that $B_1B_1 = 0$.

We have $f(B_1B_1, B_1^{\perp}) = -f(B_1, B_1B_1^{\perp}) = 0$ but f is non-degenerate on $B_1^{\perp}$.

Thus $B_1B_1 = 0$. Moreover, $f(B_1B_1, B_1) = -f(B_1, B_1B_1) = 0$ so $B_1B_1 \subseteq B_1$.

Therefore, $f(B_1B_1, A) = 0$ and f is non-degenerate on A then $B_1B_1 = 0$.

Finally, we let $A_1 = B_1$ and $A_2 = B_1^{\perp}$ then $A = A_1 \oplus A_2$ and the theorem is proved.

Otherwise, if $LC(B_1^{\perp})$ is not isotropic, each time we get $B_i^{\perp} = B_{i+1} \oplus B_{i+1}^{\perp}$ where $1 \leq i \leq n-2$, so we continue this process by induction till we reach a certain $B_{n-1}^{\perp}$ such that $LC(B_{n-1}^{\perp})$ is isotropic so we stop because the algebra A is of a finite dimension.

We assume that $B_n = B_{n-1}^{\perp}$.

In other words, by theorem (4.0.19), there exist a sequence of non-degenerate subalgebras of A such that

$$A \supset B_1 \supset ... \supset B_n,$$

where B_i, $1 \leq i \leq n$ is non-degenerate ideal and the left center of B_n is isotropic.

Now we already have $A = B_1 \oplus B_1{}^{\perp}$ where B_1 and $B_1{}^{\perp}$ are non-degenerate ideals and $f(B_1, B_1{}^{\perp}) = 0$ is true for dimension 2. Suppose it is true for A of dimension up to $n-1$. Then by induction there exist $n-1$ non-degenerate ideals $B_2, ..., B_n$ such that $B_1{}^{\perp} = B_2 \oplus ... \oplus B_n$ and $f(B_i, B_j) = 0$, for $i \neq j$ where $2 \leq i, j \leq n$. As a result, $A = B_1 \oplus B_2 \oplus ... \oplus B_{n-1} \oplus B_n$.

Clearly, since $f(B_1, B_1^{\perp}) = 0$ then $f(B_1, B_i) = 0$, where $2 \leq i \leq n$. In general, $f(B_{i+1}, B_i) = 0$, because $B_{i+1} \subset B_i^{\perp}$ for $1 \leq i \leq n$.

Furthermore, since $f(B_iB_i, B_n) = -f(B_i, B_iB_n) = 0$ and f is non-degenerate on B_n. Therefore, $B_iB_i = 0$, for $1 \leq i \leq n-1$.
Now we let $A_1 = B_1 \oplus ...B_{n-1}$ and $A_2 = B_n$ then we're done.

□

Proposition 6.0.29. *Let (A, f) be decomposable pseudo-Riemannian algebra whose left center equals the center.If the left center is isotropic, then there exist non-degenerate ideals A_1 and A_2 such that $A = A_1 \oplus A_2$ is orthogonal.*

Proof. Since A is decomposable, we have $A = A_1 \oplus A_2$, where $f|A_i \times A_i$, $i = 1, 2$ are non-degenerate. Then $A = A_1 \oplus A_1^{\perp}$ and $A_1A_1^{\perp} = 0$.
Now we want to prove that $A_1^{\perp}$ is an ideal.

We let $x = x_1 + x_2$, where $x \in A_1^\perp$, $x_1 \in A_1$, $x_2 \in A_2$. Since both A_1 and A_2 are ideals, we have $f(yx_1, z) = f(y(x-x_2), z) = f(yx, z) - f(yx_2, z) = 0 - f(yx_2, z) = 0$, for any $y, z \in A_1$, since $yx_2 \in A_1A_2 = 0$.

As a result,

$$f(yx_1, z) = -f(x_1, yz) = f(yx_2, z) = -f(x_2, yz) = 0.$$

Since $f(yx_1, z) = 0$ and $f|A_1 \times A_1$ is non-degenerate. Therefore $yx_1 = 0$, for all $y \in A_1$. Thus $A_1x_1 = 0$. Moreover, it is obvious that $A_2x_1 = 0$.

Namely, $x_1 \in LC(A) = Z(A)$. Then $xy = (x_1 + x_2)y = 0$ for any $y \in A_1$, i.e., $A_1^\perp A_1 = 0$.

It follows that $A_1^\perp$ is an ideal since $f(AA_1^\perp, A_1) = -f(A_1^\perp, AA_1) = 0$ because A_1 is an ideal ($AA_1 \subseteq A_1$) and $f(A_1^\perp, A_1) = 0$. Furthermore, $f(A_1^\perp A, A_1) = -f(A, A_1^\perp A_1) = 0$. Likewise, $A_2^\perp$ is an ideal.

□

Theorem 6.0.30. *Let (A, f) be a pseudo-Riemannian algebra whose left center equals the center and whose left center is isotropic, and let*

$$A = A_1 \oplus A_2 \oplus ..... \oplus A_n,$$

$$A = A_1' \oplus A_2' \oplus ..... \oplus A_m'$$

be orthogonal decompositions of A. Here A_i, A_j', $1 \leq i \leq n$, $1 \leq j \leq m$, are indecomposable non-degenerate ideals of A. Then we have:

1. $n = m$.

2. *Changing the subscripts if necessary, we can get*

$$dimA_j = dimA'_j,$$

$$A_jA_j = A_jA'_j = A'_jA_j = A'_jA'_j,$$

$$A_jA'_k = A'_jA_k = 0, j \neq k.$$

3. *The projections* $\pi_i : A_i \longmapsto A'_i$ *are isomorphisms and preserve the bilinear form, so* $\pi = (\pi_1, ..., \pi_n)$ *is an isometry of* A*, that is, the decomposition is unique up to an isometry.*

Proof. Same proof as theorem (5.0.27).

□

Theorem 6.0.31. *Let* (A, f) *be a pseudo-Riemannian algebra whose left center equals the center and whose left center is not isotropic. If the decomposition* $A = A_1 \oplus A_2$ *is orthogonal such that* A_1 *and* A_2 *are non-degenerate,* $LC(A_1)$ *is isotropic and* $A_2 \subseteq LC(A)$*, then the decomposition is unique up to an isometry.*

Proof. Let $A = A'_1 \oplus A'_2$ be another such decomposition. Then we have

$$AA = A_1A_1 = A'_1A'_1 = A_1A'_1.$$

Now we want to prove that $LC(A_1) \subseteq LC(A_1)^{\perp} = A_1A_1 = A'_1A'_1$.

We let $x \in LC(A_1)$. Since the left center of A_1 is isotropic, this implies that for all $x, y \in LC(A_1)$ such that $f(x, y) = 0$. Moreover, we know that $LC(A_1)^{\perp} = \{x \in A_1 | \, f(x, y) = 0 \; \forall y \in LC(A_1)\}$ therefore $x \in LC(A_1)^{\perp}$. As a result, $LC(A_1) \subseteq LC(A_1)^{\perp}$. According to Proposition (3.0.14), we have $LC(A_1)^{\perp} = ((A_1A_1)^{\perp})^{\perp} = A_1A_1$. Consequently,

$$LC(A_1) \subseteq LC(A_1)^{\perp} = A_1A_1 = A_1'A_1'.$$

Now since $A_1A_1 = A_1'A_1'$ this implies that $LC(A_1)^{\perp} = LC(A_1')^{\perp}$. Namely

$$LC(A_1') = LC(A_1).$$

By Proposition (3.0.14), we have

$$dimA_1' = dimA_1'A_1' + dim(A_1'A_1')^{\perp}$$

and recall that $(A_1'A_1')^{\perp} = LC(A_1')^{\perp}$. Likewise for $dimA_1$.

As a result we get $dimA_1 = dimA_1'$ and then $dimA_2 = dimA_2'$.

Let $\{e_1, ..., e_k, ..., e_n, ..., e_{n+k}\}$ be a basis of A_1 such that $LC(A_1) = L(e_1, ..., e_k)$, $A_1A_1 = L(e_1, ..., e_n)$, and

$$f(e_i, e_j) = \delta_{ij}, \; k+1 \leq i, j \leq n,$$

$$f(e_i, e_{n+j}) = \delta_{ij}, \; 1 \leq i, j \leq k,$$

$$f(e_i, ej) = 0, \; 1 \leq i, j \leq k,$$

$$f(e_i, e_j) = 0, \; n+1 \leq i, j \leq n+k.$$

Now consider the projections

$$\pi_1 : A_1 \longmapsto A_1',$$

$$\pi_2 : A_2 \longmapsto A_2',$$

which are isomorphisms by theorem (6.0.30) . We have $\pi_1|_{A_1A_1} = id$ and $f(\pi_1(e_i), \pi_1(e_j)) = f(e_i, e_j)$ for $1 \le i \le n+k$ and $1 \le j \le n$.

Assume that $e_p = e_{p_3} + e_{p_4}$ and $e_q = e_{q_3} + e_{q_4}$ for $n+1 \le p \le n+k$, where $e_{p_3},\ e_{q_3} \in A_1'$ and $e_{p_4},\ e_{q_4} \in A_2'$. For $n+1 \le q \le n+k$, we have $0 = f(e_p, e_q) = f(e_{p_3} + e_{p_4}, e_{q_3} + e_{q_4})$ $= f(e_{p_3}, e_{q_3}) + f(e_{p_3}, e_{q_4}) + f(e_{p_4}, e_{q_3}) + f(e_{p4}, e_{q4})$ but $f(A_1', A_2') = 0$. Therefore

$$0 = f(e_p, e_q) = f(e_{p_3}, e_{q_3}) = f(e_{p_4}, e_{q_4}).$$

Let $b_{pq} = f(e_{p_4}, e_{q_4})$ for $p \neq q$, $2b_{pp} = f(e_{p_4}, e_{p_4})$ and $e'_{p_3} = e_{p_3} + \sum_{l=p}^{n+k} b_{pl}\, e_{l-n}$, it is easy to see that

$$f(e'_{p_3}, e'_{p_3}) = f(e_{p_3}, e_{p_3}) + 2b_{pp} = 0,\ n+1 \le p \le n+k;$$

$$f(e'_{p_3}, e'_{q_3}) = f(e_{p_3}, e_{q_3}) + b_{pq} = 0,\ n+1 \le p \le q \le n+k.$$

Define $\pi_1' : A_1 \longmapsto A_1'$ by

$$\pi_1'(e_j) = e_j,\ 1 \le j \le n;$$

$$\pi_1'(e_j) = e_{j_3},\ n+1 \le j \le n+k.$$

It is easy to check that π_1' is also an isomorphism from A_1 onto A_1' preserves the bilinear form. Then $\pi = (\pi_1', \pi_2)$ is an isometry of A.

□

By Theorems (6.0.30) and (6.0.31), we have:

Theorem 6.0.32. *Let (A, f) be a pseudo-Riemannian algebra whose left center equals the center. Then the orthogonal decomposition of A into indecomposable non-degenerate ideals is unique up to an isometry.*

If the algebra is anti-commutative, i.e.,

$$ab = -ba, \ for\ all\ a, b \in A,$$

then $LC(A) = Z(A)$ and

$$f(ab, c) = -f(b, ac) = f(b, ca) = f(a, bc), \ for\ all\ a, b, c \in A. \tag{6.1}$$

Lemma 6.0.33. *([4]) Let (A, f) be an anti-commutative pseudo-Riemannian algebra. If H is an ideal of A, then $H^{\perp}$ is an ideal of A. Furthermore, assume that H is non-degenerate, then $H^{\perp}$ is also non-degenerate and $A = H \oplus H^{\perp}$.*

Proof. Assume that H is an ideal of A, this means that $HA \subseteq H$ and $AH \subseteq H$. Since A is anticommutative so we have (6.1). Then $f(H^{\perp}A, H) = f(H^{\perp}, AH) = 0$, because $AH \subseteq H$ and $f(H^{\perp}, H) = 0$. Also we have f is pseudo-Riemannian then $f(AH^{\perp}, H) = -f(H^{\perp}, AH) = -f(H^{\perp}, H) = 0$. Consequently, $H^{\perp}$ is an ideal of A. Assume that H is non-degenerate. Moreover, A can be written as $A = A_1 \oplus A_2$. Now we want to prove that $H^{\perp}$ is also non-degenerate. We let $x \in H^{\perp}$, such that $f(x, H^{\perp}) = 0$. But $x \in H^{\perp}$, then $f(x, H) = 0$. Therefore $f(x, A) = 0$ and $x \in A$ but f is non-degenerate on A. Then $x = 0$. Consequently, $H^{\perp}$ is also non-degenerate.

□

It follows that:

Proposition 6.0.34. *Let (A, f) be an anti-commutative pseudo-Riemannian algebra. Then A is indecomposable if and only if A is irreducible.*

Thus, we have:

Theorem 6.0.35. *Let (A, f) be an anti-commutative pseudo-Riemannian algebra. Then the orthogonal decomposition of A into irreducible non-degenerate ideals is unique up to an isometry.*

Now we will introduce the definition of quadratic Lie algebra that will be used in the corollary.

Definition 6.0.36. *([10]) A quadratic Lie algebra $(\mathfrak{g}, f)$ is a vector space $\mathfrak{g}$ equipped with a non-degenerate symmetric bilinear form f and a Lie algebra structure on $\mathfrak{g}$ such that f is invariant (that means, $f([x, y], z) = f(x, [y, z])$, for all $x, y, z \in \mathfrak{g}$.)*

By Theorem (6.0.35) and the identity (6.1), we have the following result on the uniqueness of the decomposition of quadratic Lie algebras.

Corollary 6.0.37. *([11]) Let $\mathfrak{g}$ be a quadratic Lie algebra. Then the orthogonal decomposition of $\mathfrak{g}$ into irreducible non-degenerate ideals is unique up to an isometry.*

Bibliography

[1] Z. Chen, K. Liang and F. Zhu, *Algebras with pseudo-riemanniens bilinear forms,* J. Korean Math. Soc. **48** (2011), no. 1, 1-12.

[2] P. Imani, *Introduction To Lie algebras,* Notes and Handouts, University of Hamburg. 1-7.

[3] A. Aubert and A. Medina, *Groupes de Lie pseudo-riemanniens plats,* Tohoku Math. J. (2) **55** (2003), no. 4, 487-506.

[4] M. Bordemann, *Nondegenerate invariant bilinear forms on nonassociative algebras,* Acta Math. Univ. Comenian. (N.S.) **66** (1997), no. 2, 151-201.

[5] M. Boucetta, *Poisson manifolds with compatible pseudo-metric and pseudo-Riemannian Lie algebras,* Dierential Geom. Appl. **20** (2004), no. 3, 279-291.

[6] Z. Chen and F. Zhu, *Bilinear forms on fermionic Novikov algebras,* J. Phys. A **40** (2007), no. 18, 4729-4738.

[7] G. Favre and L. J. Santharoubane, *Symmetric, invariant, nondegenerate bilinear form on a Lie algebra,* J. Algebra **105** (1987), no. 2, 451-464 .

[8] J. Milnor, *Curvatures of left invariant metrics on Lie groups,* Advances in Math. **21** (1976), no. 3, 293-329.

[9] A. A. Sagle, *Nonassociative algebras and Lagrangian mechanics on homogeneous spaces,* Algebras Groups Geom. **2** (1985), no. 4, 478-494 .

[10] M.T.Duong , *A new invariant of quadratic Lie algebras and quadratic Lie superalgebras,* General Mathematics. University of Bourgogne, (**2011**), 1-145.

[11] F. Zhu and L. Zhu, *The uniqueness of the decomposition quadratic Lie algebras,* Comm.Algebra **29** (2001), no. 11, 5145-5154.