A GENERALIZED ITERATIVE METHOD TO COMPUTE THE INVERSE OF AN INVERTIBLE MATRIX

Marie-Reine Azar

Thesis Advisor: Dr. Ziad Rashed

THESIS

Submitted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics in the Department of Mathematics and Statistics in the Faculty of Natural and Applied Sciences of Notre Dame University-Louaize
Lebanon
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Approved by:

Dr. Ziad Rashed

Dr. George Eid

Dr. Bassem Ghalayini

Contents

Acknowledgement

"All truths are easy to understand once they are discovered; the point is to discover them." Galileo Galilei - *Italian astronomer & physicist (1564 - 1642)*

I have been working hard on accomplishing this project during the course of the year. I was able to do it through diligent effort and with the help of many.

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I thank my family for their big support in this year of accomplishments.
One big salute goes to my class mates for this wonderful academic year.

Thank you all

Abstract

The origin of mathematical matrices lies with the study of systems of simultaneous linear equations. Today, they are used not simply for solving systems of simultaneous linear equations, but also for describing the quantum mechanics of atomic structure, and even for designing computer game graphics.

Matrices are very useful due to the fact that they can be easily manipulated.

We use the notation A^{-1} to denote the inverse of a matrix A.

One of the major uses of inverses is to solve a system of linear equations.

You can write a system in matrix form as $AX = B$, then $X = A^{-1}B$.

Inverses are also used in communication through coded messages. The use of coding has become particularly significant in recent years (due to the explosion of internet for example). One way to code a message is to use matrices and their inverses.

Indeed, consider a fixed invertible matrix A. Convert the message into a matrix B such that AB is possible to perform.

Send the message generated by AB.

At the recipient end, they will need to know A^{-1} in order to decode the message sent.

Indeed, we have $A^{-1}(AB) = B$, which is the original message.

There are two classes of methods for finding the inverse matrices.

Direct methods; a finite number of arithmetic operations leads to an exact solution. Examples of such direct methods include Gauss elimination, Gauss-Jordan elimination, the matrix inverse method and LU factorization.

Methods of the second type are called *Iterative methods*. Iterative methods start with an arbitrary first approximation to the unknown solution. These methods are used for finding the inverse matrices of large systems of equations.

In this thesis, we will work on Matrix, Matrix norm, Norms and Matrix Inversion. We did some examples on how to find the inverse of a matrix using direct method.

We recalled the definition of order of convergence throughout an example.

Then in chapter 2, we showed that the following iteration method

$$X_{n+1} = \sum_{i=0}^{p} X_n(I - AX_n)^i$$

converges to A^{-1} under the assumption $\|I - AX_0\| < 1$. We also showed that the order of convergence is p+1. Then we found an upper bound on the norm of difference between m's iteration and the exact value of the Matrix inverse.

This generalizes the results presented in [15], where the author only considered the case $p = 1$.

First, we dealt with case $p = 2$, then the case $p = 3$. We also illustrate with a simple numerical analysis how the iteration works. Afterwards, we generalize the results presented in [15] for arbitrary p.

Chapter 1

The Basic Elements for Matrices and their Inverse; Order of Convergence

1.1 Introduction

In this section, we will introduce some topics that will be used later with the aim of helping us in explaining the iterative method when finding the inverse of a matrix. The material may be found in any numerical analysis textbook such as [1],[2],[3],[4],[5].

1.1.1 Matrix

Let **A** = $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ be a matrix with ***m*** rows and ***n*** columns.

It is called an $m \times n$ matrix.

If *m=n* then A is said to be an **$n \times n$ matrix**, an ***n*-square matrix** or **a square matrix of order *n*.**

Square matrices play a major role in our topic since some of them will be invertible.

Addition:

The **sum A + B** of 2 $m \times n$ matrices A and B is calculated entry wise:

$(A + B)_{ij} = A_{ij} + B_{ij}$ where $1 \le i \le m$

$1 \le j \le n$

Scalar Multiplication:

$(c.\ A)_{ij} = c.\ A_{ij}$

Transpose:

We simply mean that the rows of A become its columns: $(A^T)_{ij} = A_{ji}$
It satisfies the following properties:

- $(c\,A)^T = c\,(A^T)$
- $(A + B)^T = A^T + B^T$
- $(A^T)^T = A$
- $(A\,B)^T = B^T A^T$

Remarks:

1. Matrix multiplication is not commutative.
2. In order to multiply a matrix A by a matrix B, the number of columns of A should be equal to the number of rows of B.
3. Identity Matrix: I_n of size n is the $n \times n$ matrix in which all the elements on the main diagonal are equal to 1 and all other elements are equal to 0.

   Example: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
4. 2 matrices are equal if they have same size and if they agree entry by entry.

Square matrix:

It is a matrix with the same number of rows and columns. **Ex:** $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$

A square matrix A is called invertible if ∃ a matrix B such that $AB = I_n$ if and only if $BA = I_n$.
If B exists then it is unique and is called the inverse matrix of A denoted A^{-1}.

Trace of a matrix:

The trace of a square matrix A is the sum of its diagonal entries denoted by

$$\mathrm{tr}\,(A) = \sum_{i=1}^{n} a_{ii}$$

We have:

- $\mathrm{tr}\,(AB) = \mathrm{tr}\,(BA)$
- $\mathrm{tr}\,(A) = \mathrm{tr}\,(A^T)$
- $\mathrm{tr}\,(A+B) = \mathrm{tr}\,(A) + \mathrm{tr}\,(B)$
- $\mathrm{tr}\,(cA) = c\,\mathrm{tr}\,(A)$

We have: **- Diagonal matrix:** $\begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$

- Lower triangular matrix: $\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$

- Upper triangular matrix: $\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

- If $A = A^T \Rightarrow$ A is a symmetric matrix.

1.1.2 Matrix norm

A distance between matrices A and B is the norm between matrices $\|A - B\|$
A norm on the set of all $n \times n$ matrices is a function $\|.\|$ from this set to R satisfying these properties:

i. $\|A\| \geq 0$
ii. $\|A\| = 0$ iff $A = 0_{n\times n}$
iii. $\|\alpha A\| = |\alpha| \, \|A\|$
iv. $\|A + B\| \leq \|A\| + \|B\|$
v. $\|AB\| \leq \|A\| \, \|B\|$

Examples:

1) $\|A\|_1 = \max_{k=1\ldots n} \sum_{j=1}^{n} |a_{jk}|$

2) $\|A\|_\infty = \max_{j=1\ldots n} \sum_{k=1}^{n} |a_{jk}|$

3) $\|A\|_2 \leq \left(\sum_{j,k=1}^{n} |a_{jk}|^2\right)^{1/2}$

Remarks:

1) Check whether $\|A\|= \sum_{j=1}^{n}\sum_{i=1}^{n}|a_{ij}|$ is a norm:

We have: **i)** $\|A\| \geq 0$ **since** $|a_{ij}| \geq 0 \Rightarrow \sum_{j=1}^{n}\sum_{i=1}^{n}|a_{ij}| \geq 0$

ii) $\|A\| = 0$ **iff** $\sum_{j=1}^{n}\sum_{i=1}^{n}|a_{ij}| = 0$ **iff** $a_{ij} = 0\ \forall$ **i=1...n** $\Rightarrow$ **A = 0**
j=1...n

iii) $\|\alpha A\| = \sum_{j=1}^{n}\sum_{i=1}^{n}|\alpha a_{ij}| = |\alpha| \sum_{j=1}^{n}\sum_{i=1}^{n}|a_{ij}| = |\alpha|\|A\|$

iv) $\|A + B\| = \sum_{j=1}^{n}\sum_{i=1}^{n}|a_{ij} + b_{ij}| \leq \sum_{j=1}^{n}\sum_{i=1}^{n}(|a_{ij}| + |b_{ij}|)$
$= \sum_{j=1}^{n}\sum_{i=1}^{n}|a_{ij}| + \sum_{j=1}^{n}\sum_{i=1}^{n}|b_{ij}|$
$\leq \|A\| + \|B\|$

v) $\|AB\| \leq \|A\|\|B\|$
<u>Ex:</u> Let **A** = $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and **B** = $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$\|A\| = \sum_{j=1}^{n}\sum_{i=1}^{n}|a_{ij}| = 4$

$\|B\| = \sum_{j=1}^{n}\sum_{i=1}^{n}|b_{ij}| = 4$

Now **AB** = $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ and $\|AB\| = 8$

Since **8 ≤ 4×4 = 16** $\Rightarrow \|AB\| \leq \|A\|\|B\|$

2) Check whether $\|A\| = \max_{1\leq i,j\leq n}|a_{ij}|$ is a norm:

We have: **i)** $\|A\| = 0$ **iff** $\max_{1\leq i,j\leq n}|a_{ij}| = 0$ **iff** $a_{ij} = 0$ for **1≤ i, j ≤n iff A = 0**

ii) $\|A\| \geq 0$ **iff** $\max_{1\leq i,j\leq n}|a_{ij}| \geq 0$

iii) $\|\alpha A\| = \max_{1\leq i,j\leq n}|\alpha a_{ij}| = |\alpha| \max_{1\leq i,j\leq n}|a_{ij}| = |\alpha|\|A\|$

iv) $\|A+B\| = \max_{1\le i,j\le n}|a_{ij} + b_{ij}|$

$$\le \max_{1\le i,j\le n}(|a_{ij}| + |b_{ij}|)$$

$$= \max_{1\le i,j\le n}|a_{ij}| + \max_{1\le i,j\le n}|b_{ij}|$$

$$\le \|A\| + \|B\|$$

v) $\|AB\| \le \|A\|\|B\|$?

We take a counter example:

Let $A = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$; $B = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$; $AB = \begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix}$

Now $\|A\| = \max |a_{ij}| = 1$

$\|B\| = 1$

$\|AB\| = 2$

Then $\|AB\| \ge \|A\|\|B\|$

1.1.3 Norms

Let X be a complex (or real) linear space (vector space)

A function $\|.\|: X \rightarrow \mathbf{R}$ satisfying:

a) $\|x\| \geq 0$ **(positivity)**
b) $\|x\| = 0$ iff $x = 0$ **(definiteness)**
c) $\|\alpha x\| = |\alpha|\|x\|$ **(homogeneity)**
d) $\|x + y\| \leq \|x\| + \|y\|$ **(triangle inequality)**

For all $x \epsilon X$, $y \epsilon Y$ and all $\alpha \epsilon \mathbb{C}$ (or R) is called a norm on X.

Examples:

Norms on R^n and $\mathbb{C}^n$ are given by:

l_1**-norm:** $\|x\|_1 = \sum_{j=1}^{n} |x_j|$
l_2**-norm (Euclidean norm):** $\|x\|_2 = (\sum_{j=1}^{n} |x_j|^2)^{½}$
l_∞ **-norm (maximum norm):** $\|x\|_\infty = \max_{j=1 \ldots n} |x_j|$

These 3 norms are special cases of the l_p-norm:
$(*)$ $\|x\|_p = (\sum_{j=1}^{n} |x_j|^p)^{1/P}$ defined for $\mathbf{p} \geq \mathbf{1}$
The l_∞**-norm** is the limiting case of $(*)$ as $\mathbf{p} \rightarrow \infty$

1.1.4 Matrix inversion

Let **A** be an **n×n** matrix. **A** is said to be invertible or non-singular if ∃ an **n×n** matrix **B** such that;

$$AB = BA = I_n$$

In this case we use the notation $B=A^{-1}$ with the understanding that A^{-1} is just a notation, it is not $\frac{1}{A}$

Example:

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the matrix $B = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ satisfies $AB = BA = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

In this case $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ provided $ad - bc \neq 0$ which means $\det A \neq 0$

Accordingly, $A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Example:

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $A^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$

By **det A** we simply mean a scalar associated with a square matrix **A.**

This clearly implies that the determinant is very useful in determining whether a matrix is invertible or not.

In particular, if $A = \begin{bmatrix} 1 & 2 \\ 4 & -2 \end{bmatrix}$, $\det A = -2 - 8 = -10 \neq 0$, then A^{-1} exists and it is:

$$A^{-1} = \frac{1}{-10}\begin{bmatrix} -2 & -2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{10} & \frac{2}{10} \\ \frac{4}{10} & \frac{-1}{10} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{-1}{10} \end{bmatrix}$$

Basic properties of matrix inversion:

Let **A** be an **n×n** matrix then:

a) $(A^{-1})^{-1} = A$

b) $(kA)^{-1} = \frac{1}{k} A^{-1}$; $k \neq 0$

c) $(A^T)^{-1} = (A^{-1})^T$

d) $(AB)^{-1} = B^{-1}A^{-1}$

e) $(A^p)^{-1} = (A^{-1})^p$

Inverse matrices can be very useful for solving matrix equations.
But, given a matrix, how do you invert it?
How do you find the inverse?
The technique for inverting matrices is kind of clever.
For a given matrix **A** and its inverse A^{-1}, we know we have $A^{-1}A = I$.
We're going to use the identity matrix **I** in the process for inverting a matrix.

Example 1

- Find the inverse of the following matrix.

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

First, I write down the entries of the matrix **A**, but I write them in a double-wide matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & & & \\ 1 & 4 & 3 & & & \\ 1 & 3 & 4 & & & \end{array}\right]$$

In the other half of the double-wide, I write the identity matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

Now I'll do matrix row operations to convert the left-hand side of the double-wide into the identity. (As always with row operations, there is no one "right" way to do this. What follows are just the steps that happened to occur to me. Your calculations could easily look quite different.)

$$\begin{bmatrix} 1 & 3 & 3 & \vdots & 1 & 0 & 0 \\ 1 & 4 & 3 & \vdots & 0 & 1 & 0 \\ 1 & 3 & 4 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[-R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & 3 & 3 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 0 & \vdots & -1 & 1 & 0 \\ 0 & 0 & 1 & \vdots & -1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-3R_2+R_1} \begin{bmatrix} 1 & 0 & 3 & \vdots & 4 & -3 & 0 \\ 0 & 1 & 0 & \vdots & -1 & 1 & 0 \\ 0 & 0 & 1 & \vdots & -1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-3R_3+R_1} \begin{bmatrix} 1 & 0 & 0 & \vdots & 7 & -3 & -3 \\ 0 & 1 & 0 & \vdots & -1 & 1 & 0 \\ 0 & 0 & 1 & \vdots & -1 & 0 & 1 \end{bmatrix}$$

Now that the left-hand side of the double-wide contains the identity, the right-hand side contains the inverse. That is, the inverse matrix is the following:

$$\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Note that we can confirm that this matrix is the inverse of A by multiplying the two matrices and confirming that we get the identity:

$$\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 7*1-3*1-3*1 & 7*3-3*4-3*3 & 7*3-3*3-3*4 \\ -1*1+1*1+0*1 & -1*3+1*4+0*3 & -1*3+1*3+0*4 \\ -1*1+0*1+1*1 & -1*3+0*4+1*3 & -1*3+0*3+1*4 \end{bmatrix}$$

$$= \begin{bmatrix} 7-3-3 & 21-12-9 & 21-9-12 \\ -1+1+0 & -3+4+0 & -3+3+0 \\ -1+0+1 & -3+0+3 & -3+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example2:

Let $\mathbf{A}\vec{X} = \vec{b}$ be a linear system of **3** equations in **3** unknowns, **if A^{-1} exists (det $A \neq 0$)** then

$\mathbf{A}\vec{X}=\vec{b}$ has a unique solution given by $\vec{X}=\mathbf{A}^{-1}\vec{b}$.

$$\begin{aligned} X_1 - X_2 + 2X_3 &= 2 \\ X_1 + X_3 &= 2 \qquad (*) \\ 3X_1 + X_2 - X_3 &= 3 \end{aligned}$$

To get A^{-1}; we should pass from $[\mathbf{A} \vdots \mathbf{I}] \rightarrow [\mathbf{I} \vdots \mathbf{A}^{-1}]$ using **Gauss-Jordan elimination.**

$$\begin{bmatrix} 1 & -1 & 2 & \vdots & 1 & 0 & 0 \\ 1 & 0 & 1 & \vdots & 0 & 1 & 0 \\ 3 & 1 & -1 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 2 & \vdots & 1 & 0 & 0 \\ 0 & 1 & -1 & \vdots & -1 & 1 & 0 \\ 0 & 4 & -7 & \vdots & -3 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 1 & \vdots & 0 & 1 & 0 \\ 0 & 1 & -1 & \vdots & -1 & 1 & 0 \\ 0 & 0 & -3 & \vdots & 1 & -4 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 1 & \vdots & 0 & 1 & 0 \\ 0 & 1 & -1 & \vdots & -1 & 1 & 0 \\ 0 & 0 & 1 & \vdots & -\frac{1}{3} & \frac{4}{3} & -\frac{1}{3} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & \vdots & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \vdots & -\frac{4}{3} & \frac{7}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & \vdots & -\frac{1}{3} & \frac{4}{3} & -\frac{1}{3} \end{bmatrix}$$

The solution of **(*)** is $\mathbf{X} = \mathbf{A}^{-1}\mathbf{b} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{4}{3} & \frac{7}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{4}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

We can use another way to find the inverse of a matrix:

Adjoint method:

$A^{-1} = \frac{1}{\det A}$ (adjoint A) or $A^{-1} = \frac{1}{\det A}$ (cofactor matrix of A)^T

Example:

In **(*)**, we have **A** = $\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 3 & 1 & -1 \end{bmatrix}$ **det A = -3.**

The cofactor matrix for **A** is $\begin{bmatrix} -1 & 4 & 1 \\ 1 & -7 & -4 \\ -1 & 1 & 1 \end{bmatrix}$, so the adjoint is $\begin{bmatrix} -1 & 1 & -1 \\ 4 & -7 & 1 \\ 1 & -4 & 1 \end{bmatrix}$.

Since **det A = -3,** we get $A^{-1} = \frac{1}{-3}\begin{bmatrix} -1 & 1 & -1 \\ 4 & -7 & 1 \\ 1 & -4 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{4}{3} & \frac{7}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{4}{3} & -\frac{1}{3} \end{bmatrix}$

1.2 Order of convergence

A convergent sequence **(x_v)** from a normed space with limit **x** is said to be convergent of order **p ≥ 1** if there exists a constant **C>0** such that:
$\|x_{v+1} - x\| \leq C\|x_v - x\|^p$, **v=1, 2 ...**

Remark:
In general, a sequence with a high order of convergence converges more rapidly than a sequence with a lower order.
The constant **"C"** affects the speed of convergence but is not as important as the order.

Example:

We consider the sequences $\{p_n\}$ and $\{\widetilde{p_n}\}$ and we suppose that $\{p_n\}$ and $\{\widetilde{p_n}\}$ converge to zero and that $\{p_n\}$ is linear with $\lim_{n\to\infty} \frac{|p_{n+1}|}{|p_n|} = 0.5$ and $\{\widetilde{p_n}\}$ is quadratic with the same asymptotic error constant, $\lim_{n\to\infty} \frac{|\widetilde{p}_{n+1}|}{|\widetilde{p}_n|^2} = 0.5$

Now for the linearly convergent scheme, this assumption means that

$$|p_n - 0| = |p_n| \approx 0.5\,|p_{n-1}| \approx (0.5)^2\,|p_{n-2}| \ldots\ldots \approx (0.5)^n\,|p_0|$$

Whereas the quadratically convergent procedure has:

$$|\widetilde{p}_n - 0| = |\widetilde{p}_n| \approx 0.5\,|\widetilde{p}_{n-1}|^2 \approx (0.5)\,[0.5\,|\widetilde{p}_{n-2}|^2]^2 = (0.5)^3\,|\widetilde{p}_{n-2}|^4$$

$$\approx (0.5)^3\,[(0.5)\,|\widetilde{p}_{n-3}|^2\,]^4 = (0.5)^7\,|\widetilde{p}_{n-3}|^8 \approx \ldots\ldots\ldots \approx (0.5)^{2^n-1}|\widetilde{p}_0|^{2^n}$$

• The table below illustrates the relative speed of convergence of the sequences to zero when

$|p_0| = |\widetilde{p_0}| = 1$

N	Linear convergence Sequence $\{p_n\}$ $(0.5)^n$	Quadratic convergence Sequence $\{\widetilde{p_n}\}$ $(0.5)^{2^n-1}$
1	5.0000×10^{-1}	5.0000×10^{-1}
2	2.5000×10^{-1}	1.2500×10^{-1}
3	1.2500×10^{-1}	7.8125×10^{-3}
4	6.2500×10^{-2}	3.0518×10^{-5}
5	3.1250×10^{-2}	4.6566×10^{-10}
6	1.5625×10^{-2}	1.0842×10^{-19}
7	7.8125×10^{-3}	5.8775×10^{-39}

The quadratically convergent sequence is within $\mathbf{10^{-38}}$ of zero by the seventh term.
At least **126 terms** are needed to ensure this accuracy for the linearly convergent sequence.
Quadratically convergent sequences generally converge much more quickly than those that converge only linearly, but many techniques that generate convergent sequences do so only linearly.

Chapter 2

A Variation of an Iterative Method to compute the Inverse of an invertible Matrix

Iterative methods occur in many topics in numerical analysis such as solving systems of linear and nonlinear equations [6]—[14].

We consider the following iteration method

$$X_{n+1} = X_n + X_n (I - AX_n) + X_n (I - AX_n)^2 + \dots + X_n (I - AX_n)^p \quad n = 0, 1 \dots$$

Our aim is to show that it **converges to A^{-1}** under the assumption $\|I - AX_0\| < 1$ and to show that the order of convergence is **p** + 1

2.1 Iteration Method where the Order of Convergence is 3

We consider the following iteration method

$$X_{n+1} = 3X_n (I - AX_n) + X_n AX_n AX_n, \quad n = 0, 1 \dots$$

We will show that it converges to A^{-1} under the assumption $\|I - AX_0\| < 1$

We also show that the **order of convergence is 3**. This result can be regarded as a variation of the following iteration

$$X_{n+1} = X_n (2I - AX_n), \quad n = 0, 1 \dots$$

which was proved to **converge quadratically to A^{-1}** under the assumption $\|I - AX_0\| < 1$.

2.1.1 Introduction

Most topics in numerical analysis (linear and nonlinear systems, eigenvalues and eigenvectors, initial and boundary-value problems...) involve exact and iterative methods.

The most well-known exact method for computing the inverse of an invertible matrix A is to form the augmented matrix (A|I) where I is the identity matrix of same size as A and then reduce it by applying elementary row operations to $(I|A^{-1})$.

This method appears in every linear algebra text book. However, little is known about iterative methods.

The following iteration method

$$X_{n+1} = X_n (2I - AX_n), n = 0, 1 ...$$

was introduced and proved to **converge quadratically to A^{-1}** under the assumption $\|I - AX_0\| < 1$. Our aim in this note is to accelerate the rate of convergence, of course as usual, at the cost of computational complexity.

At the end, we provide some insights into possible directions for future work.

2.1.2 Main Result

We consider the following iteration method

$$X_{n+1} = 3X_n (I - AX_n) + X_nAX_nAX_n \quad n = 0, 1 ...$$

We will show that it **converges to A^{-1}** under the assumption $\|I - AX_0\| < 1$.

We also show that the order of convergence is 3. Note that the iteration could be written in the following equivalent form which is more convenient for our purpose.

$$X_{n+1} = X_nI + X_n (I - AX_n) + X_n (I - AX_n)^2, n = 0, 1 ...$$

First, we will prove the following lemmas.

- **Lemma 1**

$$(I - AX_1) = (I - AX_0)^3$$

Proof:

$$X_1 = X_0 + X_0(I - AX_0) + X_0(I - AX_0)^2$$

$$= 3X_0 - 3X_0AX_0 + X_0(AX_0)^2$$

Hence,

$$AX_1 = 3AX_0 - 3(AX_0)^2 + (AX_0)^3$$

Therefore

$$I - AX_1 = I - 3AX_0 + 3(AX_0)^2 - (AX_0)^3 = (I - AX_0)^3$$

- **Lemma 2**

$$(\mathbf{I} - \mathbf{AX_n}) = (\mathbf{I} - \mathbf{AX_0})^{3^n}$$

Proof:

By mathematical induction; for n = 1, it's true by the previous lemma. Suppose the result is true for n. Let us prove it for n+1.

$$\begin{aligned} I - AX_{n+1} &= I - AX_n - AX_n(I - AX_n) - AX_n(I - AX_n)^2 \\ &= I - 3AX_n + 3(AX_n)^2 - (AX_n)^3 \\ &= (I - AX_n)^3 \\ &= [(I - AX_0)^{3^n}]^3 \\ &= (I - AX_0)^{3^{n+1}} \end{aligned}$$

❖ **Lemma 3**

$$\mathbf{X_n = X_0 + X_0(I - AX_0) + X_0(I - AX_0)^2 + \dots + X_0(I - AX_0)^{3^n - 1}}$$

Proof:

By mathematical induction; for n = 1, $X_1 = X_0 + X_0(I - AX_0) + X_0(I - AX_0)^2$.
Suppose the result is true for n.
Let us prove it for n+1.

$$\begin{aligned} X_{n+1} &= X_n + X_n (I - AX_n) + X_n (I - AX_n)^2 \\ &= X_n + X_n(I - AX_0)^{3^n} + X_n [(I - AX_0)^{3^n}]^2 \\ &= X_0 + X_0(I - AX_0) + X_0(I - AX_0)^2 + \dots + X_0(I - AX_0)^{3^n - 1} \\ &\quad + X_0(I - AX_0)^{3^n} + X_0(I - AX_0)^{3^n + 1} + \dots + X_0(I - AX_0)^{2.3^n - 1} \\ &\quad + X_0(I - AX_0)^{2.3^n} + X_0(I - AX_0)^{2.3^n + 1} + \dots + X_0(I - AX_0)^{2.3^n + 3^n - 1} \end{aligned}$$

Therefore

$$X_{n+1} = X_0 + X_0(I - AX_0) + X_0(I - AX_0)^2 + \dots + X_0(I - AX_0)^{3^{n+1} - 1}$$

We further need the following elementary fact which could be found in any linear algebra textbook.

❖ **Lemma 4**

If **C** is a square matrix with norm strictly less than 1, then the series $I+C+C^2+\dots$ is convergent and its inverse is given by $(I - C)^{-1}$.

Our **first main result** is the following.

Theorem 1

The following iteration method

$$\mathbf{X_{n+1} = X_n + X_n(I - AX_n) + X_n(I - AX_n)^2 , \; n = 0, 1 \dots}$$

Converges to A^{-1} under the assumption $\|I - AX_0\| < 1$

Proof:

By lemma 3, we have

$$X_{n+1} = X_0 + X_0(I - AX_0) + X_0(I - AX_0)^2 + \dots + X_0(I - AX_0)^{3^{n+1}-1}$$

Therefore

$$X_{n+1} = X_0 \left(I + (I - AX_0) + (I - AX_0)^2 + \dots + (I - AX_0)^{3^{n+1}-1}\right)$$

Let $C = I - AX_0$. Our assumption implies that $\|C\| < 1$.
Thus, by the previous lemma, as $n \to \infty$

$$X_{n+1} \to X_0(I - (I - AX_0))^{-1} = X_0(AX_0)^{-1} = X_0X_0^{-1}A^{-1} = A^{-1}$$

Our next main result is that the convergence is of order 3.
We first need the following lemma.

❖ **Lemma 5**

$$\mathbf{I - AX_{n+1} = (I - AX_n)^3}$$

Proof:

$$AX_{n+1} = AX_n + AX_n (I - AX_n) + AX_n (I - AX_n)^2$$
$$= 3AX_n - 3(AX_n)^2 + (AX_n)^3$$

Therefore

$$I - AX_{n+1} = I - 3AX_n + 3(AX_n)^2 - (AX_n)^3 = (I - AX_n)^3$$

Theorem 2

The order of convergence is 3

Proof:

$$I - AX_{n+1} = (I - AX_n)^3$$

Hence

$$A\,(A^{-1} - X_{n+1}) = (A\,(A^{-1} - X_n))^3$$

Therefore

$$\| A^{-1} - X_{n+1} \| \leq \|A\|^2 \, \| A^{-1} - X_n \|^3$$

Numerical Example:

Consider the matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 3 & 0 & 4 \end{pmatrix}$. The exact inverse is $A^{-1} = \begin{pmatrix} 4 & 0 & -1 \\ 1.5 & 0.5 & -0.5 \\ -3 & 0 & 1 \end{pmatrix}$.

Let the initial guess be $X_0 = \begin{pmatrix} 2.3 & -0.5 & -1.5 \\ 1 & 0.2 & -1 \\ -2 & 0.2 & 1.6 \end{pmatrix}$ such that $\|\boldsymbol{I} - \boldsymbol{A}\boldsymbol{X}_0\| < 1$

Using $\mathbf{X_{n+1} = X_n I + X_n(I - AX_n) + X_n(I - AX_n)^2}$ **p=2**

Solving the equation in MATLAB, we get the following results:

For n = 0: $X_1 = \begin{pmatrix} 3.2440 & 0.6520 & -1.6280 \\ 1.2880 & 0.2480 & -1.0000 \\ -2.6640 & 0.3360 & 1.6880 \end{pmatrix}$

For n = 1: $X_2 = \begin{pmatrix} 3.8920 & -0.1881 & -1.5073 \\ 1.5628 & 0.4744 & -0.9147 \\ -3.0835 & 0.0392 & 1.5786 \end{pmatrix}$

For n = 2: $X_3 = \begin{pmatrix} 4.1817 & 0.0934 & -1.3191 \\ 1.6661 & 0.5888 & -0.7734 \\ -3.2313 & -0.1236 & 1.3811 \end{pmatrix}$

For n = 3: $X_4 = \begin{pmatrix} 4.0712 & 0.0397 & -1.1088 \\ 1.5613 & 0.5342 & -0.5937 \\ -3.0855 & -0.0476 & 1.1306 \end{pmatrix}$

For n = 4: $X_5 = \begin{pmatrix} 4.0029 & 0.0016 & -1.0045 \\ 1.5025 & 0.5014 & -0.5038 \\ -3.0035 & -0.0020 & 1.0053 \end{pmatrix}$

For n = 5: $X_6 = \begin{pmatrix} 4.0000 & 0.0000 & -1.0000 \\ 1.5000 & 0.5000 & -0.5000 \\ 3.0000 & -0.0000 & 1.0000 \end{pmatrix}$

Then for p = 2, after 5 iterations the result converges to A^{-1}

On the other hand, for **p=3**:

$$\mathbf{X_{n+1} = X_nI + X_n(I - AX_n) + X_n(I - AX_n)^2 + X_n(I - AX_n)^3}$$

For n = 0: $X_1 = \begin{pmatrix} 2.7800 & -0.9272 & -0.6200 \\ 0.8016 & -0.0144 & -0.1296 \\ -2.0080 & 0.7168 & 0.4816 \end{pmatrix}$

For n = 1: $X_2 = \begin{pmatrix} 3.6072 & -0.2622 & -0.6274 \\ 1.2145 & 0.3203 & -0.1716 \\ -2.6006 & 0.2516 & 0.5425 \end{pmatrix}$

For n = 2: $X_3 = \begin{pmatrix} 3.9004 & -0.0556 & -0.8479 \\ 1.4142 & 0.4522 & -0.3690 \\ -2.8804 & 0.0667 & 0.8174 \end{pmatrix}$

For n = 3: $X_4 = \begin{pmatrix} 3.9977 & -0.0013 & -0.9966 \\ 1.4981 & 0.4989 & -0.4970 \\ -2.9973 & 0.0015 & 0.9959 \end{pmatrix}$

For n = 4: $X_5 = \begin{pmatrix} 4.0000 & -0.0000 & -1.0000 \\ 1.5000 & 0.5000 & -0.5000 \\ -3.0000 & 0.0000 & 1.0000 \end{pmatrix}$

Then for p=3, after 4 iterations the result converges to A^{-1}

Using our output results from MATLAB, we note that for p=3; we reach our desired result faster using less iteration.

2.2 Iteration Method Where the Order of Convergence is 4

Since our aim is to accelerate the rate of convergence, we will consider another iteration method and we will show that it **converges to A^{-1}** under the assumption

$\|I - AX_0\| < 1.$

We will also show that **the order of convergence is 4.**

2.2.1 Main Result

We consider the following iteration method

$$X_{n+1} = X_n I + X_n (I - AX_n) + X_n (I - AX_n)^2 + X_n (I - AX_n)^3, \; n = 0, 1 \ldots$$

We will show that it **converges to A^{-1}** under the assumption $\|I - AX_0\| < 1.$ We also show that the order of convergence is 4.

First, we will prove the following lemmas.

- **Lemma 6**

$$(I - AX_1) = (I - AX_0)^4$$

<u>Proof:</u>

$$X_1 = X_0 + X_0(I - AX_0) + X_0(I - AX_0)^2 + X_0 (I - AX_0)^3$$

$$= 4X_0 - 6X_0AX_0 + 4X_0 (AX_0)^2 - X_0 (AX_0)^3$$

Hence,

$$AX_1 = 4AX_0 - 6(AX_0)^2 + 4(AX_0)^3 - (AX_0)^4$$

Therefore

$$I - AX_1 = I - 4AX_0 + 6(AX_0)^2 - 4(AX_0)^3 + (AX_0)^4 = (I - AX_0)^4$$

- **Lemma 7**

$$(\mathbf{I} - \mathbf{AX_n}) = (\mathbf{I} - \mathbf{AX_0})^{4^n}$$

Proof:

By mathematical induction; for n = 1, it's true by the previous lemma.
Suppose the result is true for n. Let us prove it for n+1.

$$\begin{aligned} I - AX_{n+1} &= I - AX_n - AX_n(I - AX_n) - AX_n(I - AX_n)^2 - AX_n(I - AX_n)^3 \quad n = 0, 1, \ldots \\ &= I - 4AX_n + 6(AX_n)^2 - 4(AX_n)^3 + (AX_n)^4 \\ &= (I - AX_n)^4 \\ &= [(I - AX_0)^{4^n}]^4 \\ &= (I - AX_0)^{4^{n+1}} \end{aligned}$$

- **Lemma 8**

$$\mathbf{X_n = X_0 + X_0(I - AX_0) + X_0(I - AX_0)^2 + \ldots + X_0(I - AX_0)^{4^n-1}}$$

Proof:

By mathematical induction; for n =1, $X_1 = X_0 + X_0(I - AX_0) + X_0(I - AX_0)^2 + X_0(I - AX_0)^3$
Suppose the result is true for n. Let us prove it for n+1.

$$\begin{aligned} X_{n+1} &= X_n + X_n(I - AX_n) + X_n(I - AX_n)^2 + X_n(I - AX_n)^3 \\ &= X_n + X_n(I - AX_0)^{4^n} + X_n[(I - AX_0)^{4^n}]^2 + X_n[(I - AX_0)^{4^n}]^3 \\ &= X_0 + X_0(I - AX_0) + X_0(I - AX_0)^2 + \ldots + X_0(I - AX_0)^{4^n-1} \\ &\quad + X_0(I - AX_0)^{4^n} + X_0(I - AX_0)^{4^n+1} + \ldots + X_0(I - AX_0)^{2.4^n-1} \\ &\quad + X_0(I - AX_0)^{2.4^n} + X_0(I - AX_0)^{2.4^n+1} + \ldots + X_0(I - AX_0)^{3.4^n+1} \\ &\qquad + \ldots + X_0(I - AX_0)^{4.4^n-1} \end{aligned}$$

Therefore

$$X_{n+1} = X_0 + X_0(I - AX_0) + X_0(I - AX_0)^2 + \ldots + X_0(I - AX_0)^{4^{n+1}-1}$$

We further need the following elementary fact which could be found in any linear algebra textbook.

❖ Lemma 9

If **C** is a square matrix with norm strictly less than 1, then the series $I+C+C^2+\dots$ is convergent and its inverse is given by $(I - C)^{-1}$.

Our **first main result** is the following.

Theorem 3

The following iteration method

$$\mathbf{X_{n+1} = X_n + X_n(I - AX_n) + X_n(I - AX_n)^2 + X_n(I - AX_n)^3 \quad , n = 0, 1 \dots}$$

Converges to A^{-1} under the assumption $\|I - AX_0\| < 1$

Proof:

By lemma 8, we have

$$X_{n+1} = X_0 + X_0(I - AX_0) + X_0(I - AX_0)^2 + \dots + X_0(I - AX_0)^{4^{n+1}-1}$$

Therefore

$$X_{n+1} = X_0\,(I + (I - AX_0) + (I - AX_0)^2 + \dots + (I - AX_0)^{4^{n+1}-1})$$

Let $C = I - AX_0$. Our assumption implies that $\|C\| < 1$.
Thus, by the previous lemma, as $n \to \infty$

$$X_{n+1} \to X_0(I - (I - AX_0))^{-1} = X_0(AX_0)^{-1} = X_0X_0^{-1}A^{-1} = A^{-1}$$

Our next main result is that the convergence is of order 4.
We first need the following lemma.

❖ Lemma 10

$$\mathbf{I - AX_{n+1} = (I - AX_n)^4}$$

Proof:

$$AX_{n+1} = AX_n + AX_n (I - AX_n) + AX_n (I - AX_n)^2 + AX_n (I - AX_n)^3 \quad , n=0, 1\ldots$$
$$= 4AX_n - 6(AX_n)^2 + 4(AX_n)^3 - (AX_n)^4$$

Therefore

$$I- AX_{n+1} = I - 4AX_n + 6(AX_n)^2 - 4(AX_n)^3 + (AX_n)^4 = (I - AX_n)^4$$

Theorem 4

The order of convergence is 4

Proof:

$$I - AX_{n+1} = (I - AX_n)^4$$

Hence

$$A (A^{-1} - X_{n+1}) = (A (A^{-1} - X_n))^4$$

Therefore

$$\| A^{-1} - X_{n+1} \| = \| A^{-1}(A(A^{-1} - X_n))^4 \|$$
$$\leq \|A^{-1}\| \, \|A\|^4 \, \| A^{-1} - X_n \|^4$$
$$= K(A) \, \|A\|^3 \, \| A^{-1} - X_n \|^4$$

2.3 Iteration Method Where the Order of Convergence is p+1

$$X_{n+1} = \sum_{i=0}^{p} X_n \, (I - AX_n)^i$$
$$X_{n+1} = X_n + X_n (I - AX_n) + X_n (I - AX_n)^2 + \dots + X_n (I - AX_n)^p \quad n = 0, 1\dots$$

We will show that it **converges to A^{-1}** under the assumption $\|I - AX_0\| < 1$.
We also show that the order of convergence is p+1.

❖ **Lemma 11**

$$\mathbf{I - AX_0 - AX_0 (I - AX_0) - AX_0 (I - AX_0)^2 - \dots - AX_0 (I - AX_0)^p = (I - AX_0)^{p+1}}$$

Proof:

By mathematical induction;
For p =1, $I - AX_0 - AX_0 (I - AX_0) = I - 2AX_0 + (AX_0)^2 = (I - AX_0)^2$
Suppose the result is true for p. Let us prove it for p+1.

$$\begin{aligned} I - AX_0 - AX_0 (I - AX_0) - \dots - AX_0 (I - AX_0)^p - AX_0 (I - AX_0)^{p+1} \\ = (I - AX_0)^{p+1} - AX_0 (I - AX_0)^{p+1} \\ = (I - AX_0)^{p+1} (I - AX_0) \\ = (I - AX_0)^{p+2} \end{aligned}$$

❖ **Lemma 12**

$$\mathbf{I - AX_1 = (I - AX_0)^{p+1}}$$

Proof:

$$X_1 = \sum_{i=0}^{p} X_0 \, (I - AX_0)^i$$
$$A X_1 = \sum_{i=0}^{p} AX_0 \, (I - AX_0)^i$$
$$AX_1 = AX_0 + AX_0 (I - AX_0) + \dots + AX_0 (I - AX_0)^p$$

Therefore $I - AX_1 = I - AX_0 - AX_0 (I - AX_0) - \dots - AX_0 (I - AX_0)^p$
Therefore $I - AX_1 = (I - AX_0)^{p+1}$

❖ Lemma 13

$$I - AX_n = (I - AX_0)^{(p+1)^n}$$

Proof:

By mathematical induction;

for n = 0, $(I - AX_0) = (I - AX_0)^{(p+1)^0} = (I - AX_0)^1 = (I - AX_0)$

Suppose the result is true for n. Let us prove it for n+1.

$X_{n+1} = X_n + X_n (I - AX_n) + X_n (I - AX_n)^2 + \ldots + X_n (I - AX_n)^p$

$I - AX_{n+1} = I - AX_n - AX_n (I - AX_n) - AX_n (I - AX_n)^2 - \ldots - AXn (I - AX_n)^p$

$= (I - AX_n)^{p+1}$

$= \left[(I - AX_0)^{(p+1)^n}\right]^{p+1}$

$= (I - AX_0)^{(p+1)^{n+1}}$

❖ Lemma 14

$$X_n = X_0 + X_0 (I - AX_0) + X_0 (I - AX_0)^2 + \ldots + X_0 (I - AX_0)^{(p+1)^n - 1}$$

Proof:

By mathematical induction; for n = 0, $X_0 = X_0$.

Suppose the result is true for n. Let us prove it for n+1.

$X_{n+1} = X_n + X_n (I - AX_n) + X_n (I - AX_n)^2 + \ldots + X_n (I - AX_n)^p$

$= X_n + X_n (I - AX_0)^{(p+1)^n} + X_n (I - AX_0)^{2(p+1)^n} + \ldots + X_n (I - AX_0)^{p(p+1)^n}$

$= X_0 + X_0 (I - AX_0) + X_0 (I - AX_0)^2 + \ldots + X_0 (I - AX_0)^{(p+1)^n - 1} + X_0 (I - AX_0)^{(p+1)^n}$

$+ X_0 (I - AX_0)^{(p+1)^n + 1} + \ldots + X_0 (I - AX_0)^{2(p+1)^n - 1} + X_0 (I - AX_0)^{2(p+1)^n} + \ldots +$

$X_0 (I - AX_0)^{3(p+1)^n - 1} + \ldots + \ldots + X_0 (I - AX_0)^{p(p+1)^n + (p+1)^n - 1}$

$= X_0 + X_0 (I - AX_0) + X_0 (I - AX_0)^2 + \ldots + X_0 (I - AX_0)^{(p+1)^n - 1} + X_0 (I - AX_0)^{(p+1)^n}$

$+ X_0 (I - AX_0)^{(p+1)^n + 1} + \ldots + X_0 (I - AX_0)^{2(p+1)^n - 1} + X_0 (I - AX_0)^{2(p+1)^n} + \ldots +$

$X_0 (I - AX_0)^{3(p+1)^n - 1} + \ldots + \ldots + X_0 (I - AX_0)^{(p+1)^{n+1} - 1}$

Theorem 5

The following iteration method

$$X_{n+1} = \sum_{i=0}^{p} X_n \, (I - AX_n)^i$$

Converges to A^{-1} under the assumption $\|I - AX_0\| < 1$.

Proof:

By lemma **14**, we have

$$X_{n+1} = X_0 + X_0(I - AX_0) + X_0(I - AX_0)^2 + \ldots + X_0(I - AX_0)^{(p+1)^{n+1}-1}$$

Therefore

$$X_{n+1} = X_0 \left(I + (I - AX_0) + (I - AX_0)^2 + \ldots + (I - AX_0)^{(p+1)^{n+1}-1}\right)$$

Let $C = I - AX_0$. Our assumption implies that $\|C\| < 1$.
As $n \to \infty$

$$X_{n+1} \to X_0(I - (I - AX_0))^{-1} = X_0(AX_0)^{-1} = X_0X_0^{-1}A^{-1} = A^{-1}$$

Our next main result is that the convergence is of order p+1.

Theorem 6

The order of convergence is p+1

Proof:

$$\begin{aligned}
I - AX_{n+1} &= (I - AX_n)^{p+1} \\
A(A^{-1} - X_{n+1}) &= [A(A^{-1} - X_n)]^{p+1} \\
A^{-1} - X_{n+1} &= A^{-1}[A(A^{-1} - X_n)]^{p+1} \\
\|A^{-1} - X_{n+1}\| &\leq \|A^{-1}\| \, \|A\|^{p+1} \, \|A^{-1} - X_n\|^{p+1} \\
&= K(A) \, \|A\|^{p} \, \|A^{-1} - X_n\|^{p+1}
\end{aligned}$$

With $K(A) = \|A^{-1}\| \|A\|$ the condition number

Remark

Significance of the Condition Number

The condition number of an n × n matrix A is

$$\textbf{Cond (A) = K (A)} = \|A\|\|A^{-1}\|$$

This number tells us how accurate we can expect the vector x when solving a system of equations A · x = b. We assume that there is an error in representing the vector b, call it € and otherwise the solution is given to absolute accuracy. That is we solve A · x = b + € and get a solution x + δ where x is the solution of Ax = b. How does the condition number help estimate the number δ? We note that

$$x + \delta = A^{-1}(b + €) = A^{-1}b + A^{-1}€$$

Since $A^{-1}b = x$, this gives us the following equation for δ.

$\delta = A^{-1} \cdot €$

$||\delta|| \le \|A^{-1}\| \, \|€\|$

So, the condition number for the magnitude of the absolute error δ for such a calculation is just the operator norm, $\|A^{-1}\|$. On the other hand, the relative error is given by $\frac{\|\delta\|}{\|x\|}$.

For the relative error we simply divide the above inequality by the norm of x to get the following inequality:

$\frac{\|\delta\|}{\|x\|} \le \frac{\|A^{-1}\| \ \|\varepsilon\|}{\|x\|}$

However, from the definition of the norm of A, $\frac{\|A.x\|}{\|x\|} \le \|A\|$ and Ax = b.

So, $\|b\| \le \|A\| \cdot \|x\|$. Thus, combining these inequalities we get the following.

$\frac{\|\delta\|}{\|x\|} \le \frac{\|A^{-1}\| \ \|\varepsilon\|}{\|x\|} \le \frac{\|A\| \ \|A^{-1}\| \ \|\varepsilon\|}{\|A\| \ \|x\|} = \text{cond (A)}. \frac{\|\varepsilon\|}{\|b\|}$

So, in solving the equation Ax = b, the relative error in the solution divided by the relative error in the right-hand-side vector is given by the condition number of A. The following rule of thumb is a useful way to express the above estimate. It states that if m $=\log_{10}(cond(A))$, then m is the number of digits accuracy lost in solving the system of equations Ax = b. There is typically additional error due to the many calculations needed in solving the equations. The estimate for additional losses is given by $\log_{10}(n)$ if the matrix A is n × n.

Chapter 3

Error bound on the norm of the error

Suppose A is a $r \times r$ matrix such that I - A is non-singular then the identity

$$(I - A)^{-1} = I + A + A^2 + \dots + A^{m-1} + A^m (I - A)^{-1}$$

Holds since

$$(I - A)^{-1} = I + A + A^2 + \dots + A^{m-1} + A^m + A^{m+1} + \dots$$
$$(I - A)^{-1} = I + A + A^2 + \dots + A^{m-1} + A^m (I + A + A^2 + \dots)$$
$$(I - A)^{-1} = I + A + A^2 + \dots + A^{m-1} + A^m (I - A)^{-1}$$

Now suppose that $N(A) \leq k < 1$. In the sequel, we use the Frobenius norm

$$N(A) = \sqrt{\sum_{j=1}^{r} \sum_{i=1}^{r} a_{ij}^2}$$

Taking the norm and using: $N(A+B) \leq N(A) + N(B)$

$N(AB) \leq N(A)\, N(B)$

$N(A^m) \leq [N(A)]^m$

We have:

$$N[(I - A)^{-1}] \leq r^{1/2} + k + k^2 + \dots + k^{m-1} + k^m N[(I - A)^{-1}]$$

Clearly, $N(I) = \sqrt{r}$.
Since $k < 1$; we may solve for $N[(I - A)^{-1}]$.

We obtain: $N[(I - A)^{-1}] \leq \frac{r^{1/2} - 1}{1 - k^m} + \frac{1}{1-k}$ for every $m > 0$

<u>Proof:</u>

$$N[(I - A)^{-1}] \leq r^{1/2} + k + k^2 + \dots + k^{m-1} + k^m N[(I - A)^{-1}]$$

$$N[(I - A)^{-1}] - k^m N[(I - A)^{-1}] \leq r^{1/2} - 1 + 1 + k + k^2 + \dots + k^{m-1}$$

$$(1 - k^m)\, N[(I - A)^{-1}] \leq r^{1/2} - 1 + \frac{1 - k^m}{(1-k)}$$

Therefore $$N[(I-A)^{-1}] \le \frac{r^{1/2}-1}{1-k^m} + \frac{1}{1-k}$$

If $m \to \infty$ then $k^m \to 0$ since $k < 1$. Therefore in the limit when m becomes infinite; we find: $$N[(I-A)^{-1}] \le r^{1/2} - 1 + \frac{1}{1-k} \quad ; \quad N(A) \le k < 1$$

Let X_0 be an approximation to the inverse of a matrix A.

Consider the following sequence of operations.

Calculate: $X_1 = X_0(2I - AX_0)$ (*)

$X_2 = X_1(2I - AX_1)$

.

.

.

$X_{m+1} = X_m(2I - AX_m)$ $\quad m = 0, 1, 2 \ldots$

Let us inquire as to the conditions under which the sequence of matrices X_m converges to A^{-1}, the maximum error that may be committed in stopping at any stage, and the rate of convergence.

Suppose that X_0 is an approximation to A^{-1} to make the roots of the matrix

$$D = I - AX_0 \qquad (**)$$

all less than unity in absolute value.

Then increasing powers of D approach zero, and the convergence of X_m to A^{-1} will follow from the relation $$X_m = A^{-1}(I - D^{2^m})$$

Proof:

By mathematical induction;

For m = 1; we have $X_1 = A^{-1}(AX_0)(I + D)$

$= A^{-1}(I - D)(I + D)$

$= A^{-1}(I - D^2)$

Since, from (**) we have $D - I = -AX_0$ then $I - D = AX_0$ then $X_0 = A^{-1}(I - D)$

Now in (*), $X_1 = X_0(2I - AX_0)$

$= A^{-1}(I - D)(2I - A A^{-1}(I - D))$

$= A^{-1}(I - D)(2I - I + D)$

$= A^{-1}(I - D)(I + D)$

$= A^{-1}(I - D^2)$

Suppose it is true for m and let us prove it for m+1:

$$
\begin{aligned}
X_{m+1} &= X_m (2I - AX_m) \\
&= A^{-1} (I - D^{2^m}) (2I - AA^{-1} (I - D^{2^m})) \\
&= A^{-1} (I - D^{2^m}) (I + D^{2^m}) \\
&= A^{-1} (I - D^{2^{m+1}})
\end{aligned}
$$

Therefore $\quad X_{m+1} = A^{-1} (I - D^{2^{m+1}})$

This completes the induction.

Now, we derive an upper bound for the error in X_m in terms of k and N (X_0).

From $D = I - AX_0$ we have $D - I = - AX_0$

$$
\begin{aligned}
I - D &= AX_0 \\
A^{-1} (I - D) &= X_0 \\
A^{-1} &= X_0 (I - D)^{-1}
\end{aligned}
$$

Hence, by $X_m = A^{-1} (I - D^{2^m})$

$$
\begin{aligned}
X_m &= A^{-1} - A^{-1} D^{2^m} \\
X_m - A^{-1} &= - A^{-1} D^{2^m} \\
X_m - A^{-1} &= - X_0 (I - D)^{-1} D^{2^m}
\end{aligned}
$$

Therefore, using:

$$N (AB) \le N (A) N (B)$$

$$N (A^m) \le [N (A)]^m$$

And by $N [(I - A)^{-1}] \le r^{1/2} - 1 + \frac{1}{1-k}$

We have $N (X_m - A^{-1}) \le N (X_0)\, k^{2^m} \left(r^{1/2} - 1 + \frac{1}{1-k} \right)$

The previous inequality sets an upper bound for the difference between each element of X_m and the corresponding element of A^{-1}.

Let us generalize this inequality for our iteration with $p = 3$, i.e., for the iteration

$$X_{m+1} = X_m + X_m(I - AX_m) + X_m(I - AX_m)^2$$

By Lemma 2, Chapter 2, we have

$$I - AX_m = (I - AX_0)^{3^m}$$

Let $D = I - AX_0$. Then $A^{-1} = X_0(I - D)^{-1}$.

We also have

$$I - AX_m = D^{3^m}$$

Hence

$$X_m = A^{-1}\left(I - D^{3^m}\right)$$

Which can be written as $X_m = A^{-1} - A^{-1}D^{3^m}$

Or $$X_m - A^{-1} = -X_0(I-D)^{-1}D^{3^m}$$

Therefore by N (AB) ≤ N (A) N (B)

N (A^m) ≤ [N (A)]^m

And by N $[(I-A)^{-1}] \le \frac{r^{\frac{1}{2}}-1}{1-k^m} + \frac{1}{1-k}$

We have N $(X_m - A^{-1}) \le$ N $(X_0)\, k^{3^m} \left(\frac{r^{\frac{1}{2}}-1}{1-k^m} + \frac{1}{1-k}\right)$

Our aim is to generalize this inequality for arbitrary p, i.e., for the iteration

$$X_{m+1} = \sum_{i=0}^{p} X_m\,(I - AX_m)^i$$

By Lemma 13, we have

$$I - AX_m = (I - AX_0)^{(p+1)^m}$$

Let $D = I - AX_0$. Then $A^{-1} = X_0(I-D)^{-1}$.

We also have

$$I - AX_m = D^{(p+1)^m}$$

Hence

$$X_m = A^{-1}\left(I - D^{(p+1)^m}\right)$$

Which can be written as

$$X_m = A^{-1} - A^{-1}D^{(p+1)^m}$$

Or

$$X_m - A^{-1} = -X_0(I-D)^{-1}D^{(p+1)^m}$$

Therefore by N (AB) ≤ N (A) N (B)

N (A^m) ≤ [N (A)]^m

And by N $[(I-A)^{-1}] \le \frac{r^{\frac{1}{2}}-1}{1-k^m} + \frac{1}{1-k}$

We have N $(X_m - A^{-1}) \le$ N $(X_0)\, k^{(p+1)^m} \left(\frac{r^{\frac{1}{2}}-1}{1-k^m} + \frac{1}{1-k}\right)$

Chapter 4

Conclusion and Future Work

In [15], the author showed that the iteration method

$$X_{n+1} = X_n(2I - AX_n), n = 0,1, \dots$$

Converges quadratically to the inverse of the matrix A under the assumption that

$$\|I - AX_0\| < 1$$

He also showed that

$$N(X_m - A^{-1}) \leq N(X_0)k^{2^m}\left(\sqrt{r} - 1 + \frac{1}{1-k}\right)$$

Where $N(A)$ is the Frobenius norm of an $r \times r$ matrix A which satisfies $N(A) \leq k < 1$.

In this work, we generalized the results presented in [15] by considering the iteration

$$X_{m+1} = \sum_{i=0}^{p} X_m\,(I - AX_m)^i$$

Clearly, the iteration presented in [15] is a particular case of our iteration for the value $p = 1$. We showed that our iteration converges to the inverse of the matrix A under the assumption that

$$\|I - AX_0\| < 1$$

We also showed that the convergence is of order $p + 1$, and

$$N(X_m - A^{-1}) \leq N(X_0)k^{(p+1)^m}\left(\sqrt{r} - 1 + \frac{1}{1-k}\right)$$

One possible direction for future work is to determine the optimal p. Clearly, as p increases, the computational cost increases but the number of iterations decreases (since the order of convergence increases). We have no idea currently how to address this issue.

Another possible direction is to generalize the results presented in [16] where the author considered the iteration in [15] in the context of generalized inverses.

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