HOLOMORPHIC DYNAMICS IN ONE COMPLEX VARIABLE

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THESIS

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Abstract

The paper presents the study of the iterations of rational fractions, that is, the behavior of $z_0, z_1 = f(z_0), ..., z_{n+1} = f(z_n), ...$ We illustrate the conditions needed for this function to behave as its linear part and prove the existence of Siegel and Cremer points. The study extends to a description of Fatou and Julia sets.

Dedication

I dedicate my dissertation work to my family. A special feeling of gratitude to my late grandmother, Janette Hannouche, her support has sustained me throughout my life.
A special thanks to my parents Richard and Noha Atallah, my brother Richard and my sister Stephanie for their support, encouragement and constant love.

Acknowledgements

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Introduction

0.1 Aims and Objectives

Iterations of a rational function on the Riemann sphere is one of the most attractive topics in the theory of dynamical systems.
Let $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be a rational function where $\widehat{\mathbb{C}}$ denotes the Riemann sphere obtained by adding the point ∞ to $\mathbb{C}$. We will be interested in the iterations of such f at a point z_0; that is, the sequence

$$z_0, z_1 = f(z_0), ..., z_{n+1} = f(z_n), ...$$

Regarding the dynamics of f, we first distinguish between whether the points on the sphere are "stable" or not, i.e. if nearby points behave similarly under iterations. That is, whether or not arbitrary close points behave differently. The set of stable points is called the **Fatou set**, Ω_f, and its complement is called the **Julia set**, J_f. The Fatou set can be defined as the largest open set in which the iteration of f is normal. The dynamic of f on J_f is "chaotic" whether on Ω_f it is much more regular.
The starting point of the chaos theory is the three-body problem. It consists of studying the dynamic of three bodies in gravitational interaction such as the system: Sun, Earth, Moon; supposed to be isolated from the rest of the universe. Is the solar system "stable" in the long run or might one of the bodies strike another body one day or will it be ejected form the solar system towards infinity? This remains an open question which requires further investigation.
In the first chapter of this thesis, we define all types of points: superattractive, attractive, repulsive, and indifferent. We study whether the iterations of f near each of these points behave as the iterations of its linear map.
In the second chapter, we discover Fatou and Julia sets and illustrate some fascinating fractal shapes.

Iterations of rational maps is a large subject. Good places to start reading further are "Dynamics in one complex variable" by John Milnor [1], "Iteration of rational functions" by Alan Beardon [2] and "Riemann surfaces, dynamics and geometry" by Curtis MCMullen [3].

0.2 State of the Art

Complex dynamics has quite a long history starting with the work of Ernst Schroder, Gabriel Koenigs, Hermann A. Schwarz and many others in the 19^{th} century. During that time, the local study of iterated holomorphic mappings in a neighborhood of a fixed point was well developed [4].
Progress came early in the new century with Pierre Fatou's Comptes Rendus notice where he examined the iteration of two kinds of rational functions: the family $f_k(z) = \frac{z^k}{z^k+2}$, where k is a natural number greater then 1, has a single attracting orbit consisting of a fixed point at the origin, and $g(z) = \frac{z^2+z}{2}$ that has two attracting orbits, one a fixed point at the origin and the other the point at infinity. The first Julia set appeared in Fatou's note [1906d] when Gaston Julia was 13 and before normal families were invented.
Modern day interest in Julia sets began in the 1920's with Gaston Julia. Julia first introduced the modern idea of a Julia set in his best known paper Memoire sur l'iteration des fonctions rationelles [5]. Interest in the subject flourished over the next 10 years and many other well known mathematicians began to study Julia sets such as Carl L. Siegle and Paul Montel. The Fatou-Julia theory was one of the first applications of the concept of normal families introduced by Montel. Despite the lack of computing power available at that time, Harald Cramer was able to become the first man to approximate the image of a Julia set.
After that, the subject went to sleep until the late seventies when it has undergone explosive growth due to computer graphics. It was not until Benoit Mandelbrot began studying iteration in the 1970's that Julia sets re-emerged. By then, computing facilities were available and much more detailed images could be produced. The advance in computer triggered the work of many mathematicians such as John Milnor, Adrien Douady, and John Hubbard.

Preliminaries

0.3 Topological Properties

We introduce in this section some topological properties used for studying the structure of Fatou and Julia sets.

Definition 0.3.1. *A metric space is an ordered pair (M,d) where M is a set and d is a metric on M, i.e, a function* $d : M \times M \longrightarrow \mathbb{R}$ *such that for any* $x, y, z \in M$ *the following holds:*
$1 \cdot d(x,y) \geq 0$
$2 \cdot d(x,y) = 0$ *iff* $x = y$
$3 \cdot d(x,y) = d(y,x)$
$4 \cdot d(x,z) \leq d(x,y) + d(y,x)$

Example 0.3.1. *The set of complex numbers* $\mathbb{C}$ *with the metric* $d(z,w) = |z - w|$ *where* $z, w \in \mathbb{C}$ *is a metric space. The* $|\cdot|$ *represents the modulus of a complex number.*

Definition 0.3.2. *A subset K of a metric space (M,d) is said to be open if, given any point* $z \in K$*, there exists a real number* $\varepsilon > 0$ *such that* $B(z,\varepsilon) = \{z'; d(z,z') < \varepsilon\}$ *lies entirely in K.*

Definition 0.3.3. *A subset K of a metric space (M,d) is said to be closed if its complement is open. Equivalently, if there exists* $z \in K$ *such that for any* $r > 0$, $B(z,r) \supseteq$ *points from K and points from outside K.*

Definition 0.3.4. *Let (M,d) be a metric space and* $E \subseteq M$*. We say that the family of open sets in M,* $\{G_\alpha\}_\alpha$*, is an open covering of E if and only if* $E \subseteq \cup G_\alpha$*.*

Definition 0.3.5. *A subset K of a metric space (M,d) is said to be compact if every open covering of K contains a finite subcovering of K.*

Theorem 0.3.1. *A closed subset of a compact set is compact.*

Theorem 0.3.2. *A subset K of $\mathbb{C}^n$ is compact if and only if it is closed and bounded.*

Definition 0.3.6. *Given a metric space (M,d), a sequence $(x_n)_{n\in\mathbb{N}}$ in M is said to be a Cauchy sequence if for every positive real number ε, there is a positive integer N such that for all integers $m, n > N$, $d(x_m, x_n) < \varepsilon$; that is, a sequence whose elements become very close as $n \longrightarrow \infty$.*

Definition 0.3.7. *A metric space M is said to be complete if every Cauchy sequence of points in M has a limit that is also in M, or, alternatively, if every Cauchy sequence in M converges in M.*

Theorem 0.3.3. *In a compact metric space (M,d), every sequence in M has a convergent subsequence whose limit is in M. Therefore, any compact metric space is complete.*

Definition 0.3.8. *A metric space (M,d) is said to be connected if no two disjoint open or closed sets cover M.*

Definition 0.3.9. *The maximal connected subsets of a nonempty metric space (M,d) are called the connected components of M. The connected components of M form a partition of M: they are disjoint, nonempty, and their union is the whole space M.*

Definition 0.3.10. *An open set D of (M,d) is simply connected if it is connected and if any closed path in D is holomorphic to a point in D; that is, D is a set with no holes; that is, all interior points to a closed simple path in D are in D.*

Example 0.3.2. *Spaces A,B,C and D are connected whereas space E (made of subsets E_1, E_2, E_3, E_4) is not connected. Furthermore, A and B are also simply connected while C and D are not.*

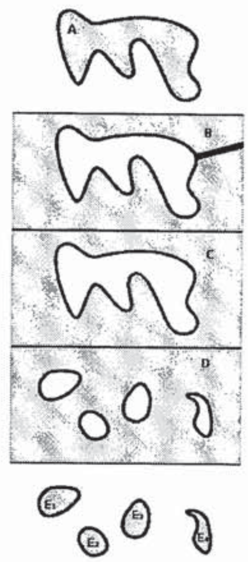


Figure 1: Connected and Disconnected Subspaces of R^2

Definition 0.3.11. *The closure of a subset S in a metric space (M,d) consists of all points in S plus the limit points of S: it is the union of S and its boundary. That is, it is the smallest closed set containing S. The closure of S is denoted by* $\overline{S}$.

Definition 0.3.12. *The interior of S is the union of all open sets contained in S; that is, it is the largest open set contained in S.*

Definition 0.3.13. *A point z of a metric space M is called an isolated point of a subset S of M if z belongs to S and there exists in M a neighborhood of z not containing other points of S.*

Definition 0.3.14. *A subset S of a metric space M is called dense in M if* $\overline{S} = M$.

Definition 0.3.15. *The subset S is said to be nowhere dense if its closure has empty interior.*

Definition 0.3.16. *A perfect subset S is a closed set with no isolated points; every point of the set is an accumulation point of the set.*

0.4 Riemann Sphere and Holomorphic Functions

In this section, we state important results in complex analysis.

Definition 0.4.1. *Let S^2 denote the unit sphere $x^2+y^2+z^2=1$ in $\mathbb{R}^3$. The stereographic projection is the function*

$$\Pi : S^2 - \{N\} \to \mathbb{C} : M \to P$$

$$\Pi(a,b,c) = \frac{a+ib}{1-c}$$

where N(0,0,1) is the "north pole" of S^2 and P is the point of intersection of the line connecting MN and the xy-plane.

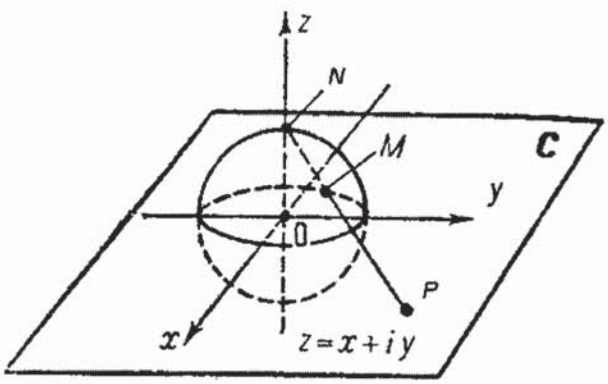


Figure 2: Stereographic projection

Proposition 0.4.1. *The stereographic projection is a bijection between $S^2 - \{N\}$ and $\mathbb{C}$ and so, if we assign ∞ to N, Π can be extended to a bijection from S^2 to $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} = P^1$. Under this identification, $\widehat{\mathbb{C}}$ is known as the Riemann Sphere.*

Definition 0.4.2. *Let $V \subset \mathbb{C}$ be an open set of complex numbers. A function $f : V \longrightarrow \mathbb{C}$ is called holomorphic if the first derivative*

$$z :\longrightarrow f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

is defined and continuous as a function from V to $\mathbb{C}$.
Equivalently, if f has a power series expansion about any point $z_0 \in V$ which converges to f in some neighborhood of z_0 with positive radius of convergence:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Theorem 0.4.1. ***Inverse Function Theorem***
Let $V \subset \mathbb{C}$ be an open set and f a holomorphic function over V such that $f'(z_o) \neq 0$. Then there exists open sets $U \subset V$ and $W \subset \mathbb{C}$ such that $z_0 \in U$ and $f : U \longrightarrow W$ is one to one and onto and $f^{-1} : W \longrightarrow \mathbb{C}$ is holomorphic and for any $z \in W$, $(f^{-1})'(z) = \frac{1}{f'(f^{-1}(z))}$.

Theorem 0.4.2. ***Maximum Principal***
Let V be an open set, $a \in V$, and $f : V \longrightarrow \mathbb{C}$ a holomorphic function. If $|f(z)| \leq f(a)\ \forall z$ close to a, then f is a constant near a.

Corollary 0.4.1. *Let V be an open, bounded and connected set in $\mathbb{C}$ and $f : \overline{V} \longrightarrow \mathbb{C}$ a holomorphic function. Let M be the maximum value of $f\ |_{\partial V}$. Then:*
1) $|f(z)| \leq M\ \forall z \in \overline{V}$.
2) If $|f(a)| = M$ for $a \in V$ then f is a constant.

Theorem 0.4.3. ***Riemann Mapping Theorem***
If U is a non empty simply connected open subset of the complex plane $\mathbb{C}$, then there exists a biholomorphic (bijective and holomorphic) mapping f from U onto the open unit disc. f is said to be a conformal map.

Theorem 0.4.4. ***Schwarz Lemma***
Let f be holomorphic on the open unit disc and assume that:
$1 \cdot |f(z)| < 1\ \forall z \in D(0,1)$ *and*
$2 \cdot f(0) = 0$
Then $|f(z)| \leq |z|\ \forall |z| < 1$ and $|f'(0)| \leq 1$. If either $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$ or if $|f'(0)| = 1$, then f is a rotation; i.e, $f(z) = az$ for some complex constant a with $|a| = 1$.

Theorem 0.4.5. ***Rouche's Theorem***
Suppose f and g are holomorphic functions on and inside a simple closed path γ. Suppose that $|f(z)| > |g(z)|$ on γ. Then f and f+g have the same number of zeros inside γ where each zero is counted as many times as its multiplicity. Equivalently, if f and g are sufficiently close to each other on γ, then they have the same number of zeros inside γ.

Theorem 0.4.6. ***Cauchy's Argument Principle***
Let f(z) be a holomorphic function inside and on a simple closed path γ where f(z) has no zeros nor poles on γ. Then:

$$N_0 - N_\infty = \frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}dz$$

where N_0 is the number of zeros of f(z) inside γ and N_∞ is the number of poles of f(z) inside γ, counted as many times as its multiplicity.

Definition 0.4.3. *A collection $F = \{f_\alpha\}_{\alpha\in I}$ of holomorphic functions $f_\alpha : \Omega \longrightarrow \widehat{\mathbb{C}}$ is called a normal family if and only if every infinite sequence of functions $\{f_n\} \in F$ contains a subsequence which converges uniformly on every compact of Ω.*

Theorem 0.4.7. *Let $F = \{f_\alpha\}_{\alpha\in\mathbb{Z}}$ be a collection of holomorphic functions $f_\alpha : \Omega \longrightarrow \widehat{\mathbb{C}}$. Suppose that F is uniformly bounded on every compact included in Ω. Then F is a normal family.*

Chapter 1

Iterated Holomorphic Maps

We consider, in this chapter, holomorphic functions $f : P^1 \longrightarrow P^1$ where $f(z) = \frac{P(z)}{Q(z)}$ is a rational fraction fixing the origin (f(0)=0), with derivative $\lambda = f'(0)$. So f can be written in terms of a power series in a neighborhood of 0:

$$f(z) = \lambda z + O(z^2)$$

where $O(z^2) := \sum_{p\geq 2} a_p z^p$. We denote such a function by $f : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$.
We will be studying the iterations of rational fractions: what is the behavior of the sequence

$$z_0, z_1 = f(z_0), ..., z_{n+1} = f(z_n), ...?$$

1.1 Linear Maps

In this section, let $f : P^1 \longrightarrow P^1$ be a linear function; $f(z) = \lambda z$, $\lambda \neq 0$. The iterations of f at z_0 are

$$z_0, z_1 = f(z_0) = \lambda z_0, \cdots, z_n = f^n(z_0) = \lambda^n z_0, ...$$

and so defined by the sequence

$$z_n = \lambda^n z_0;\ n \in \mathbb{N}$$

- If $|\lambda| < 1$, $0 \leqslant |z_n| \leqslant |\lambda|^n |z_0| \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore $|z_n| \longrightarrow 0$ as $n \longrightarrow \infty$ and so the iterations of f at any z_0 go inside the circle centered at O of radius $|z_0|$ and become very close to 0.

- If $|\lambda| > 1$, $|z_n| \longrightarrow \infty$ as $n \longrightarrow \infty$ and so the iterations at any z_0 go outside the circle and converge to ∞.
- If $|\lambda| = 1$, $|z_n| = |z_0|\ \forall n$. Therefore, the iterations at any z_0 lie entirely on the circle centered at O of radius $|z_0|$. In this case, $\lambda = e^{2i\pi\alpha}$, we have a rotation of an angle α and so 2 cases arise: $\alpha \in \mathbb{Q}$ or $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

1. If $\alpha \in \mathbb{Q}$, $\alpha = \frac{p}{q}$ where $\frac{p}{q}$ is reduced to its lowest terms and $q > 0$, the iterations are periodic of period q since

$$z_q = e^{q2i\pi\frac{p}{q}} z_0 = e^{2i\pi p} z_0 = (cos2\pi p + isin2\pi p) z_0 = z_0$$

Now, $z_1 = e^{2i\pi\frac{p}{q}} z_0$, therefore $argz_1 = argz_0 + p(\frac{1}{q})2\pi$. Hence, if we decompose the circle into q equal parts, each point of the cycle $z_0 \to z_1 \to ... \to z_{q-1} \to z_q = z_0$ is distant p points from the previous point on the circle.
Suppose that $\lambda = e^{2i\pi\frac{5}{12}}$. Then, q=12, p=5 and $z_{12} = z_0$. So the circle is divided into 12 equal parts and z_i is the 5^{th} point after z_{i-1}; i=1,2,...,11.

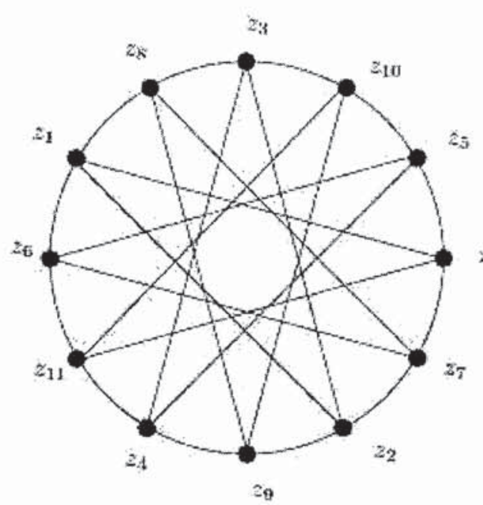


Figure 1.1: Cycle for the rotation $z \longrightarrow e^{2i\pi\frac{5}{12}} z$.

2. If $\alpha \in \mathbb{R} \backslash \mathbb{Q}$, $z_i \neq z_j\ \forall i, j$ then the iterations are dense on the circle $C_{|z_0|}$; they constitute almost the entire circle.

1.2 Linearization

Now that we understand the behavior of the iterations of a linear function, the question is: in which case the iterations of a holomorphic function $f(z) = \lambda z + O(z^2)$ near 0 behave as the iterations of its linear part $L(z) = \lambda z$ near 0?

Definition 1.2.1. *Two functions $f : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ and $g : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ are conjugates if there exists a holomorphic function $\phi : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ such that $\phi'(0) \neq 0$ and $\phi \circ f = g \circ \phi$ near 0.*

$\phi'(0) \neq 0$, by the inverse function theorem, ϕ is bijective near 0. Also, ϕ can be seen as a holomorphic change of coordinates in a neighborhood of 0. We can conclude the iterations of f from the iterations of g since:

$$\phi \circ f = g \circ \phi$$

$$f = \phi^{-1} \circ g \circ \phi$$

$$f^n = (\phi^{-1} \circ g \circ \phi)^n = \phi^{-1} \circ g \circ \phi \circ \phi^{-1} \circ g \circ \phi ...$$

$$f^n = \phi^{-1} \circ g^n \circ \phi$$

Proposition 1.2.1. *If $f : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ and $g : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ are conjugates then $f'(0) = g'(0)$.*

Proof.

$$\begin{aligned}(\phi \circ f)'(0) &= (g \circ \phi)'(0)\\ \phi'(f(0))f'(0) &= g'(\phi(0))\phi'(0)\\ \phi'(0)f'(0) &= g'(0)\phi'(0)\\ f'(0) &= g'(0)\, since\, \phi'(0) \neq 0.\end{aligned}$$

□

The problem now is in which case $f : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ and its linear part $g(z) = \lambda z$ are conjugates?

Definition 1.2.2. *A holomorphic function $f : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ is said to be linearizable if and only if it is conjugate to its linear part $g(z) = \lambda z = f'(0)z$.*

Definition 1.2.3. *Let $f : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ and $\lambda = f'(0)$. 0 is said to be a fixed*

- *superattractive point if $\lambda = 0$.*
- *attractive point if $0 < |\lambda| < 1$.*
- *repulsive point if $|\lambda| > 1$.*
- *indifferent point if $|\lambda| = 1$.*

Theorem 1.2.1. *Let $f : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ where 0 is a fixed superattractive point. f is linearizable if and only if f is constant.*

Proof. Suppose f is linearizable. Hence, $f(z) = \phi^{-1} \circ g \circ \phi(z)$ where $g(z) = \lambda z = 0\ \forall z$. Therefore, $f(z) = \phi^{-1}(0) = 0\ \forall z$ and f is the zero function.
Conversely, suppose that f is constant. Since $f(z) = \lambda z + O(z^2)$ and $\lambda = 0$ then $f(z) = 0\ \forall z$ and so f is linear. □

If 0 is an attractive or repulsive fixed point, f is always linearizable.
If 0 is an indifferent fixed point, f is sometimes linearizable and sometimes not and so different cases hold:

1. If $\lambda = e^{2i\pi\alpha}$ where $\alpha \in \mathbb{Q}$, 0 is said to be a parabolic point.
2. If $\lambda = e^{2i\pi\alpha}$ where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and f is linearizable, 0 is said to be a Siegel point.
3. If $\lambda = e^{2i\pi\alpha}$ where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and f is not linearizable, 0 is said to be a Cremer point.

In the following setions, we will show that if f is a polynomial of degree ≥ 2 and 0 is a parabolic point, f is not linearizable. We will also prove the existence of Siegel and Cremer points.

1.3 Attracting and Repelling Fixed Points

Let $B_r := \{z \in \mathbb{C} / |z| < r\}$.

Theorem 1.3.1. ***Koenigs Linearization Theorem***
If $f : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ has 0 as an attractive fixed point then f is linearizable; the following diagram commutes in the neighborhood of the origin:

$$\begin{array}{ccc} U & \xrightarrow{f} & f(U) \\ \phi \downarrow & & \downarrow \phi \\ \mathbb{C} & \xrightarrow{\lambda z} & \mathbb{C} \end{array}$$

Furthermore, ϕ is unique up to a multiplication by a non zero constant.

To prove this theorem, we will start by proving the following lemma:

Lemma 1.3.1. *If $f : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ has 0 as an attractive fixed point then $\exists r > 0$, $0 < c < 1$ such that $f(B_r) \subset B_r$, (i.e. the successive iterates $f^{\circ n}$ are all defined over B_r) and $|f^{\circ n}(z)| < c^n r$, $\forall z \in B_r$. Therefore, $\{f^{\circ n}|_{B_r}\}$ converges uniformly to 0.*

Proof. $f(z) = \lambda z + O(z^2)$ near 0. Hence $\exists r_0 > 0$ and C such that:

$$\begin{aligned} |f(z) - \lambda z| &= |c_1 z^2 + c_2 z^3 + ...| \\ &= |z^2||c_1 + c_2 z + ...| \\ &\leq C|z|^2 \ \ \forall z;\ |z| < r_0 \end{aligned}$$

So,

$$\begin{aligned} |f(z)| &= |f(z) - \lambda z + \lambda z| \\ &\leq |f(z) - \lambda z| + |\lambda z| \\ &\leq C|z|^2 + |\lambda||z| \\ &\leq (|\lambda| + C r_0)|z| \ \ \forall z;\ |z| < r_0 \end{aligned}$$

Now we can choose r as small as possible so that $|\lambda| < |\lambda| + Cr < c < 1$.
Hence $|f(z)| < (|\lambda| + Cr)|z| < c|z|;\ \forall z \in B_r$.
Then, $f(B_r) \subset B_r$ and hence

$$|f^2(z)| = |f(f(z))| < c|f(z)| < c^2|z|,\ \ \forall z \in B_r$$

By repeating the same process, we get

$$\begin{aligned} |f^{\circ n}(z)| &< c^n|z|,\ \ \forall z \in B_r \\ &< c^n r \longrightarrow 0\, as\, n \longrightarrow \infty. \end{aligned}$$

Therefore, $\{f^{\circ n}|_{B_r}\}$ converges uniformly to 0. □

Now we will prove Koenigs linearization theorem.

Proof. Consider the sequence $z_0 \to f(z_0) = z_1 \to f^2(z_0) = z_2 \to \ldots \to f^n(z_0) = z_n \to \ldots$ and the sequence $w_n = (\frac{z_n}{\lambda^n})_{n \geq 0}$. If this sequence converges uniformly in a neighborhood of the origin, we get $\phi = \lim_{n\to\infty} \frac{z_n}{\lambda^n}$.

$|w_{n+1} - w_n| = |\frac{z_{n+1}}{\lambda^{n+1}} - \frac{z_n}{\lambda^n}| = |\frac{z_{n+1} - \lambda z_n}{\lambda^{n+1}}| = \frac{1}{|\lambda^{n+1}|}|z_{n+1} - \lambda z_n|$

By the above lemma, $\exists r > 0,\ 0 < |\lambda| < c < 1$ such that all the iterates f^n are defined in B_r, $|f^n(z)| < c^n r$ and $|f(z) - \lambda z| \leq C|z|^2$, $\forall z \in B_r$. Hence

$$\begin{aligned}
|z_{n+1} - \lambda z_n| &= |f(f^{\circ n}(z)) - \lambda f^{\circ n}(z)| \\
&\leq C|f^{\circ n}(z)|^2 \\
&= C|z_n|^2 \\
&\leq C(rc^n)^2 \\
&= Cr^2c^{2n}; |z| < r
\end{aligned}$$

Hence,

$$\begin{aligned}
|w_{n+1} - w_n| &= |\frac{z_{n+1}}{\lambda^{n+1}} - \frac{z_n}{\lambda^n}| \\
&= \frac{1}{\lambda^{n+1}}|z_{n+1} - \lambda z_n| \\
&< \frac{1}{|\lambda^{n+1}|}Cr^2c^{2n} \\
&= \frac{Cr^2}{|\lambda|}(\frac{c^2}{|\lambda|})^n
\end{aligned}$$

If we take c as small as possible so that $0 < c^2 < |\lambda| < C$, then $|w_{n+1} - w_n| \longrightarrow 0\, as\, n \longrightarrow \infty$. Thus the sequence w_n converges uniformly throughout B_r to a holomorphic limit

$$\phi(z) = \lim_{n\to\infty} \frac{z_n}{\lambda^n} = \lim_{n\to\infty} \frac{f^{\circ n}(z)}{\lambda^n}$$

where:

- $\phi(0) = 0$ since $f^{\circ n}(0) = 0$.
- ϕ tangent to the identity, $\phi'(0) = 1$ since

$$\phi'(0) = \lim_{n\to\infty} \frac{(f^{\circ n}(0))'}{\lambda^n} = \lim_{n\to\infty} \frac{\lambda^n}{\lambda^n} = 1$$

.

- $\phi(f(z)) = \lim_{n\to\infty} \frac{f^{\circ(n+1)}}{\lambda^n} = \lim_{n\to\infty} \frac{f^{\circ n}}{\lambda^{n-1}} = \lambda \lim_{n\to\infty} \frac{f^{\circ n}}{\lambda^n} = \lambda\phi(z) = g(\phi(z))$ where $g(z) = \lambda z$, the linear part of f.

Suppose that there exist two functions ϕ and ψ such that $\phi \circ f \circ \phi^{-1}(z) = \lambda z$ and $\psi \circ\ f \circ \psi^{-1}(z) = \lambda z$. Consider

$$\psi \circ \phi^{-1}(\lambda w) = \psi \circ f \circ \phi^{-1}(w) = \lambda \psi \circ \phi^{-1}(w)$$

Expanding as a power series we get:

$$\psi \circ \phi^{-1}(w) = b_1 w + b_2 w^2 + ... + b_n w^n + ...$$

$$\psi \circ \phi^{-1}(\lambda w) = b_1 \lambda w + b_2 \lambda^2 w^2 + ... + b_n \lambda^n w^n + ...$$

which implies that $b_n \lambda^n = b_n \lambda$. If $b_n \neq 0$, then $\lambda = \lambda^n \Rightarrow \lambda$ is a root of 1, or $\lambda = 0$. Both cases are impossible. Therefore $b_n = 0\ \forall n \geq 2$. Hence $\psi \circ \phi^{-1}(w) = b_1(w)$.
Let $z = \phi^{-1}(w)$ then $\psi(z) = b_1\phi(z) \Rightarrow \psi = b_1\phi$. □

Theorem 1.3.2. *If $f : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ has 0 as a fixed repulsive point then f is linearizable.*

Proof. Consider $f^{-1} : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$. By the inverse function theorem, f^{-1} is well defined and holomorphic near 0 since $\lambda = f'(0) \neq 0$. f^{-1} has 0 as a fixed attractive point since:

$$(f^{-1})'(0) = \frac{1}{(f'(f^{-1}(0)))} = \frac{1}{f'(0)} = \frac{1}{\lambda} < 1$$

Therefore, f^{-1} is linearizable.
The function $\phi : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$, which linearizes $f^{-1} : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$, linearizes $f : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ since:

$$\phi \circ f^{-1} = g' \circ \phi\, where\, g' : z \longrightarrow \frac{z}{\lambda} = \lambda' z$$

$$\phi \circ f^{-1} \circ \phi^{-1} = g'$$

$$\phi \circ f^{-1} \circ \phi^{-1}(z) = \frac{z}{\lambda}$$

$$(\phi \circ f^{-1} \circ \phi^{-1})^{-1}(z) = (\frac{z}{\lambda})^{-1}$$

$$\phi \circ f \circ \phi^{-1}(z) = \lambda z$$

$$\phi \circ f \circ \phi^{-1} = g\, where\, g : z \longrightarrow \lambda z$$

□

- If $|\lambda| < 1$ at any point $z_0 \in \mathbb{C}$, $\lambda^n z_0 \longrightarrow 0$ as $n \longrightarrow \infty$. Hence

$$f^n(z_0) = \phi^{-1}(\lambda^n w_0) \longrightarrow 0\, as\, n \longrightarrow \infty$$

where $w_0 = \phi(z_0)$; the iterations of f approach the origin.

- If $|\lambda| > 1$ at any point $z_0 \in \mathbb{C}$, $\lambda^n z_0 \longrightarrow \infty$ as $n \longrightarrow \infty$. Hence

$$f^n(z_0) = \phi^{-1}(\lambda^n w_0) \longrightarrow \infty\, as\, n \longrightarrow \infty$$

where $w_0 = \phi(z_0)$; the iterations of f go to infinity.

1.4 Parabolic Fixed Points

We consider, in this section, holomorphic functions $f(z) = e^{2i\pi\frac{p}{q}}z + O(z^2)$ and we show that in this case, where 0 is a fixed parabolic point, f is almost always not linearizable.

Theorem 1.4.1. *Let $f : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ be a holomorphic function that has 0 as a fixed parabolic point; $f'(0) = e^{2i\pi\frac{p}{q}}$. Then f is linearizable if and only if $f^{\circ q} = Id$.*

Proof. If f is linearizable then f is conjugate to $g : z \longrightarrow e^{2i\pi\frac{p}{q}}z = \lambda z$. Hence, there exists a holomorphic function $\phi : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ such that $\phi'(0) \neq 0$ and

$$\phi \circ f = g \circ \phi$$

$$g = \phi \circ f \circ \phi^{-1}$$

$$\lambda z = \phi \circ f \circ \phi^{-1}(z)$$

$$z = \lambda^q z = (\phi \circ f \circ \phi^{-1})^{\circ q}(z) = \phi \circ f^{\circ q} \circ \phi^{-1}(z)$$

Hence

$$Id = \phi \circ f^{\circ q} \circ \phi^{-1}$$

$$Id \circ \phi(z) = \phi \circ f^{\circ q}(z)$$

$$\phi(z) = \phi(f^{\circ q}(z))$$

$$\phi^{-1}(\phi(z)) = \phi^{-1}(\phi(f^{\circ q}(z))$$

$$z = f^{\circ q}(z)$$

$$f^{\circ q}(z) = Id$$

Conversely, if $f^{\circ q} = Id$:

Let $\phi = \frac{1}{q}\sum_{n=0}^{q-1}\frac{f^{\circ n}}{\lambda^n} = \frac{1}{q}\sum_{n=0}^{q-1} e^{-2i\pi n\frac{p}{q}} f^{\circ n}$, where $f^{\circ n} = \lambda^n z + O(z^2)$. Then:

$\phi(0) = 0\, since\, f^{\circ n}(0) = 0\ \forall n$ and $\phi'(0) = 1$ since

$$\phi' = \frac{1}{q}\sum_{n=0}^{q-1} e^{-2i\pi n\frac{p}{q}}(f^{\circ n})'$$

$$\phi'(0) = \frac{1}{q}\sum_{n=0}^{q-1} e^{-2i\pi n\frac{p}{q}} e^{2i\pi n\frac{p}{q}} = \frac{1}{q}\sum_{n=0}^{q-1} 1 = \frac{1}{q} q = 1$$

Also $\phi \circ f = \frac{1}{q}\sum_{n=0}^{q-1}\frac{f^{\circ(n+1)}}{\lambda^n} = \frac{1}{q}\sum_{n=1}^{q}\frac{f^{\circ n}}{\lambda^{n-1}} = \frac{1}{q}\sum_{n=1}^{q}\lambda\frac{f^{\circ n}}{\lambda^n}$.

Hence $\phi \circ f = \lambda\frac{1}{q}\sum_{n=0}^{q-1}\frac{f^{\circ n}}{\lambda^n} = \lambda\phi = g\circ\phi$ where $g : z \longrightarrow \lambda z$.

Hence f is linearizable. □

Corollary 1.4.1. *If f is a rational fraction of degree $\geqslant 2$, that has 0 as a fixed parabolic point, then f is not linearizable.*

Proof. The rational fraction $f^{\circ q}$ is a rational fraction of degree d^q, where d is the degree of f. If $f^{\circ q} = Id$ near 0 then $d^q = 1$ which is not possible only if $d = 1$ and so f will be equal to the identity everywhere. Therefore, f is not of degree $\geqslant 2$. For this reason, $f^{\circ q}$ cannot be equal to the identity near 0, or else it will be equal to the identity everywhere. □

1.5 Siegel Points

First, in this section, we consider the case where 0 is a fixed indifferent point and then when α is an irrational number, i.e, $f(z) = \lambda z + O(z^2)$ with $\lambda = e^{2i\pi\alpha}$ and $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ in order to prove the existence of Siegel points.

Definition 1.5.1. *Let $f : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ be a holomorphic function such that $f'(0) = \lambda$ and $|\lambda| = 1$. The dynamic of f is said to be stable near 0 if and only if $\exists R > r > 0$ such that the iterations $f^{\circ n}\mid_{D_r}$ are defined $\forall n \geq 0$*

and $f^{\circ n}(D_r) \subset D_R$. Geometrically, the dynamic of f is stable near the fixed point 0 if the iterations of any 2 points z_1, z_2 close to 0 remain very close in a certain neighborhood of the origin.

Theorem 1.5.1. ***Stability and Linearizability***
Let $f : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ be a holomorphic function such that $|f'(0)| = |\lambda| = 1$. The dynamic of f is stable near 0 if and only if f is linearizable.

Proof. If f is linearizable then there exist $r > 0$, $R > r$ and a holomorphic function $\phi : D_r \longrightarrow D_R$ such that $\phi'(0) \neq 0$ and $\phi \circ f = g \circ \phi$ near 0 where $g(z) = \lambda z \ \forall z \in D_r$ and so

$$\phi \circ (f(z)) = \lambda\phi(z)$$

$$f(z) = \phi^{-1}(\lambda\phi(z))$$

Now $\phi(z) \in D_R \Rightarrow \lambda\phi(z) \in D_R \Rightarrow \phi^{-1}(\lambda\phi(z)) \in D_r \Rightarrow f(D_r) \subset D_r$.
Therefore $f^{\circ n}(D_r) \subset D_r \subset D_R$ and so the dynamic of f is stable near 0 and $f^{\circ n}(D_r) \subset D_R$.
Reciprocly, if the dynamic of f is stable near 0, consider $\phi_n : D_r \longrightarrow D_R$ defined by $\phi_n := \frac{1}{n}\sum_{i=0}^{n-1} \frac{1}{\lambda^i} f^{\circ i}$.

$\phi_n'(0) = \frac{1}{n}\sum_{i=0}^{n-1} \frac{1}{\lambda^i}(f^{\circ i})'(0) = \frac{1}{n}\sum_{i=0}^{n-1} \frac{1}{\lambda^i}\lambda^i = \frac{1}{n}\sum_{i=0}^{n-1} 1 = 1.$

Also, $\phi_n \circ f = \frac{1}{n}\sum_{i=0}^{n-1} \frac{1}{\lambda^i} f^{\circ(i+1)} = \frac{\lambda}{n}\sum_{i=0}^{n-1} \frac{1}{\lambda^{i+1}} f^{\circ(i+1)} = \frac{\lambda}{n}\sum_{i=1}^{n} \frac{1}{\lambda^i} f^{\circ i}$
$= \frac{\lambda}{n}(\sum_{i=0}^{n-1} \frac{1}{\lambda^i} f^{\circ i} + \frac{1}{\lambda^n} f^{\circ n} - Id) = \lambda\phi_n + \frac{\lambda}{n}(\frac{1}{\lambda^n} f^{\circ n} - Id).$
Since $\phi_n(D_r) \subset D_R$ the sequence $\{\phi_n\}_n$ constitutes a uniformly bounded sequence and so, by theorem 0.4.7, $\{\phi_n\}_n$ is a normal family. Therefore, we can extract a subsequence $\{\phi_{n_j}\}_{n_j}$ which converges uniformly in D_r to ϕ and so $\phi_{n_j} \circ f = \lambda\phi_{n_j} + \frac{\lambda}{n_j}(\frac{1}{\lambda^{n_j}} f^{\circ n_j} - Id)$.
For any $z \in D_r$, $\| f^{n_j}(z) \| \leq R$ then

$$\| \frac{1}{\lambda^{n_j}} f^{\circ n_j}(z) - z \| \leqslant \frac{1}{|\lambda|^{n_j}} \| f^{\circ n_j}(z) \| + \| z \| \leqslant R + R = 2R$$

Hence, as $n_j \longrightarrow \infty$, $\frac{\lambda}{n_j}(\frac{1}{\lambda^{n_j}} f^{\circ n_j} - Id) \longrightarrow 0$ and $\phi \circ f = \lambda\phi$.
For any n_j, $\phi_{n_j}(0) = 0$, $\phi'_{n_j}(0) = 1$ therefore $\phi(0) = 0$, $\phi'(0) = 1$ and so ϕ is a bijection near 0 that linearizes f. □

The following theorem is equivalent to the stability theorem.

Theorem 1.5.2. *Let $f : D(0, R) \longrightarrow \mathbb{C}$ be a holomorphic function fixing the origin such that $|\lambda| = |f'(0)| = 1$. Let $K = \{z \in D_R \backslash f^n(z) \in D(0, R); \forall n \geq 0\}$. f is linearizable if and only if $0 \in \mathring{K}$.*

Proof. Let U be the connected component of $\mathring{K}$ containing 0. $f(K) \subset K$ and, by the maximum principle, $f(\mathring{K}) \subset \mathring{K}$. Therefore $f(U) \subset U$. Since U is simply connected, by the conformal representation, there exists an isomorphism $\phi : U \longrightarrow D(0, 1)$ such that $\phi(0) = 0$. To show that this ϕ linearizes f, we will check the product $g := \phi \circ f \circ \phi^{-1}$.
$g : D(0, 1) \longrightarrow D(0, 1)$ is a holomorphic function fixing the origin and $g'(0) = f'(0) = \lambda$ with $|\lambda| = 1$. By Schwarz lemma, $|g(z)| \leq |z|$ and, since $|g'(0)| = 1$, then $g(z) = \lambda z$.
Reciprocly, if f is linearizable then, by the stability theorem, $0 \in \mathring{K}$. □

Corollary 1.5.1. *Let $f : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ be a holomorphic function such that 0 is a Cremer point. Then $\exists R > 0$ such that $\forall r < R$,*
a) we can find a point z and an integer n such that $|z| < r$ and $|f^{\circ n}(z)| > R$.
b) we can find a point z' and an integer n' such that $|z'| > R$ and $|f^{\circ n}(z')| < r$.

Proof. a) Since 0 is a Cremer point then f is not linearizable. Hence, by the above theorem, the dynamic of f is not stable and $\exists R > 0$ such that $\forall r < R$ and $|z| < r$, $|f^{\circ n}(z)| > R$.
b) We choose $R > 0$ such that $f^{-1} : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ be defined over D_R. f^{-1} is not linearizable and so the dynamic of f^{-1} is not stable and $\forall r < R$, we can find a point z and an integer n such that $|z| < r$ and $|(f^{-1})^n(z)| > R$.
If we set $z' := (f^{-1})^n(z)$, then $|z'| > R$ and $|f^{\circ n}(z')| < r$. □

Example 1.5.1. *Consider the quadratic polynomial P_λ defined by $P_\lambda(z) = \lambda z + z^2$; $|\lambda| = 1$. We define the filled-in-Julia set by*

$$K_\lambda := \{z \in \mathbb{C};\ (P_\lambda^{\circ n}(z))_n \text{ bounded}\}.$$

By theorem 1.5.2, P_λ is linearizable and 0 is a Siegle point if and only if $0 \in \mathring{K}_\lambda$. In this case, there exists a connected component U of $\mathring{K}$, containing 0. This connected component is said to be the Siegel disk.

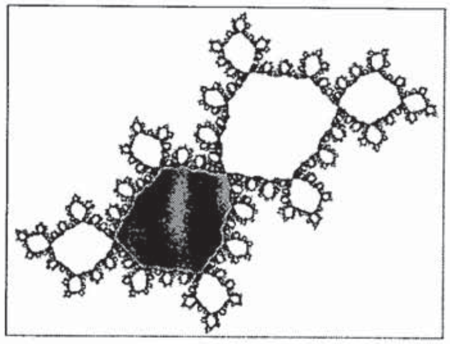


Figure 1.2: Filled-in-Julia set for $z^2 + e^{2i\pi\xi}z$ with $\xi = \sqrt[3]{\frac{1}{4}} = 0.62996....$ It has a Siegel disk.

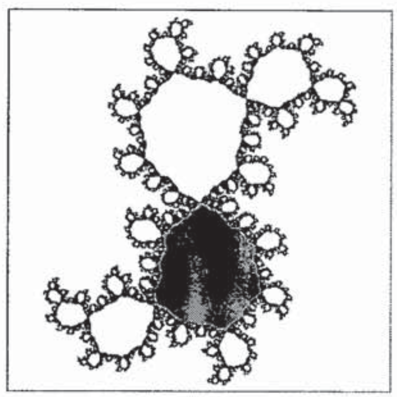


Figure 1.3: Filled-in-Julia set for $z^2 + e^{2i\pi\xi}z$ with $\xi = 0.7870595....$ It has a Siegel disk.

If $\lambda = e^{2i\pi\frac{p}{q}}$, 0 is a parabolic point and, in this case, P_λ is not linearizable. The point 0 is on the boundary of the filled-in-Julia set.

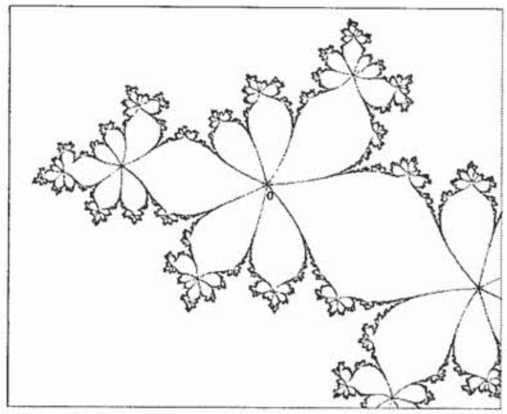


Figure 1.4: Filled-in-Julia set for $z \longrightarrow z^2 + e^{2\pi i t}z$ with $t = \frac{3}{7}$ which is not linearizable. 0 is on the boundary of K_λ.

1.6 Cremer Points

To show that Cremer points exist, we use Baire's theorem.

Theorem 1.6.1. ***Baire's Theorem***
Let X be a complete metric space. X is a Baire space [the union of any countable collection of closed sets $(F_n)_{n\geq 1}$ with empty interior has empty interior].

Proof. We want to show that $\cup F_n$ has empty interior. For this reason, we negate the following statement: $\exists x_0 \in \cup F_n$, $\exists r_0 > 0$ such that $\forall x \in X$ if $d(x_0, x) < r_0$ then $x \in \cup F_n$, which is equivalent to: $\forall x_0 \in X$, $\forall r_0 > 0$, $\exists x \in X$, $x \notin \cup F_n$ such that $d(x_0, x) < r_0$.
Let $x_0 \in X$ and $r_0 > 0$. We construct by recurrence $x_n \in X \setminus F_n$ and $r_n > 0$ such that $d(x_{n-1}, x_n) < r_n$ and $r_n < min(\frac{r_{n-1}}{2}, d(x_n, F_n))$. Since each F_n has empty interior, let

$$\begin{cases} x_1 \in X \setminus F_1 \, such \, that \, d(x_0, x_1) < r_1 < min(\frac{r_0}{2}, d(x_1, F_1)) < r_0 \\ x_2 \in X \setminus F_2 \, such \, that \, d(x_1, x_2) < r_2 < min(\frac{r_1}{2}, d(x_2, F_2)) < r_1 \\ x_3 \in X \setminus F_3 \, such \, that \, d(x_2, x_3) < r_3 < min(\frac{r_2}{2}, d(x_3, F_3)) < r_2 \end{cases}$$

Therefore, the terms are getting closer to each other. The sequence $(x_n)_{n\geq 0}$ is a Cauchy sequence and hence has a limit $x \in X$ (since X is a complete space). Hence, $\forall n \geq 0$, $d(x_n, x) < r_n$ and $x \notin F_n$, i.e., $x \notin \cup F_n$ and $d(x_0, x) < r_0$. □

Corollary 1.6.1. *Let X be a complete metric space with no isolated points. The intersection of any countable collection of open dense sets is a dense uncountable set.*

Proof. Let $(O_n)_{n\in\mathbb{N}}$ be the sequence of open dense sets. $F_n := X \setminus O_n$ is a closed set and has empty interior. By Baire's theorem, $\cup F_n$ has empty interior. Hence $O := \cap O_n = \cap(X \setminus F_n) = X \setminus \cup F_n$ is dense.
If O is countable, we can find an increasing sequence of finite sets $A_n \subset O$ such that $\cup A_n = O$. Hence $(O_n \setminus A_n)_{n\in\mathbb{N}}$ is a sequence of open dense sets since X has no isolated points and $\cap(O_n \setminus A_n) = \cap O_n \setminus \cup A_n = O \setminus O = \emptyset$. This is impossible since $\cap(O_n \setminus A_n)$ is dense. Therefore O is uncountable. □

Definition 1.6.1. *The intersection of open dense sets is called a residual set (dense and uncountable).*

Definition 1.6.2. *A function $f : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ has small cycles if $\forall r > 0$, there exists a periodic orbit $< z_0 \longrightarrow z_1 = f(z_0) \longrightarrow ... \longrightarrow z_n = f^n(z_0) \longrightarrow z_0 = f^{n+1}(z_0) >$ contained in the punctured disk D_r^* [any neighborhood of 0 contains a periodic cycle].*

Theorem 1.6.2. ***Cremer***
Let $P_\alpha(z) = e^{2i\pi\alpha}z + O(z^2)$, $\alpha \in \mathbb{R}$ be a polynomial of degree ≥ 2. Then the set of $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, such that P_α has small cycles, is a residual set.

We prove first the following lemma:

Lemma 1.6.1. *Let $\alpha_0 = \frac{p}{q}$ be a rational number and $r > 0$ a real number. $\exists\varepsilon > 0$ such that for $0 < |\alpha - \alpha_0| < \varepsilon$, P_α has a periodic orbit in the punctured disc D_r^*.*

Proof. $P_{\alpha_0}(z) = e^{2i\pi\alpha_0}z + O(z^2)$
$P_{\alpha_0}^{\circ q}(z) = e^{2i\pi\alpha_0 q}z + O(z^2) = e^{2i\pi\frac{p}{q}q}z + O(z^2) = e^{2i\pi p}z + O(z^2) = z + O(z^2)$
$P_{\alpha_0}^{\circ q} - z = O(z^2)$. Let $g(z) = P_{\alpha_0}^{\circ q}(z) - z$, a polynomial of degree > 1, and 0 is the only root of multiplicity $m \geq 2$ inside a certain path $\gamma = \partial D_r$. By Cauchy's Argument Principle

$$m = \frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}dz$$

Now, $P_\alpha^{\circ q}(z) = (e^{2i\pi\alpha})^q z + O(z^2) = \lambda^q z + O(z^2)$
$P_\alpha^{\circ q}(z) - z = (\lambda^q - 1)z + O(z^2)$

Let $g_\lambda(z) = P_\alpha^{\circ q}(z) - z$, a polynomial of degree > 1. Given $\varepsilon > 0$, if $0 < |\alpha - \alpha_0| < \varepsilon$, the 2 polynomials g(z) and $g_\lambda(z)$ are close to each other and, by Rouche's theorem, they have the same number of zeros inside γ. Let $r' > 0$, $\varepsilon > 0$ sufficiently small in such a way that if $|\alpha - \alpha_0| < \varepsilon$, $g_\lambda(z)$ has only zeros inside $\gamma = \partial D_{r'}$, i.e. $g_\lambda(z) \neq 0$ ($P_\alpha^{\circ q}(z) \neq z$), $\forall |z| = r'$ and, for $|z| < r'$, $|P_\alpha^i(z)| < r$, $i = 0, ..., q-1$.
$g_\lambda(z) = (\lambda^q - 1)z + O(z^2)$ is a polynomial of degree ≥ 2 that has at least m roots inside γ. 0 is a root and $\frac{g_\lambda(z)}{z} = (\lambda^q - 1) + O(z)$. $\lambda = e^{2i\pi\alpha}$ and α very close to $\alpha_0 = \frac{p}{q}$ so α is an irrational number and $\lambda^q = (e^{2i\pi\alpha})^q$ cannot be equal to 1 and so $\lambda^q - 1 \neq 0$. Therefore 0 is not a root for the polynomial $\frac{g_\lambda(z)}{z}$ and 0 is a simple root for $g_\lambda(z)$.
Hence, $g_\lambda(z)$ has m-1 roots different from 0 inside γ. If z_0 is a root then

$$g_\lambda(z_0) = P_\alpha^{\circ q}(z_0) - z_0 = 0$$

$$P_\alpha^{\circ q}(z_0) = z_0$$

Hence P_α has (m-1) cycles inside D_r^*.
Comment: $P_{\alpha_0}^q(z) = z + O(z^2)$ has 0 a fixed point of multiplicity $m \geq 2$ inside γ. By a small perturbation and for $\alpha \approx \alpha_0$, $P_\alpha^q(z) = \lambda^q(z) + O(z^2)$ still has m fixed points near 0. □

Now we prove Cremer's theorem:

Proof. $\forall n > 0$, let $U_n = \{\alpha \in \mathbb{R};\, P_\alpha \text{ has a cycle in } D^*_{\frac{1}{n}}\}$. U_n is an open set since $U_n = \bigcup_{\mathbb{Q}}]\frac{p}{q} - \varepsilon, \frac{p}{q} + \varepsilon[$. U_n is dense in $\mathbb{R}$ since $\mathbb{Q} \subset U_n$ and $\mathbb{Q}$ dense in $\mathbb{R}$. Let $G = decreasing \cap U_n$, by Baire's theorem, G is a residual set.

$$\begin{aligned} G &= \{\alpha \in \mathbb{R} \backslash P_\alpha\, has\, a\, cycle\, in\, all\, D^\alpha_{\frac{1}{n}}\} \\ &= \{\alpha \in \mathbb{R} \backslash P_\alpha\, has\, a\, cycle\, near\, 0\} \end{aligned}$$

where $\{D^\alpha_{\frac{1}{n}}\}_n$ consists of all neighberhoods of 0. □

Conclusion

Suppose P_α has a cycle $< z_0 \longrightarrow z_1 = P_\alpha(z_0) \longrightarrow ... \longrightarrow z_n = P_\alpha^n(z_0) \longrightarrow z_0 = P_\alpha^{n+1}(z_0) >$ and P_α linearizable then:

$$\phi \circ P_\alpha = g \circ \phi\, where\, g : z \longrightarrow \lambda z; \lambda = e^{2i\pi\alpha}, \alpha \in \mathbb{R} \setminus \mathbb{Q}$$

$$P_\alpha^{\circ q} = \phi^{-1} \circ g^{\circ q} \circ \phi$$

$$\phi^{-1} \circ g^{\circ q} \circ \phi(z_0) = z_0$$
$$g^{\circ q} \circ \phi(z_0) = \phi(z_0)$$

Let $w = \phi(z_0)$

$$g^{\circ q}(w) = w$$
$$\lambda^q w = w$$
$$\lambda^q = 1$$

This is a contradiction since $\lambda = e^{2i\pi\alpha}$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Hence f is not linearizable for certain $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and Cremer points exist.

Chapter 2

Fatou and Julia Sets

The Fatou set, Ω_f, and Julia set, J_f, are fascinating sets named after the Mathematicians Pierre Fatou and Gaston Julia. They are 2 dimensional images with fractal shapes. Ω_f and J_f form a partition for the Riemann Sphere. J_f is when the dynamic is chaotic and Ω_f is when the dynamic is regular. In this chapter, we will show some important results and will discover examples of these mathematical visualizations.

2.1 Dynamics on the Riemann Sphere

Definition 2.1.1. *The Fatou set Ω_f of the function f is the largest open set of the complex plane where the family $\{f^{\circ n}\}_n$ forms a normal family, $\Omega_f = \{z \in \mathbb{C} \backslash \exists U$ neighborhood of z; the family $f^n : U \longrightarrow P^1$ normal$\}$. The complement of the Fatou set is called the Julia set J_f.*

Proposition 2.1.1. *The Julia set and Fatou set are completely invariant by f: $f(J_f) = J_f = f^{-1}(J_f)$, $f(\Omega_f) = \Omega_f = f^{-1}(\Omega_f)$.*

Proof. Since the Julia set is the complement of the Fatou set, it is enough to prove the result for the Fatou set. Lets prove that $f^{-1}(\Omega_f) = \Omega_f = f(\Omega_f)$.
We prove first that $f^{-1}(\Omega_f) \subset \Omega_f$:
Let $z_0 \in f^{-1}(\Omega_f) \Rightarrow w_0 = f(z_0) \in \Omega_f$. Then the sequence $f^{\circ(n-1)}(w_0) = f^{\circ n}(z_0)$ is a normal family near w_0 and so near z_0. Therefore $z_0 \in \Omega_f$ and $f^{-1}(\Omega_f) \subset \Omega_f$.
We now prove that $f(\Omega_f) \subset \Omega_f$:
Let $z_0 \in f(\Omega_f) \Rightarrow z_0 = f(w_0)$, $w_0 \in \Omega_f$. Then the sequence $f^{\circ n}(w_0)$ is a

normal family near w_0.
$f^{\circ n}(w_0) = f^{\circ(n-1)}(f(w_0)) = f^{\circ(n-1)}(z_0)$ which is a normal family near z_0. Therefore $z_0 \in \Omega_f$ and $f(\Omega_f) \subset \Omega_f$.
We proved that $f^{-1}(\Omega_f) \subset \Omega_f$ then $\Omega_f \subset f(\Omega_f)$ and since $f(\Omega_f) \subset \Omega_f$, therefore $f(\Omega_f) = \Omega_f$.
Also, $f(\Omega_f) \subset \Omega_f$ then $\Omega_f \subset f^{-1}(\Omega_f)$ and since $f^{-1}(\Omega_f) \subset \Omega_f$, therefore $\Omega_f = f^{-1}(\Omega_f)$. □

Theorem 2.1.1. ***Montel's Theorem (Fundamental Normality Test)*** *A family $F = \{f_\alpha : U \longrightarrow P^1\}_{\alpha \in A}$ of holomorphic functions which omits at least 3 different values in P^1, (i.e, there exist distinct points $a, b, c \in P^1$ so that $f_\alpha(U) \subset P^1 \setminus \{a, b, c\}$ for every $f_\alpha \in F$) is normal.*

Definition 2.1.2. *Two rational fractions are said to be conjugates if there exists a holomorphic function $h : P^1 \longrightarrow P^1$ such that $h \circ f = g \circ h$.*

Proposition 2.1.2. *If $h : P^1 \longrightarrow P^1$ is a Moebius transformation then $h(\Omega_f) = \Omega_g$ and $h(J_f) = J_g$.*

Proof. The family $\{f^{\circ n} : U \longrightarrow P^1\}$ is normal if and only if the family $\{h \circ f^{\circ n} \circ h^{-1} : h(U) \longrightarrow P^1\}$ is normal. □

When the degree of $f < 2$, f is either a constant or the Moebius transformation $z \longrightarrow \frac{az+b}{cz+d}$. In these cases, the study of holomorphic dynamics is very simple.

Theorem 2.1.2. *Suppose f is a Moebius transformation different from the identity ($f(z) \neq z$), $f : z \longrightarrow \frac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$.*
Then f has either one fixed point in P^1 and is so conjugate to a translation, or two fixed points in P^1 and is so conjugate to a similarity or to a rotation.

Proof. Let $f(z) = \frac{az+b}{cz+d}$.
Note that $f(\infty) = \begin{cases} \frac{a}{c} & c \neq 0 \\ \infty & c = 0 \end{cases}$
Suppose z_0 is a fixed point of f; $f(z_0) = z_0$. $\frac{az_0+b}{cz_0+d} = z_0$ is a polynomial of degree 2. Then either $\Delta = 0$ and so we have 1 fixed point or $\Delta \neq 0$ and so we have 2 fixed points.
<u>**Case 1:**</u> if f has only one fixed point $\alpha \in P^1$:

- If $\alpha = \infty$ then $c = 0$ and $f(z) = \frac{az+b}{d}$ but since it is the only one then $a = d$. Hence $f(z) = z + k$, $k \neq 0$ is a translation and $f^n(z_0) = z_0 + nk \longrightarrow \infty\, as\, n \longrightarrow \infty\ \forall z_0$. Therefore, $\Omega_f = P^1$ and $J_f = \emptyset$.

- If $\alpha \neq \infty$, let $\phi(z)$ be a holomorphic function that takes f to g where $g(\infty) = \infty$ (g is a translation).

$$\phi \circ f \circ \phi^{-1}(\infty) = \infty$$

$$f \circ \phi^{-1}(\infty) = \phi^{-1}(\infty)$$

$$\Rightarrow \alpha = \phi^{-1}(\infty)$$

$$\phi(\alpha) = \infty$$

$$\phi(z) = \frac{1}{z - \alpha}$$

In this case $f^n(z_0) \longrightarrow \alpha$ as $n \longrightarrow \infty\ \forall z_0$ and $J_f = \emptyset$.

<u>**Case 2:**</u> if f has exactly 2 fixed points:

- If $\alpha_1 = 0$, $\alpha_2 = \infty$, then $f(z) = kz$ and

$$\begin{aligned} |f^n(z_0)| = |k^n z_0| &\longrightarrow 0\, as\, n \longrightarrow \infty \forall z_0,\ if\, |k| < 1 \\ &\longrightarrow \infty\, as\, n \longrightarrow \infty \forall z_0,\ if\, |k| > 1 \\ &= |z_0|,\ the\, iterations\, remain\, in\, S^1. \end{aligned}$$

In this case, $J_f = S^1$.

- If $\alpha_1, \alpha_2 \neq 0, \infty$, let ϕ be a holomorphic function that takes f to g where $g(0) = 0$, $g(\infty) = \infty$ and so $g(z) = kz$.

$$\phi \circ f \circ \phi^{-1}(\infty) = \infty$$

$$f \circ \phi^{-1}(\infty) = \phi^{-1}(\infty)$$

$$\Rightarrow \alpha_2 = \phi^{-1}(\infty)$$

$$\phi(\alpha_2) = \infty$$

and

$$\phi \circ f \circ \phi^{-1}(0) = 0$$

$$f \circ \phi^{-1}(0) = \phi^{-1}(0)$$

$$\Rightarrow \alpha_1 = \phi^{-1}(0)$$
$$\phi(\alpha_1) = 0$$
$$\Longrightarrow \phi(z) = \frac{z - \alpha_1}{z - \alpha_2}$$

In this case, by proposition 2.1.2, $J_f = S^1$.

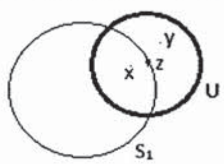
□

Next, we will be interested in rational fractions of degree ≥ 2. For certain polynomials, the dynamic is easily understood.

Theorem 2.1.3. *Let $f(z) = z^d$, where $d \geqslant 2$ or $d \leqslant -2$. The Julia set of f J_f is $S^1 := \{z \in \mathbb{C}; |z| = 1\}$, the unit circle of $\mathbb{C}$.*

Proof.
- $\forall z$; $|z| < 1$, $f^{\circ n}(z) = z^{d^n}$ and so as $n \longrightarrow \infty$, $|f^{\circ n}(z)| \longrightarrow 0$ and $z \in \Omega_f$.
- $\forall z$; $|z| > 1$, $|f^{\circ n}(z)| \longrightarrow \infty$ as $n \longrightarrow \infty$ and $z \in \Omega_f$.

Hence $J_f \subset S^1$. We still have to show that $S^1 \subset J_f$.
Now let $z \in S^1$ and U a small neighborhood of z then there exist $x, y \in U$ where x inside the disc D and y outside the disc D.



$|f^{\circ n}(z)| \longrightarrow 1$ as $n \longrightarrow \infty$.
$|f^{\circ n}(x)| \longrightarrow 0$ as $n \longrightarrow \infty$.
$|f^{\circ n}(y)| \longrightarrow \infty$ as $n \longrightarrow \infty$.
Therefore, the sequence $f^{\circ n}(z)$ does not converge and $z \in J_f$. Hence $J_f = S^1$. □

Remark 2.1.1. *Another reason to say that $J_f \subset S^1$ using Montel's theorem is that $f^n(P^1 \setminus S^1) \subset P^1 \setminus S^1$ and so the sequence $f^n|_{P^1 \setminus S1}$ omits at least 3 points. Hence $P^1 \setminus S^1 \subset \Omega_f$ and $J_f \subset S^1$.*

Definition 2.1.3. *The Chebyshev polynomials $f_d(z)$ of the first kind of degree d are polynomials satisfying $f_d(cos x) = cos(dx)$. They satisfy the recurrence relation $f_{d+1}(z) = 2zf_d(z) - f_{d-1}(z)$, $d \geq 1$.*
An example for d=2 is $f_2(z) = 2z^2 - 1$.

Proposition 2.1.3. *If f is a Chebyshev polynomial of degree d, then the set of points where the orbit is bounded is the interval [-1,1].*

Proof. We have z=a+ib. Hence, the proof is divided into 2 cases:
<u>Case 1:</u> b=0, i.e. $z \in \mathbb{R}$.

- If $|z| \leq 1$, then z can be written as cos x. Hence
  $f_d(z) = f_d(\cos x) = cos(dx)$
  $f_d^2(z) = f_d^2(\cos x) = f_d(\cos dx) = cos(d^2x)$
  $\vdots$
  $f_d^n(z) = f_d^n(\cos x) = cos(d^n x) \in [-1, 1]$
- If $|z| > 1$, $f_d(z)$ is a polynomial of degree d: $f_d(z) = a_d z^d + a_{d-1}z^{d-1} + \cdots + a_2z^2 + a_1z + a_0$ and so $f_d^n(z)$ is a polynomial of degree d^n which converges to ∞ as $n \longrightarrow \infty$.

<u>Case 2:</u> $b \neq 0$, i.e. $z \in \mathbb{C}$.
Any $z \in \mathbb{C}$ can be written as: $z = cosw = cos(a + ib) = cosacoshb - isinasinhb$. Hence $f_d^n(z) = f_d^n(cosw) = cos(d^n w)$ where

$$cos(d^n w) = cos(d^n a)cosh(d^n b) - isin(d^n a)sinh(d^n b)$$

$$\begin{aligned}
|cos(d^n w)|^2 &= cos^2(d^n a)cosh^2(d^n b) + sin^2(d^n a)sinh^2(d^n b) \\
&= cos^2(d^n a)cosh^2(d^n b) + (1 - cos^2(d^n a))sinh^2(d^n b) \\
&= cos^2(d^n a)(cosh^2(d^n b) - sinh^2(d^n b)) + sinh^2(d^n b) \\
&= cos^2(d^n a) + sinh^2(d^n b) \\
&= (\frac{e^{id^n a} + e^{-id^n a}}{2})^2 + (\frac{e^{d^n b} - e^{-d^n b}}{2})^2 \\
&\longrightarrow \infty \, as \, n \longrightarrow \infty
\end{aligned}$$

Hence $f_d^n(z)$ converges to ∞ as $n \longrightarrow \infty$. Therefore, $\Omega_f = P^1$ and $J_f = \emptyset$. □

In most cases, the J_f has a fractal shape.

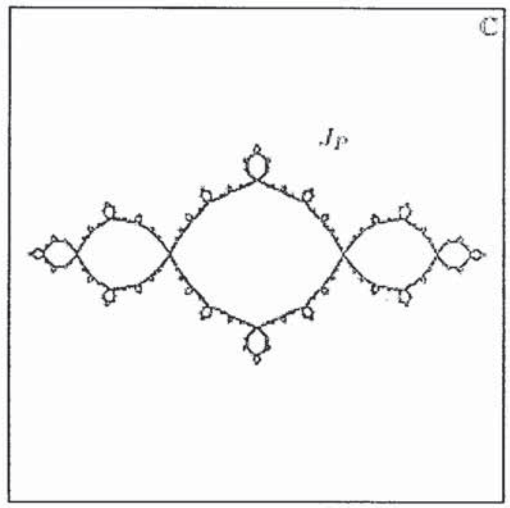


Figure 2.1: Julia Set of the Quadratic Polynomial $P(z) = z^2 - 1$ in $\mathbb{C}$.

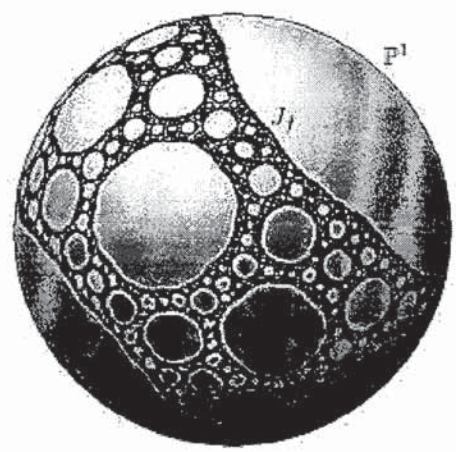


Figure 2.2: Julia Set of the Rational Fraction $f(z) = \frac{1}{2}(z^2 - \frac{1}{z^2})$ in P^1.

2.2 Periodic Points

We introduce in this setion the definition of periodic points and their relation with Fatou and Julia sets.

Definition 2.2.1. *A periodic point of f is a point α such that $f^{\circ p}(\alpha) = \alpha$ for $p \geq 1$. In this case we say that $\{\alpha, f(\alpha), ..., f^{\circ(p-1)}(\alpha)\}$ is a cycle (periodic orbit). If $p \geq 1$ is the smallest integer such that $f^{\circ p}(\alpha) = \alpha$, then p is called the period of α.*
If the cycle does not contain ∞, the product of the derivatives of f throughout the cycle is called the multiplier of the periodic orbit:

$$\lambda := [f^{\circ p}]'(\alpha) = \prod_{i=0}^{p-1} f'(f^{\circ i}(\alpha))$$

The periodic orbit is said to be

- *superattractive if $\lambda = 0$;*
- *attractive if $0 < |\lambda| < 1$;*
- *indifferent if $|\lambda| = 1$;*
- *repulsive if $|\lambda| = 1$.*

Proposition 2.2.1. *The multiplier of a cycle is invariant by analytic conjugacy. In fact, if $h : P^1 \longrightarrow P^1$ is an isomorphism that conjugates $f : P^1 \longrightarrow P^1$ to $g : P^1 \longrightarrow P^1$ and if α is a periodic point of f, then $\beta = h(\alpha)$ is a periodic point of g.*

Proof. $f^{\circ p}(\alpha) = \alpha$ and $g^{\circ p} = h \circ f^{\circ p} \circ h^{-1}$ Hence

$$g^{\circ p}(h(\alpha)) = h \circ f^{\circ p} \circ h^{-1}(h(\alpha)) = h \circ f^{\circ p}(\alpha) = h(\alpha)$$

Also, h is a bijection between the orbit $\{\alpha, f(\alpha), ..., f^{\circ(p-1)}(\alpha)\}$ and the orbit $\{\beta, g(\beta), ..., g^{\circ(p-1)}(\beta)\}$. Therefore, the two cycles have the same period p. Finally, $[g^{\circ p}]'(\beta) = [g^{\circ p}]'(h(\alpha)) = \frac{[g^{\circ p} \circ h]'(\alpha)}{h'(\alpha)} = \frac{[h \circ f^{\circ p}]'(\alpha)}{h'(\alpha)} = \frac{h'(f^{\circ p}(\alpha))[f^{\circ p}]'(\alpha)}{h'(\alpha)} = \frac{h'(\alpha)[f^{\circ p}]'(\alpha)}{h'(\alpha)} = [f^{\circ p}]'(\alpha)$. Hence the multipliers are equal. □

Proposition 2.2.2. *Let $f : P^1 \longrightarrow P^1$ be a rational fraction and let α a periodic point of f. If α is superattractive or attractive then $\alpha \in \Omega_f$.*
If α is repulsive or indifferent then $\alpha \in J_f$.

Proof.

- If α is (super)attractive, $\exists U$ neighborhood of α such that $f(U) \subset U$ and so the family of iterations $(f^n|_U)_n$ is uniformly bounded. Therefore $\alpha \in \Omega_f$.

- If α is repulsive, $(f^{\circ pn})'(\alpha) = \lambda^n \longrightarrow \infty$ as $n \longrightarrow \infty$ and so f^{pn} cannot converge to a holomorphic function near α and therefore $\alpha \in J_f$.

- If α is indifferent then, composing with a Moebius transformation, we may assume that the indifferent fixed point is 0. Then there is some iterate $f^{\circ n}$ that has power series expansion $z + a_m z^m + a_{m+1} z^{m+1} + \ldots$ near 0 where $a_m \neq 0$ and $m \geq 2$. The iterates $f^{\circ(jn)} = (f^{\circ n})^j$ have the power series expansion $z + j a_m z^m + \ldots$ near 0. Then the m^{th} derivative of $f^{\circ(jn)}$ at 0 is equal to $j a_m m!$ and hence converges to ∞ as $j \longrightarrow \infty$. This implies that the sequence of iterations $f^{\circ jn}$ cannot converge to a holomorphic function near 0. Therefore $0 \in J_f$.

□

2.3 Description of the Julia Set

In this section, we give some important properties concerning the topology of Julia set.

Proposition 2.3.1. *The Fatou set is an open set.*

Proof. Let $z_0 \in \Omega_f$. Then $\exists r > 0$ such that $f^{\circ n}|_{B_r(z_0)}$ is normal, i.e., it has a subsequence that converges uniformly at any z near z_0. Therefore, $B_r(z_0) \subset \Omega_f$. □

Proposition 2.3.2. *The Julia set is a closed nonempty and compact set.*

Proof. If the Julia set was empty then $\Omega_f = P^1$. Hence, the family $\{f^{\circ n}\}_n$ forms a normal family over P^1 and there exists a subsequence $f^{\circ m_i}$ that converges uniformly on P^1. Let f_0 be its limit. The degree of f_0 is finite. For large i, the degree of the map $f^{\circ m_i}$ is equal to the degree of f_0. However, the degree of $f^{\circ m_i}$ is d^{m_i} which converges to ∞ as $n \longrightarrow \infty$. Contradiction. The Fatou set is open. Hence its complement, the Julia set, is closed in P^1 which is compact. Therefore the Julia set is a compact set. □

Proposition 2.3.3. *Let E be a completely invariant set. Then E is either infinite or contains a maximum of two points.*

Let's prove first the following lemma:

Lemma 2.3.1. *If a is a multiple root of $f(z) = 0$ then a is a critical point of f; a is a root of $f'(z) = 0$.*

Proof. Suppose a is a root of $f(z) = 0$ of multiplicity 2 then $f(z) = (z-a)^2 g(z)$. Hence $f'(z) = 2(z-a)g(z) + (z-a)^2 g'(z)$ and so a is also a root of $f'(z) = 0$. Hence a is a critical point of f. □

We will now prove proposition 2.3.3:

Proof. Let E be a completely invariant set, i.e., $f(E) = E = f^{-1}(E)$. Suppose that E is a finite set. Since f is onto, each point of E has at least one preimage and two distinct points of E have distinct preimages. Hence the number of points in $f^{-1}(E)$ is greater than or equal to the number of points in E. Since $f^{-1}(E) = E$ each point of E has exactly one preimage. Hence f is one-to-one. Therefore $f : E \longrightarrow E$ is bijective; it is a permutation of points in E.

Since each point of E has only one preimage, and since the degree of f is ≥ 2, this preimage is a multiple root of degree d for f. Hence E consists only of critical points for f.

If $f : E \longrightarrow E$ has a fixed point, it can be placed in a coordinate where ∞ is this point. Since ∞ has only itself as a preimage then the expression of f in this coordinate is a polynomial.

If E has another fixed point, f can be placed in a coordinate where 0 is this point. Since f(0) has only 0 as a preimage then $f(z) = az^d + f(0)$, $a \neq 0$ and since f(0) must be a critical point of f, then $f(z) = az^d$. In this case, E contains 2 points.

If E contains an orbit of period 2, f can be placed in a coordinate where the two points of the orbit are 0 and ∞. Since 0 is the only preimage of ∞ and ∞ is the only preimage of 0 then $f(z) = \frac{a}{z^d}$, $a \neq 0$. In this case also f has no other critical points and E contains only 2 points.

Finally, if E contains an orbit of period ≥ 3, lets suppose that it contains $1 \longrightarrow 0 \longrightarrow \infty$. In this case, since 1 is the only preimage of 0 and 0 is the only preimage of ∞, we have $f(z) = \frac{a(z-1)^\alpha}{z^\alpha}$ where $a \neq 0$. Now $f(\infty) = a$ which is impossible since ∞ is a critical point of f. □

Definition 2.3.1. *The exceptional set E_f is the largest finite set completely invariant by f.*

Corollary 2.3.1. *The exceptional set E_f is nonempty in precisely 2 cases:*

- *when f is conjugate to $z \longrightarrow z^d$, $|d| \geq 2$; $E_f = \{0, \infty\}$.*
- *when f is conjugate to a polynomial but not to z^d; $E_f = \{\infty\}$.*

Since the exceptional set is formed by superattractive orbits $(\lambda = [f^{\circ p}]'(\alpha) = 0)$, then it is in the Fatou set.

Proposition 2.3.4. *The Julia set is the smallest closed completely invariant set containing at least 3 points.*

Proof. The Julia set is closed and completely invariant (already proved). If it didn't contain at least 3 points, it would be finite and so included in the exceptional set which is in the Fatou set. Contradiction.
Also, if E is a closed completely invariant set containing at least 3 points, then its complement $\Omega = P^1 \setminus E$ is an open completely invariant set omiting at least 3 points. By Montel's theorem, the family of iterations $f^{\circ n}|_\Omega$ is normal. Hence $\Omega \subset \Omega_f$ and $J_f \subset E$. □

Corollary 2.3.2. *If 2 rational fractions $f : P^1 \longrightarrow P^1$ and $g : P^1 \longrightarrow P^1$ are conjugates by a homeomorphism h, then $h(\Omega_f) = \Omega_g$ and $h(J_f) = J_g$.*

Proof. The homeomorphism h sends completely invariant sets by f to completely invariant sets by g since $h \circ f = g \circ h$ and it sends closed sets to closed sets. Hence, the smallest closed set completely invariant by f containing at least 3 points is sent to the smallest closed set completely invariant by g containing at least 3 points. Hence $h(J_f) = J_g$. Taking the complementary we get $h(\Omega_f) = \Omega_g$. □

Proposition 2.3.5. *$\forall z \notin E_f$, we have $J_f \subset \overline{\bigcup_{n\geq 1} f^{-n}(z)}$. This means that there exists an exceptional set E_f depending only on f such that $J_f \subset \overline{\mathbb{C}} \setminus E_f \subset \bigcup_{n\geq 1} f^n(U)$ where U is any open set intersecting J_f.*

Proof. Let $z_0 \in J_f$ and U a neighborhood of z_0. We need to show that $z_0 \in \overline{\bigcup_{n\geq 1} f^{-n}(z)}$, i.e., that U contains points from $\bigcup_{n\geq 1} f^{-n}(z)$. We set $\Omega = \bigcup_{n\in\mathbb{N}} f^{\circ n}(U)$ and to prove that $z \in \Omega$, we show that $E = P^1 \setminus \Omega$ is in E_f.
In fact, Ω is an open invariant set that omits at most 2 points, since otherwise, the family of iterations $f^{\circ n}|_\Omega$ is normal. Hence, E contains at most 2 points and $f^{-1}(E) = E = f(E)$. Therefore $E \subset E_f$. □

Corollary 2.3.3. *If $z \in J_f$ then $J_f = \overline{\underset{n\geq 1}{\cup} f^{-n}(z)}$.*

Proof. If $z \in J_f$, then $z \notin E_f$, therefore $J_f \subset \overline{\underset{n\geq 1}{\cup} f^{-n}(z)}$. On the other hand, since $z \in J_f$ then $f^{-1}(z) \subset J_f$ and so $\underset{n\geq 1}{\cup} f^{-n}(z) \subset J_f$. Therefore $\overline{\underset{n\geq 1}{\cup} f^{-n}(z)} \subset \overline{J_f} = J_f$ (since the Julia set is closed). □

Theorem 2.3.1. *J_f has either an empty interior or is equal to the entire Riemann Sphere.*

Proof. Suppose that J_f contains interior points and let $U \subset J_f$ an open set. Then the family of iterations $\{f^{\circ m} : U \longrightarrow P^1\}$ omits at most 2 points a and b. Therefore

$$P^1 \setminus \{a,b\} \subset \underset{n\geq 1}{\cup} f^{\circ n}(U) \subset J_f$$

$$\overline{P^1 \setminus \{a,b\}} \subset \overline{J_f} = J_f$$

$$P^1 \subset J_f$$

Therefore $J_f = P^1$. □

Theorem 2.3.2. *The Julia set is nonempty and perfect; it has no isolated points.*

Proof. The Julia set is infinite since the only possibilities for finite completely invariant sets are (up to conjugacy) the sets $\{\infty\}$ or $\{\infty, 0\}$ and they are contained in the Fatou set.
To prove that J_f is perfect, we show that any point $a \in J_f$ is not isolated, i.e. there exists U a neighborhood of a containing points of J_f other then a.
<u>**Case 1:**</u> if a is not periodic, $f^{\circ p}(a) \neq a$, $\forall p \in \mathbb{N}$:

$$\begin{aligned} a \in J_f &\Rightarrow a \in \underset{n\geq 1}{\cup} f^n(U) \\ &\Rightarrow a \in f^N(U)\, for\, a\, certain\, N \\ &\Rightarrow a = f^N(b),\, b \neq a,\, b \in U \\ &\Rightarrow b = f^{-N}(a) \in J_f \end{aligned}$$

Then $U \cap J_f \supset \{a, b\}$. Hence J_f is a perfect set.
<u>**Case 2:**</u> if a is periodic, $f^{\circ p}(a) = a$ for certain p:
Since $J_f = J_{f^n}$ we can replace f by f^n and suppose that f(a)=a and degree

of f is ≥ 4. We have to prove that $a = f^N(b)$, $a \neq b$. a is not a critical point since otherwise it would be contained in Ω_f and so $a = f(a)$ is not a critical value and it has at least 3 different preimages b_1, b_2, $b_3 \neq a$. Since the degree of f is ≥ 4, therefore at least one of the $b_i's$ is not in E_f. Suppose that $b_1 \notin E_f$, we can proceed as in case 1, there exists $b \in U$; $f^N(b) = b_1$, $b \neq a$. □

2.4 Julia Set and Periodic Points

Proposition 2.4.1. *The Julia set is contained in the closure of the set of periodic points.*

Proof. Let $z_0 \in J_f$ and U a neighborhood of z_0. We prove that U contains a periodic point of f.
Since J_f is a perfect set, we can assume that z_0 is not a critical value for $f^{\circ 2}$ (without loss of generality, since there are only a finite set of critical values).
Let U the neighborhood of z_0 sufficiently small so that 3 distinct branches of f^{-2} are defined. Denote these by $g_1 : U \longrightarrow U_1$, $g_2 : U \longrightarrow U_2$ and $g_3 : U \longrightarrow U_3$ where U_1, U_2, U_3 are disjoint.
Suppose (for a contradiction) that U contains no periodic points of f. For each $z \in U$, set

$$h_m(z) := \frac{f^{\circ m}(z) - g_1(z)}{f^{\circ m}(z) - g_2(z)} / \frac{g_3(z) - g_1(z)}{g_3(z) - g_2(z)}$$

If $f^{\circ m}(z) \neq g_i(z)$, i=1,2,3, then $h_m(z) \neq 0, 1, \infty$ for $z \in U$ (else f would have a periodic point). So by Montel's theorem, $\{h_m\}$ forms a normal family. Hence $\{f^{\circ m}\}$ is normal since

$$f^{\circ m} = \frac{g_1(g_3 - g_2) - g_2(g_3 - g_1)h_m}{(g_3 - g_2) - (g_3 - g_1)h_m}$$

Contradiction since $z_0 \in J_f$.
Hence, in each neighborhood of z_0, $\exists$ solutions of $f^{\circ m}(z) = g_i(z)$ for an i and an m, hence solutions for $f^{\circ(m+2)}(z) = z$. □

Theorem 2.4.1. ***Fatou-Shishikura inequality*** *[7]*
A rational fraction has only a finite number of non repelling orbits.

Corollary 2.4.1. *The Julia set is equal to the closure of the set of all repelling periodic points of f.*

Proof. $J_f \subset \overline{\{periodic\, points\}} = \overline{\{repulsive\, points\}} \cup \overline{\{non\, repulsive\, points\}}$ where the set of non repulsive points is finite. Since J_f has no isolated points, $J_f \subset \overline{\{periodic\, repulsive\, points\}}$. on the other hand, $\overline{\{periodic\, repulsive\, points\}} \subset J_f$. Hence $J_f = \overline{\{periodic\, repulsive\, points\}}$. □

Appendix

This appendix will describe some theorems regarding Fatou and Julia sets.

Definition 2.4.1. *Consider the polyomial $P : \mathbb{C} \longrightarrow \mathbb{C}$. The filled Julia set of P is the set K_P of points having a bounded orbit:*

$$K_P = \{z \in \mathbb{C} \mid (P^{\circ n}(z))_{n \in \mathbb{N}} \, is \, bounded\}$$

Proposition 2.4.2. *The filled Julia set K_P is a nonempty compact set. The Julia set is the boundary of K_P.*

Proposition 2.4.3. *The complementary of the filled Julia set is connected.*

Theorem 2.4.2. *A (super)attractive basin always contains at least one critical point.*

Theorem 2.4.3. ***Fatou***
A connected component of Ω_f is either a (super)attractive basin, a parabolic basin, a Siegel disc or a Herman ring.

Proposition 2.4.4. *Every polynomial of degree 2 is conjugate by an affine function to a unique polynomial of the form $P_c : z \longrightarrow z^2 + c$.*

Definition 2.4.2. *We denote by J_c and K_c the Julia set and filled Julia set of the polynomial $P_c : z \longrightarrow z^2 + c$.*

Theorem 2.4.4. *If $c \in K_c$ then the sets J_c and K_c are connected. If $c \notin K_c$ then $J_c = K_c$ is a Cantor set.*

Definition 2.4.3. *The Mandelbrot set M is the set*

$$M := \{c \in \mathbb{C};\, c \in K_c\}$$

Proposition 2.4.5. *If P_c has an attractive orbit, then c is in the interior of the Mandelbrot set.*

Conjecture 2.4.1. *If c is in the interior of the Mandelbrot set, P_c has a periodic attractive point.*

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