INTEGRAL EVALUATION THROUGH THE BINOMIAL FORMULA AND DIFFERENTIATION TECHNIQUES

A Thesis

presented to

the Faculty of Natural and Applied Sciences

at Notre Dame University-Louaize

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Mathematics

by

GHEWA ZGHEIB

JUNE 2024

Notre Dame University-Louaize,
Zouk Mosbeh, Lebanon

Department of Mathematics and Statistics

Ghewa Zgheib

We, the thesis committee for the above candidate for the Master of Science degree, hereby recommend acceptance of this thesis.

Roger Nakad – Thesis Advisor
Department of Mathematics and Statistics

George Eid – First Reader
Department of Mathematics and Statistics

Roger Nakad
Dean of the Faculty of Natural and
Applied Sciences

Abstract of the Thesis

Integral Evaluation through the Binomial Formula and Differentiation Techniques

by

Ghewa Zgheib

Master of Science

in

Mathematics

Notre Dame University-Louaize,

Zouk Mosbeh,
Lebanon

2024

In the realm of mathematics, the evaluation of integrals plays a crucial role in various fields such as physics, engineering, and economics. Integrals represent the accumulation of quantities over a continuous range and are essential for understanding the behavior of functions and solving complex problems. Over the years, mathematicians and researchers have developed numerous techniques to evaluate integrals efficiently and accurately. This thesis consists of two parts. In the first part, we present some binomial identities,

special numbers and polynomials as well as basic formulas related to Euler's transformation of series, Handamard's series multiplication theorem and several transformation formulas with example and applications. These theoretical tools will be used to gain a better understanding of the second part of this thesis which explores two different techniques for evaluating integrals. The first technique uses a special formula to transform integrals to series. The resulting series involves binomial transforms with the Taylor coefficients of the integral. The second technique is by differentiation with respect to a parameter.

To my father...

Contents

Acknowledgements

I am truly at a loss for words to adequately convey the depth of my gratitude towards my advisor, Dr. Roger Nakad. During my academic journey, for the past five years at NDU, it is impossible to fully capture the exceptional qualities that make Dr. Nakad not only an outstanding advisor but also an extraordinary individual. His insightful encouragement, feedbacks and advices have been instrumental in shaping this thesis and will remain unforgettable life lessons throughout my future.

Besides my advisor, I would like to thank Dr. Georges Eid as the first reader of this thesis whose huge experience impacted my journey in this field. I would also like to thank everyone who taught me a class at NDU, it was an honor learning from all of you.

I thank my friends for supporting me all the way.

Last but not least, thank you to all my family, my parents, my uncle and my sisters for never leaving my side. Everything I do stemmed from you, and is for you.

Chapter 1

Mathematical Preliminaries

In this chapter, we recall basic definitions of binomial transforms. For more details, we refer to [1].

1.1 Some Binomial Identities

The binomial coefficients formula is given by

$$\binom{n}{k} = \begin{cases} \frac{p(p-1)\cdots(p-k+1)}{k!} & k \geq 0, \\ 0 & k < 0, \end{cases}$$

where p does not need to be an integer.

Definition 1.1.1. *Let $\{a_k\}_k$ be a sequence where $k = 0, 1, 2, \cdots$. Its binomial transform is the new sequence $\{b_n\}_n$ where $n = 0, 1, 2, \cdots$ generated by the formula:*

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k \tag{1.1.1}$$

Proposition 1.1.2. *Let $\{a_k\}_k$ be a sequence where $k = 0, 1, 2, \cdots$, and denote by $\{b_n\}_n$ Its binomial transform. The inversion of $\{b_n\}_n$ is given by*

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_k, \tag{1.1.2}$$

which can also be written as,

$$(-1)^n a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k b_k$$

Proof. We have $b_k = \sum_{j=0}^{k} \binom{k}{j} a_j$, then,

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_k &= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \left\{ \sum_{j=0}^{k} \binom{k}{j} a_j \right\} \\
&= (-1)^n \sum_{j=0}^{n} a_j \left\{ \sum_{k=j}^{n} \binom{n}{k} \binom{k}{j} (-1)^k \right\} \\
&= (-1)^n \sum_{j=0}^{n} a_j (-1)^j \delta_{nj} \\
&= a_n,
\end{aligned}$$

where we used the convolution identity

$$\sum_{k=j}^{n} \binom{n}{k} \binom{k}{j} (-1)^k = (-1)^j \delta_{nj} \tag{1.1.3}$$

with δ_{nj} being the Kronecker symbol. □

Proposition 1.1.3. *Let $\{b_k\}$ be the sequence defined by:*

$$b_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k. \tag{1.1.4}$$

We have:

$$\frac{b_{n+1}-a_0}{n+1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k+1} \frac{a_{k+1}}{k+1} \tag{1.1.5}$$

Proof. We have

$$\begin{aligned}
b_{n+1} &= \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k a_k \\
&= \sum_{k=1}^{n+1} \binom{n+1}{k} (-1)^k a_k + a_0 \\
b_{n+1} - a_0 &= \sum_{l=0}^{n} \binom{n+1}{l+1} (-1)^{l+1} a_{l+1} \qquad ;where\ \ l = k-1 \\
b_{n+1} - a_0 &= \sum_{l=0}^{n} \frac{n+1}{l+1} \binom{n}{l} (-1)^{l+1} a_{l+1} \qquad ;where \binom{n+1}{l+1} = \frac{n+1}{l+1}\binom{n}{l} \\
\frac{b_{n+1}-a_0}{n+1} &= \sum_{l=0}^{n} \binom{n}{l} (-1)^{l+1} \frac{a_{l+1}}{l+1}
\end{aligned}$$

□

Remark 1. If the sequence $\{a_k\}$ is indexed from $k = 1, 2, \ldots,$ we can assume that $a_0 = 0$ and use the same previous formulas. In this case, we also have $b_0 = 0$.

2. The inversion formula of (1.1.4) is given by

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^n b_k \tag{1.1.6}$$

Here the factor $(-1)^k$ can be replaced by $(-1)^{k-1}$. In fact, the inversion formula (1.1.6) follows from (1.1.2) as we can write

$$(-1)^n a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_k$$

The binomial transform naturally appear in the theory of finite differences.

Definition 1.1.4. *Let* $\{a_k\}_{k=0}^{\infty}$ *be a sequence. Consider the forward difference operator* Δ *defined by:*

$$\Delta_k = a_{k+1} - a_k.$$

It is easy to compute:

$$\Delta^n a_0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_{n-k},$$

and because of the well-known property $\binom{n}{k} = \binom{n}{n-k}$ *this can be also written as,*

$$\Delta^n a_0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a_k \tag{1.1.7}$$

or equivalently as,

$$(-1)^n \Delta^n a_0 = \sum_{k=0}^{n} = \binom{n}{k} (-1)^k a_k \tag{1.1.8}$$

Proposition 1.1.5. *Another elementary property involves the transform of the shifted sequence. If* $b_n = \sum_{k=0}^{n} a_k$*, then*

$$\Delta b_n = b_{n+1} - b_n = \sum_{k=0}^{n} \binom{n}{k} a_{k+1} \quad for\ n \geq 1 \tag{1.1.9}$$

Proof.

$$\begin{aligned} b_{n+1} - b_n &= \sum_{k=0}^{n+1} \binom{n+1}{k} a_k - \sum_{k=0}^{n} \binom{n}{k} a_k \\ &= \sum_{k=0}^{n+1} a_k \left\{ \binom{n+1}{k} - \binom{n}{k} \right\} \\ &= \sum_{k=0}^{n+1} a_k \binom{n}{k-1} \\ &= \sum_{j=0}^{n} \binom{n}{j}, \end{aligned}$$

after letting $j = k - 1$ □

Remark By iterating $\Delta b_n = b_{n+1} - b_n = \sum_{k=0}^{n} \binom{n}{k} a_{k+1}$, we get

$$\sum_{k=0}^{n} \binom{n}{k} a_{k+p} = \Delta^p b_n = \sum_{k=0}^{p} \binom{p}{k} (-1)^{p-k} b_{n+k} \tag{1.1.10}$$

for every $p \geq 0$. It follows from (1.1.10) that if we have a recurrence relation

$$a_{n+2} + A a_{n+1} + B a_n = 0$$

where A and B are constants, then, it transforms into the difference equation

$$\Delta^2 b_n + A \Delta b_n + B b_n = 0$$

Proposition 1.1.6. *If we iterate* $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$ *we find,*

$$\sum_{n=0}^{m} \left\{ \sum_{k=0}^{m} \binom{m}{k} a_k \right\} = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} a_k \tag{1.1.11}$$

Proof. This follows from the binomial coefficient property that is,

$$\binom{n}{m}\binom{m}{k} = \binom{n}{n}\binom{n-k}{m-k} \tag{1.1.12}$$

In fact, let any integer $n \geq m$ and $0 \leq k \leq m$. Then,

$$\begin{aligned} \binom{n}{m}\binom{m}{k} &= \frac{n!m!}{m!(n-m)!k!(m-k)!} \\ &= \frac{n!(n-k)!}{k!(n-k)!(m-k)!(n-m)!} \\ &= \binom{n}{k}\binom{n-k}{m-k} \end{aligned}$$

□

Remark The iterated symmetric transformation (1.1.6) is the identity transform. i.e. leads back to $\{a_k\}$.

1.2 Special Numbers and Polynomials

Definition 1.2.1 (Stirling numbers of the first kind)**.** *A Stirling number of first kind count how many ways to partition a set into cycle rather than subsets.*

Definition 1.2.2 (Stirling numbers of the second kind)**.** *A Stirling number of the second kind* $S(n;k)$ *counts number of ways in which* n *distinguishable* *objects can be partitioned into k distinguished subsets when each subset has to contain at least one object.*

We can count them by counting the number of onto functions from set A to set B, where $|A| = n$ and $|B| = k$

Definition 1.2.3 (Cycle)**.** *A cycle is a sort of ordered subsets. The order of elements matters, but in a circular way. A cycle of size k is a way to place k items evenly around a circle, where two cycles are considered the same if you can rotate one into the other.*

Example 1.2.4. $[1;2;3]$ *and* $[2;3;1]$ *represent the same cycle, but* $[1;2;3]$ *and* $[1;3;2]$ *represent different cycles.*

Definition 1.2.5. *The binomial transform can be recognized in many formulas involving classical numbers and polynomials. We have the representation,*

$$S(\alpha;n) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}K^{\alpha} \tag{1.2.1}$$

or

$$(-1)^n n! S(\alpha;n) = \sum_{k=0}^{n}\binom{n}{k}(-1)^{k}K^{\alpha},$$

where $Re(\alpha) > 0$ and $S(\alpha;n)$ are the Stirling functions, the generalized Stirling numbers of the second kind. When $\alpha = m$ is a positive integer, $S(m;n)$ are the usual Stirling numbers of the second kind.

The Stirling number of the second kind can be defined by the generating function

$$\frac{x^n}{(1-x)(1-2x)...(1-nx)} = \sum_{m=n}^{\infty} S(m;n)x^m \tag{1.2.2}$$

or by the exponential generating function

$$\frac{1}{n!}(e^x - 1)^n = \sum_{m=n}^{\infty} S(m;n)\frac{x^m}{m!} \tag{1.2.3}$$

The Stirling number of the first kind $S(n;k)$ are defined by the generating function

$$x(x-1)...(x-n+1) = n!\binom{x}{n} = \sum_{k=0}^{n} S(n;k)x^k \tag{1.2.4}$$

Their exponential generating function is

$$\frac{(\ln(1+x))^k}{k!} = \sum_{n=k}^{\infty} \frac{S(n;k)}{n!}x^n \tag{1.2.5}$$

Definition 1.2.6. *Many binomial transform formulas involve the harmonic numbers and the generalized Harmonic numbers; for n= 0, 1, 2...*

$$H_n = \sum_{k=1}^{n} \frac{1}{k}; \qquad H_n^{(s)} = \sum_{k=1}^{n} \frac{1}{k^s}; H_0 = H_0^{(s)} = 0 \tag{1.2.6}$$

Remark Here S is any complex number.

These numbers can be expressed in terms of the Digamma function Ψ,

$$\Psi(x) = \frac{d}{dx}\log\Gamma(x) \tag{1.2.7}$$

namely,

$$H_n = \Psi(n+1) + \gamma \tag{1.2.8}$$

where, $\gamma = -\Psi(1)$ is Euler's constant.

It is good to mention that

$$H_n = \log(n) + \gamma + \theta(\frac{1}{n}) \tag{1.2.9}$$

When $m \geq 2$ is an integer

$$H_n^{(m)} = \zeta(m) + \frac{(-1)^{(m-1)}}{(m-1)!}\Psi^{(m-1)}(n+1) \tag{1.2.10}$$

Definition 1.2.7. *The generating function of the harmonic numbers and the generalized harmonic numbers are, correspondingly,*

$$\frac{-\ln(1-t)}{1-t} = \sum_{n=1}^{\infty} H_n t^n \tag{1.2.11}$$

$$\frac{1}{1-t} Li_s(t) = \sum_{n=1}^{\infty} H_n^{(s)} t^n, \tag{1.2.12}$$

where $Li_s(t)$ is the polylogarithm function

$$Li_s(t) = \sum_{k=1}^{\infty} \frac{t^k}{k^s}. \tag{1.2.13}$$

We shall use also the skew-harmonic numbers

$$H_n^- = \sum_{k=1}^{n} \frac{(-1)^{(k-1)}}{k}; H_0^- = 0, \tag{1.2.14}$$

with the generating function

$$\frac{\ln(1+t)}{1-t} = \sum_{n=1}^{\infty} H_n^- t^n \tag{1.2.15}$$

Definition 1.2.8. *The Bernoulli polynomials $B_n(x)$; n= 0, 1, 2... are very important polynomials in analysis. They are defined by the generating function,*

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{1.2.16}$$

where $B_n = B_n(0)$ are the Bernoulli numbers.

Definition 1.2.9. *The Euler polynomials $E_n(x)$; n = 0, 1, 2,... are defined by the generating function.*

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \tag{1.2.17}$$

with $E_n(0) = 0$ when, $n = 2, 4, 6, ...$ and $E_n(1) = (-1)^n E_n(0)$. we also have

$$E_n(0) = \frac{2}{n+1}(1 - 2^{n+1})E_{n+1}$$

The Euler numbers are defined by

$$E_n = 2^n E_n(\frac{1}{2})$$

or by the generalized function

$$\frac{1}{\cosh t} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \tag{1.2.18}$$

Definition 1.2.10. *The Genocchi polynomials are defined by the generating function*

$$\frac{2te^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} \tag{1.2.19}$$

and $G_n = G_n(0)$ *are the Genocchi numbers.*

It is easy to see that

$$G_n = 2(1-2^n)B_n$$

Note also that

$$G_n(x) = nE_{n-1}(x)$$

Definition 1.2.11. *The Euler-Bernoulli functions* $\beta_n(x;\lambda)$ *are defined by the generating function*

$$\frac{te^{xt}}{\lambda e^t - 1} = \sum_{n=0}^{\infty} \beta_n(x;\lambda)\frac{t^n}{n!} \qquad (1.2.20)$$

When $\lambda = 1; \beta_n(x;1) = \beta_n(x)$ *are the Bernoulli polynomials*

When $\lambda \neq 1$ *and* $x = 0$*; the function* $\beta_n(\lambda) = \beta_n(0;\lambda)$ *are rational functions.*

Definition 1.2.12. *The exponential polynomials* $\phi_n(x); n = 0;1;2;...$ *are defined by the generating function*

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} \phi_n(x)\frac{t^n}{n!}$$

or by the characteristic property

$$\left(x\frac{d}{dx}\right)^n e^x = \phi_n(x)e^x \qquad (n = 0;1;2;...) \qquad (1.2.21)$$

The coefficients of these polynomials are the Stirling numbers of the second kind.

$$\phi_n(x) = \sum_{k=0}^{n} S(n;k)x^k \qquad (1.2.22)$$

The value at $x = 1$

$$\phi_n(1) = \sum_{k=0}^{n} S(n;k) \tag{1.2.23}$$

are the well known Bell numbers.

Remark The Bell numbers count the possible partitions of a set.

Definition 1.2.13. *The geometric polynomials* $\omega_{n,r}(x) = 0, 1, ..., r \geq 0$ *where* $\omega_{n,0}(x) = \delta_{n,0}$ *,are defined by the generating function*

$$\frac{1}{[1 - x(e^t - 1)]^r} = \sum_{n=0}^{\infty} \omega_{n,r}(x)\frac{t^n}{n!} \tag{1.2.24}$$

or, by the property

$$\sum_{k=0}^{\infty} \binom{k+r-1}{k} k^n x^k = \frac{1}{(1-x)^r}\omega_{n,r}\left(\frac{x}{1-x}\right) \tag{1.2.25}$$

They have the representation

$$\omega_{n,r}(x) = \frac{1}{\Gamma(r)}\sum_{k=0}^{n} S(n;k)\Gamma(k+r)x^k = \sum_{k=0}^{n} S(n;k)\binom{k+r-1}{k} k!x^k \tag{1.2.26}$$

Where $r = 1$ *we write* $\omega_{n,1}(x) = \omega_n(x)$ *so that,*

$$\omega_n(x) = \sum_{k=0}^{n} S(n;k)k!x^k \tag{1.2.27}$$

Remark The numbers

$$\omega_n(1) = \sum_{k=0}^{n} S(n;k)k!$$

are known in combinatorics as the preferential arrangement numbers.

When $r = 1$ we have from (1.2.25)

$$\sum_{k=0}^{\infty} k^n x^k = \frac{1}{1-x}\omega_n\left(\frac{x}{1-x}\right) \tag{1.2.28}$$

Definition 1.2.14. *The Eulerian polynomials $A_n(x)$ are defined by the equation*

$$\sum_{k=0}^{\infty} K^n x^k = \frac{1}{(1-x)^{n+1}} A_n(x) \tag{1.2.29}$$

or by the generating function

$$\frac{1-x}{1-xe^{(t(t-x))}} = \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!} \tag{1.2.30}$$

The Eulerian polynomials are related to the geometric polynomials by the equation (6)

$$\omega_n\left(\frac{x}{1-x}\right) = \frac{A_n(x)}{(1-x)^n} \tag{1.2.31}$$

Definition 1.2.15. *The Cauchy numbers of first type c_n and the Cauchy numbers of second type d_n were defined as $c_0 = d_0 = 1$ and for $n = 1, 2, \ldots$*

$$c_n = \int_0^1 x(x-1)\ldots(x-n+1)dx \tag{1.2.32}$$

$$d_n = \int_0^1 x(x+1)\ldots(x+n-1)dx \tag{1.2.33}$$

They have exponential generating functions correspondingly,

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!} \tag{1.2.34}$$

$$\frac{-t}{(1-t)\log(1-t)} = \sum_{n=0}^{\infty} d_n \frac{t^n}{n!} \tag{1.2.35}$$

The numbers $(-1)^n d_n$ *appeared in the works of Norlund (54) and known as Norlund numbers.*

From (1.2.4) , (1.2.32) and (1.2.33) we have correspondingly (22)

$$c_n = \sum_{k=0}^{n} \frac{S(n;k)}{k+1} \tag{1.2.36}$$

$$d_n = \sum_{k=0}^{n} \frac{(-1)^{n-k} S(n;k)}{k+1}, \tag{1.2.37}$$

where $S(n;k)$ *are the Stirling numbers of the first kind 1.2.4.*

Definition 1.2.16. *The Fibonacci numbers* F_n *and the Lucas numbers* L_n *, where* $n = 0, 1, 2, ...$ *are defined by the generating functions*

$$f(t) = \frac{1}{1-t-t^2} = \sum_{k=0}^{\infty} F_k t^k \tag{1.2.38}$$

and correspondingly by,

$$l(t) = \frac{2-t}{1-t-t^2} = \sum_{k=0}^{\infty} L_k t^k \tag{1.2.39}$$

These numbers also have convenient exponential generating functions

$$\frac{e^{\phi t} - e^{\Phi t}}{\phi - \Phi} = \sum_{k=0}^{\infty} \frac{t^k}{k!} F_k \tag{1.2.40}$$

$$e^{\phi t} + e^{\Phi t} = \sum_{k=0}^{\infty} \frac{t^k}{k!} L_k \tag{1.2.41}$$

where $\phi = \frac{1}{2}(1+\sqrt{5}); \Phi = \frac{1}{2}(1-\sqrt{5}) = \frac{-1}{\phi}$

Definition 1.2.17. *The classical Laguerre polynomials* $L_n(x), n = 0, 1, ...$ *are defined by Rodrigues formula.*

$$L_n(x) = \frac{e^x}{n!}\left(\frac{d}{dx}\right)^n (x^n e^{-x}) \tag{1.2.42}$$

or by the generating function

$$\frac{1}{1-t}e^{\left(\frac{xt}{1-t}\right)} = \sum_{n=0}^{\infty} L_n(x)t^n \tag{1.2.43}$$

These polynomials appear as a binomial transform

$$L_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-x)^k}{k!} \tag{1.2.44}$$

Definition 1.2.18. *The Hermite polynomial* $H_n(x); n = 0; 1; 2; ...$ *can be defined by the Rodrigues formula*

$$H_n(x) = (-1)^n e^{x^2}\left(\frac{d}{dx}\right)^n e^{-x^2} \tag{1.2.45}$$

or by the generating function

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \tag{1.2.46}$$

Chapter 2

Euler's Transformation of Series

2.1 Basic Formulas

The Binomial transform is closely related to Euler's series transformation. We refer in this chapter to [1].

Definition 2.1.1 (neighborhood of a point)**.** *An ϵ-neighborhood of a point z_0 in $\mathbb{C}$ is the set of all points z lying inside but not on circle centered at z_0 with radius ϵ . i.e. in the disc*

$$B(z_0;\epsilon) = \{z \in \mathbb{C}/|z - z_0| < \epsilon\}$$

Definition 2.1.2. *A function f ia said to be analytic at a point z_0 if it is differentiable at each point in some neighborhood of z_0 .*

Proposition 2.1.3. *Suppose we have a function f analytic in a neighborhood of the origin*

$$f(t) = \sum_{k=0}^{\infty} a_k t^k \tag{2.1.1}$$

Euler's series transformation formula says that

$$\frac{1}{1-t} f\left(\frac{t}{1-t}\right) = \sum_{n=0}^{\infty} t^n \left\{\sum_{k=0}^{n} \binom{n}{k} a_k\right\} \tag{2.1.2}$$

for $|t|$ small enough.

Remark Euler's formula can be used, among other things, to evaluate the binomial transform

$$b_n = \sum_{k=0}^{\infty} \binom{n}{k} a_k \qquad n = 0; 1; 2; ... \tag{2.1.3}$$

by computing the Taylor coefficients of the function on the left hand side in (2.1.2) independently and comparing coefficient.

Example 2.1.4. *If we have a convergent series*

$$s = f(1) = a_0 + a_1 + a_2 + ...$$

then with $t = \frac{1}{2}$ in (2.1.2) we find

$$s = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \left\{\sum_{k=0}^{n} \binom{n}{k} a_k\right\}$$

Proof. for $t = \frac{1}{2}$

$$\begin{aligned}
\frac{1}{1-\frac{1}{2}} f\left(\frac{\frac{1}{2}}{1-\frac{1}{2}}\right) &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \left\{\sum_{k=0}^{n} \binom{n}{k} a_k\right\} \\
\Rightarrow \quad 2f(1) &= \sum_{n=0}^{\infty} \frac{1}{2^n} \left\{\sum_{k=0}^{n} \binom{n}{k} a_k\right\} \\
\Rightarrow \quad s = f(1) &= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \left\{\sum_{k=0}^{n} \binom{n}{k} a_k\right\}
\end{aligned}$$

□

Example 2.1.5. *Using the substitution*

$$z = \frac{t}{1-t}; t = \frac{z}{z+1}$$

Equation (2.1.2) becomes

$$f(z) = \frac{1}{z+1} \sum_{n=0}^{\infty} \left(\frac{z}{z+1}\right)^n \left\{\sum_{k=0}^{n} \binom{n}{k} a_k\right\} \tag{2.1.4}$$

It is good to mention an interesting formula resulting from (2.1.4) with x, z appropriate parameters. We apply (2.1.4) to the function

$$f\left(\frac{x}{z} t\right) = \sum_{n=0}^{\infty} \left(\frac{a_n x^n}{z^n}\right) t^n$$

and the resulting is the representation

$$f(x) = \sum_{n=0}^{\infty} \frac{z^n}{(1+z)^{n+1}} \left\{\sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{z^k} a_k\right\} \tag{2.1.5}$$

In fact, we have

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

then,

$$f\left(\frac{x}{z}t\right) = \sum_{n=0}^{\infty} \frac{a_n x^n}{z^n} t^n = \sum_{n=0}^{\infty} \alpha_n t^n$$

Now, let's call $f\left(\frac{x}{z}t\right) = g(t)$*. Then, we have,* $g(z) = f\left(\frac{x}{z}z\right) = f(x)$*. Hence,*

$$\begin{aligned} f(x) = g(z) &= \frac{1}{z+1}\sum_{n=0}^{\infty}\left(\frac{z}{z+1}\right)^n \left\{\sum_{k=0}^{n}\binom{n}{k}\alpha_k\right\} \\ &= \frac{1}{z+1}\sum_{n=0}^{\infty}\left(\frac{z}{z+1}\right)^n \left\{\sum_{k=0}^{n}\binom{n}{k}\frac{a_k x^k}{z^k}\right\} \end{aligned}$$

Remark Notice that in this representation the variable (parameter) z appears only on the right hand side.

2.2 A General Theorem and Several Transformation Formulas

Lemma 2.2.1. *(Handamard's series multiplication theorem)*

Given two power series, say, (2.1.1) ;

$$f(t) = \sum_{k=0}^{\infty} a_k t^k$$

and also a second one

$$g(t) = \sum_{k=0}^{\infty} c_k t^k \tag{2.2.1}$$

We have the representation

$$\sum_{n=0}^{\infty} a_n c_n z^n = \frac{1}{2\pi i} \oint g\left(\frac{z}{\lambda}\right) f(\lambda) \frac{d\lambda}{\lambda} \tag{2.2.2}$$

Where L is an appropriate closed curve around the origin.

Theorem 2.2.2. *The following representation holds*

$$\sum_{n=0}^{\infty} c_n h(z)^n \left\{ \sum_{k=0}^{n} \binom{n}{k} a_k \right\} = \frac{1}{2\pi i} \oint g\left(h(z) \left(1 + \frac{1}{\lambda}\right)\right) f(\lambda) \frac{d\lambda}{\lambda} \tag{2.2.3}$$

Where $h(z)$ is an appropriate function for which the above expression is defined and the integral is a Cauchy type integral on a closed curve around the origin, as in (2.2.2).

Proof. We shall apply Cauchy's integral formula for the coefficients of the function $f(t)$ from (2.1.1).

According to this formula we have for $k = 0, 1, ...$

$$\sum_{k=0}^{n} \binom{n}{k} a_k = \frac{1}{2\pi i} \oint \left(1 + \frac{1}{\lambda}\right)^n f(\lambda) \frac{d\lambda}{\lambda} \tag{2.2.4}$$

Multiplying both sides in this equation by $c_n h(z)^n$ and summing for n we obtain

$$\sum_{n=0}^{\infty} c_n h(z)^n \left\{ \sum_{k=0}^{n} \binom{n}{k} a_k \right\} = \frac{1}{2\pi i} \oint g\left(h(z) \left(1 + \frac{1}{\lambda}\right)\right) f(\lambda) \frac{d\lambda}{\lambda}$$

□

Choosing $g(t)$ and $h(z)$ appropriately and combining this result by Han-

damard's theorem we shall generate various series transformation formulas. Here is the first example.

Corollary 2.2.3. *let α be a complex number. Then the following representation holds.*

$$\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^n a_n z^n = (z+1)^{\alpha}\sum_{n=0}^{\infty}\left(\frac{z}{z+1}\right)^n\binom{\alpha}{n}(-1)^n\left\{\sum_{k=0}^{n}\binom{n}{k}a_k\right\} \tag{2.2.5}$$

Proof. In Theorem 2.2.2 we choose:

$$h(z)=\frac{z}{z+1}; g(t)=(1-t)^{\alpha}=\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^n t^n; c_n=\binom{\alpha}{n}(-1)^n.$$

A simple computation shows that,

$$\begin{aligned} g\left(h(z)\left(1+\frac{1}{\lambda}\right)\right) &= g\left(\frac{z}{z+1}\left(1+\frac{1}{\lambda}\right)\right) \\ &= \left(1-\frac{z}{z+1}\left(1+\frac{1}{\lambda}\right)\right)^{\alpha} \\ &= \left(1-\frac{z}{z+1}-\frac{z}{(z+1)\lambda}\right)^{\alpha} \\ &= \frac{(z\lambda+\lambda-z\lambda-z)^{\alpha}}{(z+1)^{\alpha}\lambda^{\alpha}} \\ &= (z+1)^{-\alpha}\left(\frac{\lambda-1}{\lambda}\right)^{\alpha} \\ &= (z+1)^{-\alpha}\left(1-\frac{z}{\lambda}\right)^{\alpha} \end{aligned}$$

Now apply it in theorem 2.2.2, we get,

$$\begin{aligned}\sum_{n=0}^{\infty} c_n \left(\frac{z}{z+1}\right)^n \left\{\sum_{k=0}^{n} \binom{n}{k} a_k\right\} &= \frac{1}{2\pi i} \oint (z+1)^{-\alpha} \left(1-\frac{z}{\lambda}\right)^{\alpha} f(\lambda) \frac{d\lambda}{\lambda} \\ &= \frac{(z+1)^{-\alpha}}{2\pi i} \oint \left(1-\frac{z}{\lambda}\right)^{\alpha} f(\lambda) \frac{d\lambda}{\lambda}\end{aligned}$$

This representation yields to (2.2.5) in view of Handamard's Theorem.

When $\alpha = -1$ we have

$$\binom{-1}{n} = (-1)^n \tag{2.2.6}$$

and (2.2.5) becomes (2.1.4). □

Corollary 2.2.4. *Let the sequence a_n be defined by*

$$f(t) = \sum_{k=0}^{\infty} a_k t^k$$

Then the following exponential version of Euler's series transformation formula holds

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n = e^{-z} \sum_{n=0}^{\infty} \frac{z^n}{n!} \left\{\sum_{k=0}^{n} \binom{n}{k} a_k\right\} \tag{2.2.7}$$

Proof. In Theorem 2.2.2 we choose:

$$h(z) = z, \quad g(t) = e^t \quad c_n = \frac{1}{n!}$$

A simple computation shows that,

$$g\left(h(z)\left(1+\frac{1}{\lambda}\right)\right) = g\left(z\left(1+\frac{1}{\lambda}\right)\right) = e^{z\left(1+\frac{1}{\lambda}\right)}$$

Now apply in Theorem 2.2.2, we get,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{z^n}{n!}\left\{\sum_{k=0}^{n}\binom{n}{k}a_k\right\} &= \frac{1}{2\pi i}\oint e^{z\left(1+\frac{1}{\lambda}\right)}f(\lambda)\frac{d\lambda}{\lambda}\\ &= \frac{e^z}{2\pi i}\oint e^{\frac{z}{\lambda}}f(\lambda)\frac{d\lambda}{\lambda}\end{aligned}$$

Now from (2.2.2) we get,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{a_n}{n!}z^n &= \frac{e^{-z}e^z}{2\pi i}\oint e^{\frac{z}{\lambda}}f(\lambda)\frac{d\lambda}{\lambda}\\ \sum_{n=0}^{\infty}\frac{a_n}{n!}z^n &= e^{-z}\sum_{n=0}^{\infty}\frac{z^n}{n!}\left\{\sum_{k=0}^{n}\binom{n}{k}a_k\right\}\end{aligned}$$

□

We can replace in (2.2.7) a_n by $\frac{a_n}{\lambda}$ and z by λz (where λ is a parameter) to give the equation the more flexible form,

$$e^{\lambda z}\sum_{n=0}^{\infty}\frac{a_n}{n!}z^n = \sum_{n=0}^{\infty}\frac{z^n}{n!}\left\{\sum_{k=0}^{n}\binom{n}{k}\lambda^{n-k}a_k\right\} \tag{2.2.8}$$

In the next two applications we use natural logarithm function. In all expansions we assume that $|z|$ is small enough to secure convergence.

Corollary 2.2.5. *With*

$$f(t) = \sum_{k=0}^{\infty}a_k t^k$$

the following representation holds:

$$a_0\log(1+z) + \sum_{n=1}^{\infty}\frac{z^n}{n!}a_n = \sum_{n=1}^{\infty}\left(\frac{z}{z+1}\right)^n\frac{1}{n}\left\{\sum_{k=0}^{n}\binom{n}{k}a_k\right\} \tag{2.2.9}$$

Proof. In Theorem 2.2.2 we choose:

$$h(z) = \frac{z}{z+1}; g(t) = -\log(1-t) = \sum_{n=0}^{\infty} \frac{t^n}{n}; c_n = \frac{1}{n}$$

A simple computation shows that,

$$\begin{aligned}
g\left(h(z)\left(1+\frac{1}{\lambda}\right)\right) &= g\left(\frac{z}{z+1}\left(1+\frac{1}{\lambda}\right)\right) \\
&= -\log\left(1-\left(\frac{z}{z+1}\left(1+\frac{1}{\lambda}\right)\right)\right) \\
&= -\log\left(1-\frac{z}{z+1}-\frac{z}{(z+1)\lambda}\right) \\
&= -\log\left(\frac{1}{z+1}-\frac{z}{(z+1)\lambda}\right) \\
&= -\left(\log\left(\frac{1}{z+1}\right)+\log\left(1+\frac{\frac{-z}{(z+1)\lambda}}{\frac{1}{z+1}}\right)\right) \\
&= -\log\left(\frac{1}{z+1}\right)-\log\left(1-\frac{z}{\lambda}\right) \\
&= \log(z+1)-\log\left(1-\frac{z}{\lambda}\right)
\end{aligned}$$

Now apply in Theorem2.2.2, we get,

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{n}\left(\frac{z}{z+1}\right)^n \left\{\sum_{k=0}^{n}\binom{n}{k}a_k\right\} &= \frac{1}{2\pi i}\oint \left(\log(z+1)-\log\left(1-\frac{z}{\lambda}\right)\right) f(\lambda)\frac{d\lambda}{\lambda} \\
&= \frac{\log(z+1)}{2\pi i}\oint \frac{f(\lambda)}{\lambda}d\lambda - \frac{1}{2\pi i}\oint \log\left(1-\frac{z}{\lambda}\right)\frac{f(\lambda)}{\lambda}d\lambda
\end{aligned}$$

We know that,

$$\frac{\log(z+1)}{2\pi i}\oint \frac{f(\lambda)}{\lambda}d\lambda = a_0\log(z+1)$$

We still have to prove that,

$$\frac{-1}{2\pi i}\oint \log\left(1-\frac{z}{\lambda}\right)\frac{f(\lambda)}{\lambda}d\lambda=\sum_{n=1}^{\infty}\frac{z^n}{n}a_n$$

From (2.2.2) we get the above equality, where

$$g\left(\frac{z}{\lambda}\right)=-\log\left(1-\frac{z}{\lambda}\right)$$

Hence,

$$a_0\log(1+z)+\sum_{n=1}^{\infty}\frac{z^n}{n}a_n=\sum_{n=1}^{\infty}\left(\frac{z}{z+1}\right)^n\frac{1}{n}\left\{\sum_{k=0}^{n}\binom{n}{k}a_k\right\}$$

□

In the next corollary we present an interesting identity involving harmonic numbers.

Corollary 2.2.6. *For every p with Rep > −1 we have*

$$\begin{aligned}&\sum_{n=0}^{\infty}(H_{p+n}-H_p)\binom{p+n}{n}a_nz^n+\log(1+z)\sum_{n=0}^{\infty}\binom{p+n}{n}a_nz^n \qquad (2.2.10)\\ =\ &\frac{1}{(1+z)^{p+1}}\sum_{n=0}^{\infty}\left(\frac{z}{z+1}\right)^n(H_{p+n}-H_p)\binom{p+n}{n}\left\{\sum_{k=0}^{n}\binom{n}{k}a_k\right\}\end{aligned}$$

With $\{a_n\}$ *an arbitrary sequence of coefficients as in (2.1.1).*

Proof. We use the series expansion

$$\frac{1}{(1-z)^{m+1}}\ln\left(\frac{1}{1-z}\right)=\sum_{n=0}^{\infty}(H_{m+n}-H_m)\binom{m+n}{n}z^n$$

Now let,

$$g(t) = \frac{-\log(1-t)}{(1-t)^{p+1}} = \sum_{n=0}^{\infty} (H_{p+n} - H_p) \binom{p+n}{n} t^n$$

Where $Rep > -1$. As before take $h(z) = \frac{z}{z+1}$

$$\begin{aligned} g\left(h(z)\left(1+\frac{1}{\lambda}\right)\right) &= g\left(\frac{z}{z+1} + \frac{z}{(z+1)\lambda}\right) \\ &= \frac{\log(z+1) - \log\left(1-\frac{z}{\lambda}\right)}{\left(1 - \frac{z}{z+1} - z(z+1)\lambda\right)^{p+1}} \\ &= \frac{\log(z+1)}{\left(\frac{\lambda - z}{(z+1)\lambda}\right)^{p+1}} - \frac{\log\left(1-\frac{z}{\lambda}\right)}{\left(\frac{\lambda - z}{(z+1)\lambda}\right)^{p+1}} \\ &= \frac{\log(z+1)}{\left(\frac{\lambda-z}{\lambda}\right)^{p+1} \frac{1}{(z+1)^{p+1}}} - \frac{\log\left(1-\frac{z}{\lambda}\right)}{\left(\frac{\lambda-z}{\lambda}\right)^{p+1} \frac{1}{(z+1)^{p+1}}} \\ &= (z+1)^{p+1} \left\{ \frac{\log(z+1)}{\left(1-\frac{z}{\lambda}\right)^{p+1}} - \frac{\log\left(1-\frac{z}{\lambda}\right)}{\left(1-\frac{z}{\lambda}\right)^{p+1}} \right\} \end{aligned}$$

Hence, the right hand side of (2.2.3) becomes

$$(1+z)^{p+1} \frac{\log(1+z)}{2\pi i} \oint \frac{1}{\left(1-\frac{z}{\lambda}\right)^{p+1}} \frac{f(\lambda)}{\lambda} d\lambda - \frac{(1+z)^{p+1}}{2\pi i} \oint \frac{\log\left(1-\frac{z}{\lambda}\right)}{\left(1-\frac{z}{\lambda}\right)^{p+1}} \frac{f(\lambda)}{\lambda} d\lambda$$

At this point we use the well-known binomial expansion

$$\frac{1}{(1-z)^{p+1}} = \sum_{n=0}^{\infty} \binom{p+n}{n} z^n$$

So according to Handamard's theorem the first term above becomes

$$\begin{aligned}
& (z+1)^{p+1}\log(z+1)\frac{1}{2\pi i}\oint \frac{1}{\left(1-\frac{z}{\lambda}\right)^{p+1}}\frac{f(\lambda)}{\lambda}d\lambda \\
= & (z+1)^{p+1}\log(z+1)\frac{1}{2\pi i}\oint \sum_{n=0}^{\infty}\binom{n+p}{n}z^n\frac{f(\lambda)}{\lambda}d\lambda \\
= & (z+1)^{p+1}\log(z+1)\sum_{n=0}^{\infty}\binom{p+n}{n}z^n\frac{1}{2\pi i}\oint \frac{f(\lambda)}{\lambda}d\lambda \\
= & (z+1)^{p+1}\log(z+1)\sum_{n=0}^{\infty}\binom{p+n}{n}z^n a_n
\end{aligned}$$

While the second term becomes

$$\begin{aligned}
& (z+1)^{p+1}\frac{1}{2\pi i}\oint \frac{\log\left(1-\frac{z}{\lambda}\right)}{\left(1-\frac{z}{\lambda}\right)^{p+1}}\frac{f(\lambda)}{\lambda}d\lambda \\
= & (z+1)^{p+1}\sum_{n=0}^{\infty}(H_{p+n}-H_p)\binom{p+n}{n}a_n z^n
\end{aligned}$$

and the desire identity follows. In this case we use in (2.2.3) the coefficients

$$c_n = (H_{p+n}-H_p)\binom{p+n}{n}$$

The result in this corollary was used to evaluate in closed from the series

$$\sum_{n=0}^{\infty}(H_{p+n}-H_p)\binom{p+n}{n}n^m z^n$$

for any $m \geq 0$ and any $p \geq 0$.

When $p = 0$ then $c_n = H_n$ and we have the special case bellow. $\square$

Corollary 2.2.7. *With*

$$f(t) = \sum_{k=0}^{\infty} a_k t^k$$

the following transformation formula holds:

$$\sum_{n=0}^{\infty} H_n a_n z^n + \log(1+z) f(z) = \frac{1}{1+z} \sum_{n=0}^{\infty} \left(\frac{z}{z+1} \right)^n H_n \left\{ \sum_{k=0}^{n} \binom{n}{k} a_k \right\} \tag{2.2.11}$$

For completeness we present here one more series transformation formula involving two power series.

Proposition 2.2.8. *Given two analytic functions $f(t)$ and $g(t)$ where*

$$f(t) = \sum_{k=0}^{\infty} a_k t^k$$

and

$$g(t) = \sum_{k=0}^{\infty} c_k t^k$$

then the following representation is true

$$\sum_{n=0}^{\infty} a_n c_n t^n = \sum_{n=0}^{\infty} \frac{g^{(n)}(-t)}{n!} t^n \left\{ \sum_{k=0}^{n} \binom{n}{k} a_k \right\} \tag{2.2.12}$$

Proof. Multiplying both sides in (2.2.4) by

$$\frac{g^{(n)}(-t)}{n!} t^n$$

and summing for n we get

$$\sum_{n=0}^{\infty} \frac{g^{(n)}(-t)}{n!} t^n \left\{ \sum_{k=0}^{n} \binom{n}{k} a_k \right\} = \frac{1}{2\pi i} \oint \sum_{n=0}^{\infty} \frac{g^{(n)}(-t)}{n!} \left(\frac{t}{\lambda} + t \right)^n \frac{f(\lambda)}{\lambda} d\lambda$$

From the Taylor expansion of $g\left(\frac{t}{\lambda}\right)$ centered at $(-t)$ we get

$$\sum_{n=0}^{\infty} \frac{g^{(n)}(-t)}{n!}\left(t+\frac{t}{\lambda}\right)^{n} = g\left(\frac{t}{\lambda}\right)$$

Hence,

$$\sum_{n=0}^{\infty} a_n c_n t^n = \frac{1}{2\pi i}\oint g\left(\frac{t}{\lambda}\right)\frac{f(\lambda)}{\lambda}d\lambda$$

which is the same as (2.2.2). □

Euler's transformation works also for asymptotic series. Namely, we have this result:

Corollary 2.2.9. *Suppose the function*

$$F(\lambda) = \sum_{n=0}^{\infty} \frac{a_n}{\lambda^{n+1}}$$

is analytic in a neighborhood of infinity (or is a formal power series). Then

$$F(\lambda-1) = \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}}\left\{\sum_{k=0}^{n}\binom{n}{k}a_k\right\}$$

Proof. Let $\lambda = \frac{1}{t}$, so $t = \frac{1}{\lambda}$ Now by substituting in (2.1.2) we get

$$\begin{aligned}
\frac{1}{1-\frac{1}{\lambda}} f\left(\frac{\frac{1}{\lambda}}{1-\frac{1}{\lambda}}\right) &= \sum_{n=0}^{\infty}\left(\frac{1}{\lambda}\right)^{n}\left\{\sum_{k=0}^{n}\binom{n}{k}a_k\right\} \\
\frac{\lambda}{\lambda-1} f\left(\frac{1}{\lambda-1}\right) &= \sum_{n=0}^{\infty}\frac{1}{\lambda^{n}}\left\{\sum_{k=0}^{n}\binom{n}{k}a_k\right\}
\end{aligned}$$

Dividing both sides by λ we get

$$\begin{aligned}\frac{1}{\lambda-1}f\left(\frac{1}{\lambda-1}\right) &= \sum_{n=0}^{\infty}\frac{1}{\lambda^{n+1}}\left\{\sum_{k=0}^{n}\binom{n}{k}a_k\right\} \\ F(\lambda-1) &= \sum_{n=0}^{\infty}\frac{1}{\lambda^{n+1}}\left\{\sum_{k=0}^{n}\binom{n}{k}a_k\right\}\end{aligned}$$

□

2.3 Examples and Applications

Example 2.3.1. *In this example we show how the transformation formula (2.2.8) can be used to prove some classical properties of Bernouilli polynomials* $B_n(x)$*.*

In (2.2.8) we set $a_n = B_n(x)$ *so (2.2.8) become*

$$e^{\lambda z}\sum_{n=0}^{\infty}\frac{B_n(x)}{n!}z^n = \sum_{n=0}^{\infty}\frac{z^n}{n!}\left\{\sum_{k=0}^{n}\binom{n}{k}\lambda^{n-k}B_k(x)\right\}$$

so from (1.2.16) we get that

$$e^{\lambda z}\sum_{n=0}^{\infty}\frac{B_n(x)}{n!}z^n = \sum_{n=0}^{\infty}\frac{z^n}{n!}\left\{\sum_{k=0}^{n}\binom{n}{k}\lambda^{n-k}B_k(x)\right\} = \frac{ze^{(x+\lambda)z}}{e^z-1}$$

so that

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}\left\{\sum_{k=0}^{n}\binom{n}{k}\lambda^{n-k}B_k(x)\right\} = \frac{ze^{(x+\lambda)z}}{e^z-1} = \sum_{n=0}^{\infty}\frac{z^n}{n!}B_n(x+\lambda)$$

By comparing coefficients we get the important identity

$$\sum_{k=0}^{n} \binom{n}{k} B_k(x)\lambda^{n-k} = B_n(x+\lambda) \tag{2.3.1}$$

Where x *and* λ *are any two numbers.*

It is known that $B_n(1) = (-1)^n B_n(0) = (-1)^n B_n$*, where* B_n *are the Bernoulli numbers.*

With $x = 0$ *in (2.3.1) we find the well-known representation*

$$\sum_{k=0}^{n} \binom{n}{k} B_k\lambda^{n-k} = B_n(\lambda) \tag{2.3.2}$$

and when $\lambda = 1$ *we get*

$$\sum_{k=0}^{n} \binom{n}{k} B_k = (-1)^n B_n$$

Note that (2.3.1) can be written as binomial transform

$$\sum_{k=0}^{n} \binom{n}{k} B_k(x)\lambda^{-k} = \lambda^{-n} B_n(x+\lambda)$$

The above method can be used to compute the binomial transforms of sequences of special numbers, or polynomials which have an exponential generating function.

For Euler's polynomial $E_n(x)$ *with generating function (1.2.17) and by using (2.2.8) we compute in the same way*

$$\sum_{k=0}^{n} \binom{n}{k} E_k(x)\lambda^{n-k} = E_n(x+\lambda) \tag{2.3.3}$$

In the next example and also in several other places we shall use the lemma:

Lemma 2.3.2. *Given the power series*

$$f(t) = a_0 + a_1 t + a_2 t^2 + ...,$$

we have

$$\frac{1}{1-t} f(t) = \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} a_k \right\} t^n \tag{2.3.4}$$

Proof.

$$\begin{aligned} \frac{1}{1-t} f(t) &= (1 + t + t^2 + ...) \left(\sum_{n=0}^{\infty} a_n t^n \right) \\ &= (1 + t + t^2 + ...)(a_0 + a_1 t + a_2 t^2 + ...) \\ &= a_0 + (a_0 + a_1)t + (a_0 + a_1 + a_2)t^2 + ... \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} a_k \right\} t^n \end{aligned}$$

□

Example 2.3.3. *We use know formula (2.1.2) to find the binomial transform of the sequence*

$$a_n = \frac{(-1)^{n-1}}{n} \qquad n = 1, 2, 3...$$

The generating function of this sequence is

$$f(t) = \log(1 + t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} t^n$$

and then

$$\begin{aligned}\frac{1}{1-t}f\left(\frac{t}{1-t}\right) &= \frac{1}{1-t}\log\left(1+\frac{t}{1-t}\right)\\ &= \frac{1}{1-t}\log\left(\frac{1}{1-t}\right)\\ &= \frac{-\log(1-t)}{1-t}\\ &= \frac{1}{1-t}\sum_{n=1}^{\infty}\frac{t^n}{n}\end{aligned}$$

According to Lemma (2.3.2) this equals

$$\sum_{n=1}^{\infty}\left(1+\frac{1}{2}+...+\frac{1}{n}\right)t^n = \sum_{n=1}^{\infty}H_n t^n$$

Where H_n *are the Harmonic numbers (1.2.6).*

That is (2.1.2) yields to

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{k} = H_n \tag{2.3.5}$$

Assuming that the summation starts from zero with $a_0 = 0$ *we write*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{k} = H_n$$

and then applying (1.1.9) we find

$$H_{n+1} - H_n = \frac{1}{n+1} = \sum_{j=0}^{n}\binom{n}{j}\frac{(-1)^j}{j+1} \tag{2.3.6}$$

This is an example of a sequence $\frac{1}{j+1}$ *invariant for the symmetric binomial* *transform (1.1.3).*

By inversion in (2.3.5) we have also

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} H_n = \frac{1}{n} \tag{2.3.7}$$

Example 2.3.4. *Integrating the representation*

$$\frac{-\log(1-t)}{1-t} = \sum_{n=1}^{\infty} H_n t^n$$

we get

$$\frac{\log^2(1-t)}{2} = \sum_{n=1}^{\infty} \frac{H_n}{n+1} t^{n+1} \tag{2.3.8}$$

and then dividing by t both sides we find the generating function

$$\frac{\log^2(1-t)}{2t} = \sum_{n=1}^{\infty} \frac{H_n}{n+1} t^n \tag{2.3.9}$$

Replacing t by $-t$ we obtain also

$$\begin{aligned} \frac{\log^2(1+t)}{-2t} &= \sum_{n=1}^{\infty} \frac{H_n}{n+1} (-t)^n \\ \frac{\log^2(1+t)}{-2t} &= \sum_{n=1}^{\infty} \frac{H_n}{n+1} (-1)^n t^n \\ \frac{\log^2(1+t)}{2t} &= \sum_{n=1}^{\infty} \frac{H_n}{n+1} (-1)^{n-1} t^n \end{aligned}$$

We shall use now Euler's series transformation (2.1.2) to compute the binomial *transform of*

$$a_n = \frac{(-1)^{n-1}}{n+1} H_n$$

Applying (2.1.2) to the function in (2.3.9) we get,

$$\frac{1}{1-t}\left(\frac{1-t}{2t}\right)\log^2\left(\frac{1}{1-t}\right) = \frac{\log^2(1-t)}{2t} = \sum_{n=1}^{\infty} t^n \left\{\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{k-1} H_k}{k+1}\right\}$$

and computing this to (2.3.8) we conclude that

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{k-1} H_k}{k+1} = \frac{H_n}{n+1} \tag{2.3.10}$$

Remark Formula (2.1.2) can be put in a more flexible equivalent form

$$\frac{1}{1-\lambda t} f\left(\frac{\mu t}{1-\lambda t}\right) = \sum_{n=0}^{\infty} t^n \left\{\sum_{k=0}^{n} \binom{n}{k} \lambda^{n-k} \mu^k a_k\right\} \tag{2.3.11}$$

where λ and μ are appropriate parameters.

Proof. To show the equivalence between (2.3.11) and (2.1.2) we first write (2.1.1) in the form

$$f\left(\frac{\mu t}{\lambda}\right) = \sum_{k=0}^{\infty} a_k \left(\frac{\mu}{\lambda}\right)^k t^k$$

and then apply (2.1.2) to this function as a function of t to get

$$\frac{1}{1-t} f\left(\frac{\mu t}{1-\lambda t}\right) = \sum_{n=0}^{\infty} t^n \left\{\sum_{k=0}^{n} \binom{n}{k} \lambda^{n-k} \mu^k a_k\right\}$$

□

Lemma 2.3.5. *For sufficiently small $|t|$ the following representation holds*

$$\frac{-\log(1-\alpha t)}{1-\beta t} = \sum_{n=1}^{\infty}\left(\alpha\beta^{n-1} + \frac{1}{2}\alpha^2\beta^{n-2} + ... + \frac{1}{n}\alpha^n\right)t^n \tag{2.3.12}$$

Where α and β are small parameters.

Proof. From the expansion of the logarithm and lemma (2.3.2) we get the representation

$$\frac{-\log(1-\alpha t)}{1-t} = \sum_{n=1}^{\infty}\left(\alpha + \frac{1}{2}\alpha^2 + ... + \frac{1}{n}\alpha^n\right)t^n$$

Now replace t by βt and then replace $\alpha\beta$ by α we get

$$\frac{-\log(1-\alpha t)}{1-\beta t} = \sum_{n=1}^{\infty}\left(\alpha\beta^{n-1} + \frac{1}{2}\alpha^2\beta^{n-2} + ... + \frac{1}{n}\alpha^n\right)t^n$$

□

Example 2.3.6. *We show that for all λ and μ*

$$\sum_{k=1}^{n}\binom{n}{k}H_k\lambda^{n-k}\mu^k = (\lambda+\mu)^n H_n - \sum_{k=1}^{n}\frac{\lambda^k(\lambda+\mu)^{n-k}}{k} \tag{2.3.13}$$

Proof. We apply the transformation formula (2.3.11) to the function

$$\frac{-\log(1-t)}{1-t} = \sum_{n=1}^{\infty}H_n t^n$$

to find

$$\frac{-1}{1-\lambda t}\frac{\log\left(1-\frac{\mu t}{1-\lambda t}\right)}{1-\frac{\mu t}{1-\lambda t}} = \frac{-\log(1-(\lambda+\mu)t)}{1-(\lambda+\mu)t} + \frac{\log(1-\lambda t)}{1-(\lambda+\mu)t} \tag{2.3.14}$$

Which equals in view of (1.2.11) and the above lemma (with $\alpha = \lambda$ and $\beta = \lambda + \mu$). from (2.3.12)

$$\frac{-\log(1-\alpha t)}{1-\beta t} = \frac{-\log(1-\lambda t)}{1-(\lambda+\mu)t} = \sum_{n=1}^{\infty}\left\{\sum_{k=1}^{n}\frac{\lambda^k(\lambda+\mu)^{n-k}}{k}\right\}t^n$$

so (2.3.14),

$$\frac{-1}{1-\lambda t}\frac{\log\left(1-\frac{\mu t}{1-\lambda t}\right)}{1-\frac{\mu t}{1-\lambda t}} = \sum_{n=1}^{\infty}(\lambda+\mu)^n H_n t^n - \sum_{n=1}^{\infty}\left\{\sum_{k=1}^{n}\frac{\lambda^k(\lambda+\mu)^{n-k}}{k}\right\}t^n$$

At the same time (2.3.11) shows that (2.3.14) equals

$$\sum_{n=0}^{\infty} t^n \left\{\sum_{k=1}^{n}\binom{n}{k}H_k\lambda^{n-k}\mu^k\right\}$$

and comparing coefficients in these power series we finish the proof.

With $\lambda = \mu = 1$ in (2.3.13), we find the companion to formula (2.3.7)

$$\sum_{k=1}^{n}\binom{n}{k}H_k = 2^n\left(H_n - \sum_{k=1}^{n}\frac{1}{k2^k}\right) \tag{2.3.15}$$

With $\lambda = 1$ in (2.3.13), we have

$$\sum_{k=1}^{n}\binom{n}{k}H_k\mu^k = (1+\mu)^n H_n - \sum_{k=1}^{n}\frac{(1+\mu)^{n-k}}{k} \tag{2.3.16}$$

We apply to this equation the differential operator $\left(\mu\frac{d}{d\mu}\right)^m$ to get

$$\sum_{k=1}^{n}\binom{n}{k}H_k\mu^k k^m = \alpha(m,n,\mu)H_n - \sum_{k=1}^{n}\frac{\alpha(m,n-k,\mu)}{k} \tag{2.3.17}$$

Where

$$\alpha(m,n,\mu) = \left(\mu\frac{d}{d\mu}\right)^m (1+\mu)^n = \sum_{k=0}^{n} \binom{n}{k} k^m \mu^k \tag{2.3.18}$$
$$= \sum_{k=0}^{n} \binom{n}{k} k! S(m,k) \mu^k (1+\mu)^{n-k}$$

The second equality here follows from the obvious fact that

$$\left(\mu\frac{d}{d\mu}\right)^m \mu^k = k^m \mu^k$$

The third equality comes from the representation

$$\sum_{k=0}^{n} \binom{n}{k} k^m \mu^k = \sum_{k=0}^{n} \binom{n}{k} k! S(m,k) \mu^k (1+\mu)^{n-k} \tag{2.3.19}$$

□

Example 2.3.7. *We show here an interesting application of corollary (2.1.4) to series with central binomial coefficients. The central binomial coefficients are defined by*

$$\binom{2n}{n} = \frac{(2n!)}{(n!)^2}$$

First we note that by simple computation

$$\binom{\frac{-1}{2}}{n} (-1)^n = \frac{1}{4^n} \binom{2n}{n}$$

and then we set $\alpha = \frac{-1}{2}$ *in corollary (2.1.4) so that formula (2.2.4) turns into*

$$\sum_{n=0}^{\infty} \binom{2n}{n} a_n \frac{z^n}{4^n} = \frac{1}{\sqrt{z+1}} \sum_{n=0}^{\infty} \left(\frac{z}{z+1}\right)^n \frac{1}{4^n} \binom{2n}{n} \left\{ \sum_{k=0}^{n} \binom{n}{k} a_k \right\}$$

Substituting now by $z = 4x$ *we get*

$$\sum_{n=0}^{\infty} \binom{2n}{n} a_n x^n = \frac{1}{\sqrt{4x+1}} \sum_{n=0}^{\infty} \left(\frac{x}{4x+1}\right)^n \binom{2n}{n} \left\{ \sum_{k=0}^{n} \binom{n}{k} a_k \right\} \quad (2.3.20)$$

Setting here

$$a_n = (-1)^{n-1} H_n$$

and using equation (2.3.6) we find

$$\sum_{n=0}^{\infty} \binom{2n}{n} (-1)^{n-1} H_n x^n = \frac{1}{\sqrt{4x+1}} \sum_{n=1}^{\infty} \left(\frac{x}{4x+1}\right)^n \binom{2n}{n} \frac{1}{n} \quad (2.3.21)$$

Now we reach for the well-known expansion

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{z^n}{n} = 2\ln\left(\frac{1-\sqrt{1-4x}}{2z}\right) = 2\ln\left(\frac{2}{1+\sqrt{1-4x}}\right) \quad (2.3.22)$$

which we use to evaluate in closed from the right side in (2.3.21). Applying (2.3.22) with

$$z = \frac{x}{4x+1}$$

We find after simple computations

$$\sum_{n=0}^{\infty} \binom{2n}{n} (-1)^{n-1} H_n x^n = \frac{2}{\sqrt{4x+1}} \ln\left(\frac{2\sqrt{4x+1}}{1+\sqrt{4x+1}}\right) \quad (2.3.23)$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} \binom{2n}{n} (-1)^{n-1} H_n x^n &= \frac{1}{\sqrt{4x+1}} 2 \ln \left(\frac{2}{1+\sqrt{1-4\left(\frac{x}{4x+1}\right)}} \right) \\
&= \frac{2}{\sqrt{4x+1}} \ln \left(\frac{2}{\frac{1+\sqrt{4x+1}}{\sqrt{4x+1}}} \right) \\
&= \frac{2}{\sqrt{4x+1}} \ln \left(\frac{2\sqrt{4x+1}}{1+\sqrt{4x+1}} \right)
\end{aligned}$$

□

Remark

$$c_n = \frac{1}{n+1} \binom{2n}{n} \tag{2.3.24}$$

are the Catalan numbers.

Example 2.3.8. *In this example we present one application of formula (2.2.12) from proposition (2.2.4) to series with Hermite polynomials. Changing t to $-t$ and a_k to $(-1)^k a_k$ in that formula we write it here in the form*

$$\sum_{n=0}^{\infty} a_n c_n t^n = \sum_{n=0}^{\infty} \frac{(-1)^n g^{(n)}(t)}{n!} t^n \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k \right\} \tag{2.3.25}$$

Recall now that the Hermite polynomials $H_n(x)$ satisfy the Rodrigues equation

$$H_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx} \right)^n e^{-x^2}$$

and their generating function is

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

Corollary 2.3.9. *The following series transformation formula holds*

$$\sum_{n=0}^{\infty} a_n H_n(x) \frac{t^n}{n!} = e^{2xt-t^2} \sum_{n=0}^{\infty} (-1)^n H_n(x-t) \frac{t^n}{n!} \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k \right\} \quad (2.3.26)$$

for any sequence of coefficients $\{a_n\}$.

Proof. We use equation (2.3.25) with $g(t) = e^{2xt-t^2}$ and $c_n = \frac{H_n(x)}{n!}$ so (2.3.25) becomes

$$\sum_{n=0}^{\infty} \frac{a_n H_n(x)}{n!} t^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{d}{dt}\right)^n e^{2xt-t^2} t^n \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k \right\}$$

From the Rodrigues formula we find

$$\begin{aligned} \left(\frac{d}{dt}\right)^n e^{2xt-t^2} &= \left(\frac{d}{dt}\right)^n e^{x^2} e^{-(x-t)^2} \\ = e^{x^2} (-1)^n \left(\frac{d}{dt}\right)^n e^{-(x-t)^2} &= e^{x^2} (-1)^n \left(\frac{d}{d(x-t)}\right)^n e^{-(x-t)^2} \\ = e^{x^2} (-1)^n (-1)^n e^{-(x-t)^2} H_n(x-t) &= e^{2xt-t^2} H_n(x-t) \end{aligned}$$

That is,

$$\left(\frac{d}{dt}\right)^n g(t) = e^{2xt-t^2} H_n(x-t)$$

and (2.3.26) follows now from (2.3.25). □

Next we consider the binomial transform identity

$$(-1)^n n! S(m,n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k k^m$$

Where $S(m,n)$ are the Stirling numbers of the second kind. This is the inversion of (1.2.1), a well-known analytic representation of the numbers $S(m,n)$. Here m,n are any two non negative integers. We substitute $a_k = k^m$ in (2.3.26) to get the closed form evaluation

$$\sum_{k=0}^{\infty} k^m H_k(x) \frac{t^k}{k!} = e^{2xt-t^2} \sum_{n=0}^{m} S(m,n) H_n(x-t) t^n \tag{2.3.27}$$

The series on the right hand side truncates because $S(m,n) = 0$ for $n > m$.

Example 2.3.10. *The Chebyshev polynomials of the first kind $T_n(x)$ and the Chebyshev polynomials of the second kind $U_n(x)$ have exponential generating functions correspondingly*

$$\begin{aligned} \sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!} &= e^{xt} \cos\left(t\sqrt{1-x^2}\right) \\ \sum_{n=1}^{\infty} U_{n-1}(x) \frac{t^n}{n!} &= e^{xt} \frac{\sin\left(t\sqrt{1-x^2}\right)}{\sqrt{1-x^2}} \end{aligned}$$

Setting $x = \cos\theta$ we can rewrite these series in the form

$$\begin{aligned} e^{-t\cos\theta} \sum_{n=0}^{\infty} T_n(\cos\theta) \frac{t^n}{n!} &= \cos(t\sin\theta) \\ e^{-t\cos\theta} \sum_{n=0}^{\infty} U_{n-1}(\cos\theta) \frac{t^n}{n!} &= \frac{\sin(t\sin\theta)}{\sin\theta} \end{aligned}$$

Using now Euler's series transformation (2.2.8) we have

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \left\{ \sum_{k=0}^{n} \binom{n}{k} (-\cos\theta)^{n-k} T_k(\cos\theta) \right\} = \cos(t\sin\theta)$$
$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \left\{ \sum_{k=0}^{n} \binom{n}{k} (-\cos\theta)^{n-k} U_{k-1}(\cos\theta) \right\} = \frac{\sin(t\sin\theta)}{\sin\theta}$$

or

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \left\{ (-\cos\theta)^n \sum_{k-0}^{n} \binom{n}{k} (-1)^k \frac{T_k(\cos\theta)}{(\cos\theta)^n} \right\} = \cos(t\sin\theta)$$
$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \left\{ (-\cos\theta)^n \sum_{k-0}^{n} \binom{n}{k} (-1)^k \frac{U_{k-1}(\cos\theta)}{(\cos\theta)^n} \right\} = \frac{\sin(t\sin\theta)}{\sin\theta}$$

Proof. Let $z = t$ and $\lambda = -\cos\theta$ in (3.13) we get

$$\begin{aligned} e^{-t\cos\theta} \sum_{n=0}^{\infty} T_n(\cos\theta) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \left\{ \sum_{k=0}^{n} \binom{n}{k} (-\cos\theta)^{n-k} T_k(\cos\theta) \right\} \\ &= \cos(t\sin\theta) \end{aligned}$$

Same for $\frac{\sin(t\sin\theta)}{\sin\theta}$ □

Comparing coefficients of both sides we obtain the binomial transforms

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{T_k(\cos\theta)}{(\cos\theta)^n} = \begin{cases} 0 & (n \text{ odd}) \\ (-1)^{\frac{n}{2}} (\tan\theta)^n & (n \text{ even}) \end{cases} \tag{2.3.28}$$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{U_{k-1}(\cos\theta)}{(\cos\theta)^n} = \begin{cases} 0 & (n \text{ even}) \\ (-1)^{1+\frac{n}{2}} \frac{(\tan\theta)^{n-1}}{\cos\theta} & (n \text{ odd}) \end{cases} \tag{2.3.29}$$

Example 2.3.11. *The following representation is true*

$$\frac{-2}{1-t} Li_2\left(\frac{-t}{1-t}\right) = \sum_{n=0}^{\infty} \left(H_n^2 + H_n^{(2)}\right) t^n$$

Where Li_2 is the dilogarithm (1.2.13). We have

$$-Li_2(-t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^n}{n^2}$$

According to Euler's transformation formula (2.1.1) we have

$$\frac{-1}{1-t} Li_2\left(\frac{-t}{1-t}\right) = \sum_{n=0}^{\infty} t^n \left\{ \sum_{k=1}^{\infty} \binom{n}{k} \frac{(-1)^{k-1}}{k^2} \right\}$$

and therefore,

$$\sum_{k=1}^{\infty} \binom{n}{k} \frac{(-1)^{k-1}}{k^2} = \frac{1}{2}\left(H_n^2 + H_n^{(2)}\right)$$

Chapter 3

A Binomial Formula for Evaluating Integrals

In this chapter we will refer to [2] to present a rule for evaluating integrals in terms of series with binomial expressions.

Theorem 3.0.1. *Let $f(x)$ be a function defined and integrable on $(-r, \lambda]$ for some $r > 0$,$\lambda > 0$.*
Let also $f(x)$ be analytic in a neighborhood of the origin with Taylor series $f(x) = \sum_{n=0}^{\infty} a_n x^n$.
Then we have

$$\int_0^{\lambda} f(x)dx = \sum_{n=0}^{\infty} \left(\frac{\lambda}{\lambda+1}\right)^{n+1} \frac{1}{n+1} \sum_{m=0}^{n} b_m = \sum_{n=0}^{\infty} \left(\frac{\lambda}{\lambda+1}\right)^{n+1} \sum_{k=0}^{n} \binom{n}{k} \frac{a_k}{k+1}$$

where the sequence b_n is the binomial transform of the sequence a_n.

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$$

In particular, for $\lambda = 1$ *we have*

$$\int_0^1 f(x)dx = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}(n+1)} \sum_{m=0}^{n} b_m = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{a_k}{k+1}$$

and for $\lambda \to \infty$

$$\int_0^{\infty} f(x)dx = \sum_{n=0}^{\infty} \frac{1}{(n+1)} \sum_{m=0}^{n} b_m = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{a_k}{k+1}$$

Proof. With the substitution $x = \frac{t}{1-t}$, $t = \frac{x}{x+1}$ we get

$$\begin{aligned}
\int_0^{\lambda} f(x)dx &= \int_0^{\frac{\lambda}{\lambda+1}} \frac{1}{(1-t)^2} f\left(\frac{t}{1-t}\right) dt \\
&= \int_0^{\frac{\lambda}{\lambda+1}} \frac{1}{(1-t)} \left\{ \frac{1}{1-t} f\left(\frac{t}{1-t}\right) \right\} dt \\
&= \int_0^{\frac{\lambda}{\lambda+1}} \frac{1}{(1-t)} \left\{ \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \binom{n}{k} a_k \right\} dt \\
&= \int_0^{\frac{\lambda}{\lambda+1}} \frac{1}{(1-t)} \left\{ \sum_{n=0}^{\infty} b_n t^n \right\} dt
\end{aligned}$$

by using Euler's transformation formula

$$\frac{1}{1-t} f\left(\frac{t}{1-t}\right) = \sum_{n=0}^{\infty} t^n \left\{ \sum_{k=0}^{n} \binom{n}{k} a_k \right\} = \sum_{n=0}^{\infty} b_n t^n$$

where the sequence $\{b_n\}$ is the binomial transform of the sequence $\{a_n\}$ as described above.

Expanding $(1-t)^{-1}$ as geometric series and using Cauchy's rule for multiplication of two power series we write

$$\int_0^{\frac{\lambda}{\lambda+1}} \frac{1}{(1-t)} \left\{ \sum_{n=0}^{\infty} b_n t^n \right\} dt = \int_0^{\frac{\lambda}{\lambda+1}} \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} b_k \right\} t^n dt = \sum_{n=0}^{\infty} \left(\frac{\lambda}{\lambda+1} \right)^{n+1} \frac{1}{n+1} \sum_{k=0}^{n} b_k$$

by the property

$$\sum_{k=0}^{n} \binom{n}{k} \frac{a_k}{k+1} = \frac{1}{n+1} \sum_{k=0}^{n} b_k$$

□

Differentiating in the theorem with respect to λ we come to the following result.

Corollary 3.0.2. *Under the conditions of the theorem we have the representation*

$$f(\lambda) = \frac{1}{(\lambda+1)^2} \sum_{n=0}^{\infty} \left(\frac{\lambda}{\lambda+1} \right)^n \sum_{m=0}^{n} b_m$$

Now we give some applications of our theorem in form of examples

Example 3.0.3. *In the first example we will evaluate the integral*

$$\int_0^{\infty} \frac{\log(1+t)}{t(1+t)} dt$$

We start from the well-known series

$$\sum_{n=1}^{\infty} H_n t^n = \frac{-\log(1-t)}{1-t}$$

(Here $H_n = 1 + \frac{1}{2} + ... + \frac{1}{n}, H_0 = 0$ *are the harmonic numbers).*

Now replacing t by −t and dividing both sides by t we get

$$\begin{aligned}
\frac{-\log(1+t)}{t(1+t)} &= \sum_{n=1}^{\infty} H_n \frac{(-t)^n}{t} \\
&= \sum_{k=0}^{\infty} H_{k+1} \frac{(-t)^{k+1}}{t} && ; k = n-1 \\
\frac{\log(1+t)}{t(1+t)} &= \sum_{k=0}^{\infty} H_{k+1} (-1)^k t^k
\end{aligned}$$

and take $a_k = (-1)^k H_{k+1}$.

$$\sum_{k=0}^{n} \binom{n}{k} \frac{a_k}{k+1} = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k H_{k+1}}{k+1} = \frac{1}{(n+1)^2}$$

hence,

$$\int_0^{\infty} \frac{\log(1+t)}{t(1+t)} dt = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{a_k}{k+1} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6}$$

Example 3.0.4. *Here, we will evaluate the difficult integral*

$$\int_0^{\infty} \left(\frac{\log(1+t)}{t} \right)^2 dt$$

We will start from the well-known power series

$$\frac{\log^2(1-t)}{-2t} = \sum_{n=1}^{\infty} \frac{H_n t^n}{n+1}$$

Replacing t by $-t$ and dividing both sides by t, we get

$$\begin{aligned}
\frac{\log^2(1+t)}{-2t^2} &= \sum_{n=1}^{\infty} \frac{H_n(-t)^n}{t(n+1)} \\
\frac{\log^2(1+t)}{-2t^2} &= \sum_{n=1}^{\infty} \frac{(-1)^n H_n t^{n-1}}{n+1} \\
\frac{\log^2(1+t)}{-2t^2} &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} H_n t^{n-1}}{n+1} = \sum_{k=0}^{\infty} \frac{(-1)^k H_{k+1} t^k}{k+2} \; ; k = n-1
\end{aligned}$$

So take $a_k = \dfrac{(-1)^k H_{k+1}}{k+2}$

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} \frac{a_k}{k+1} &= \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k H_{k+1}}{(k+1)(k+2)} \\
&= \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k H_{k+1}}{k+1} - \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k H_{k+1}}{k+2} \\
&= \frac{1}{(n+1)^2} - \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k+2} \left(H_k + \frac{1}{k+1} \right)
\end{aligned}$$

We know that,

$$\begin{aligned}
-\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k+2} \left(H_k + \frac{1}{k+1} \right) &= -\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k H_k}{k+2} - \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(k+1)(k+2)} \\
&= \frac{n+H_n}{(n+1)(n+2)} - \frac{1}{n+2} \\
&= \frac{H_n - 1}{(n+1)(n+2)}
\end{aligned}$$

It is easy to see that

$$\sum_{n=0}^{\infty} \frac{H_n - 1}{(n+1)(n+2)} = \sum_{n=0}^{\infty} \frac{H_n}{(n+1)(n+2)} - \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = 1 - 1 = 0$$

and we compute

$$\begin{aligned}\int_0^\infty \frac{\log^2(1+t)}{2t^2}dt &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{a_k}{k+1} \\ &= \sum_{n=0}^{\infty}\frac{1}{(n+1)^2} \\ &= \frac{\pi^2}{6}\end{aligned}$$

Finally, $\displaystyle\int_0^\infty \left(\frac{\log(1+t)}{t}\right)^2 dt = \frac{\pi^2}{3}$

Example 3.0.5. *Using some well-known generating functions we evaluate here the integral*

$$\int_0^1 Li_2\left(\frac{t}{1+t}\right) dt$$

$Li_2(x)$ *is the dilogarithm.*

$$Li_2(x) = \sum_{n=1}^{\infty}\frac{x^n}{n^2} \quad (|x|<1)$$

We have,

$$Li_2\left(\frac{t}{1+t}\right) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}H_n t^n}{n} \quad (|t|<1)$$

So we take,

$$a_n = \frac{(-1)^{n-1}H_n}{n}$$

So

$$\frac{a_k}{k+1} = \frac{(-1)^{k-1}H_k}{k(k+1)} = \frac{(-1)^{k-1}H_k}{k} - \frac{(-1)^{k-1}H_k}{K+1}$$

And using the two binomial formulas

$$\begin{aligned}\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\frac{H_k}{k} &= \sum_{k=1}^{n}\frac{1}{k^2} = H_n^{(2)}\\ \sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\frac{H_k}{k+1} &= \frac{H_n}{k+1}\end{aligned}$$

we have

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}\frac{a_k}{k+1} &= \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{k-1}H_k}{k} - \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{k-1}H_k}{k+1}\\ &= H_n^{(2)} - \frac{H_n}{n+1}\end{aligned}$$

Here, $H_n^{(2)} = 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}\ ; H_0^{(2)} = 0$

This way,

$$\begin{aligned}\int_0^1 Li_2\left(\frac{t}{1+t}\right)dt &= \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\sum_{k=0}^{n}\binom{n}{k}\frac{a_k}{k+1}\\ &= \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\left(H_n^{(2)} - \frac{H_n}{n+1}\right)\\ &= \sum_{n=0}^{\infty}\frac{H_n^{(2)}}{2^{n+1}} - \sum_{n=0}^{\infty}\frac{H_n}{2^{n+1}(n+1)}\end{aligned}$$

These two series are easy to evaluate.

We have,

$$\sum_{n=0}^{\infty} H_n^{(2)} x^n = \frac{Li_2(x)}{1-x} \quad ; \quad \sum_{n=0}^{\infty} \frac{H_n x^n}{n+1} = \frac{\log^2(1-x)}{2x} \quad |x| < 1$$

And we compute with $x = \frac{1}{2}$

$$\begin{aligned}
\int_0^1 Li_2\left(\frac{t}{1+t}\right) dt &= \frac{1}{2}\left(\frac{Li_2\left(\frac{1}{2}\right)}{1-\frac{1}{2}} - \frac{\log^2\left(1-\frac{1}{2}\right)}{2\times\frac{1}{2}}\right) \\
&= \frac{1}{2}\left(\frac{Li_2\left(\frac{1}{2}\right)}{\frac{1}{2}} - \frac{\log^2\left(\frac{1}{2}\right)}{1}\right) \\
&= Li_2\left(\frac{1}{2}\right) - \frac{\log^2(2)}{2} \\
&= \frac{\pi^2}{12} - \frac{\log^2(2)}{2}
\end{aligned}$$

and at the end we used the well known formula

$$Li_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{\log^2(2)}{2}$$

Chapter 4

Evaluation of Integrals by Differentiation with Respect to a Parameter

We review in this chapter a special technique for evaluating challenging integrals by differentiating with respect to a parameter. This technique will be illustrated by providing a number of examples from [3, 4], as well as presenting new examples.

Example 4.0.1. *In this example we will evaluate the following integral*

$$\int_0^\infty \frac{\sin x}{x} dx$$

In order to do so, we will introduce the function

$$F(\lambda) = \int_0^\infty e^{-\lambda x} \frac{\sin x}{x} dx \qquad ; \lambda > 0$$

Now differentiate this function we get

$$\begin{aligned} F'(\lambda) &= -\int_0^\infty e^{-\lambda x} \sin x dx \\ F'(\lambda) &= \frac{-1}{1+\lambda^2} \end{aligned}$$

(Laplace transform of the sine function).

Integrating back we find

$$F(\lambda) = -\arctan(\lambda) + C$$

Setting $\lambda \to \infty$ *yields the equation*

$$0 = -\frac{\pi}{2} + C; C = \frac{\pi}{2}$$

and therefore,

$$F(\lambda) = \int_0^\infty e^{-\lambda x} \frac{\sin x}{x} dx = \frac{\pi}{2} - \arctan\lambda$$

Taking limits for $\lambda \to 0$ *we find*

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

Example 4.0.2. *We will now evaluate the following integral*

$$\int_0^1 \frac{\ln(1+x)}{1+x^2} dx$$

Consider the function

$$F(\lambda) = \int_0^1 \frac{\ln(1+\lambda x)}{1+x^2}dx \qquad ; \lambda \geq 0$$

Now differentiate this function we get

$$F'(\lambda) = \int_0^1 \frac{x}{(1+\lambda x)(1+x^2)}dx.$$

To evaluate this integral we will split the integrand into partial fractions.

$$\begin{aligned} \frac{x}{(1+\lambda x)(1+x^2)} &= \frac{A}{1+\lambda x} + \frac{Bx+C}{1+x^2} \\ &= \frac{A + Ax^2 + Bx + B\lambda x^2 + C + C\lambda x}{(1+\lambda x)(1+x^2)} \end{aligned}$$

The coefficients satisfy

$$\begin{cases} A + B\lambda = 0 \\ B + C\lambda = 1 \\ A + C = 0 \end{cases}$$

Hence, we have

$$\begin{aligned} B &= \frac{1}{1+\lambda^2} \\ A &= \frac{-\lambda}{1+\lambda^2} \\ C &= \frac{\lambda}{1+\lambda^2} \end{aligned}$$

so,

$$
\begin{aligned}
F'(\lambda) &= \int_0^1 \frac{-\lambda}{(1+\lambda x)(1+\lambda^2)}dx + \int_0^1 \frac{x}{(1+x^2)(1+\lambda^2)}dx + \int_0^1 \frac{\lambda}{(1+\lambda^2)(1+x^2)}dx \\
&= -\frac{\ln(1+\lambda)}{1+\lambda^2} + \frac{1}{2}\ln(2)\frac{1}{1+\lambda^2} + \frac{\pi}{4}\frac{\lambda}{1+\lambda^2}
\end{aligned}
$$

Integrating back we find

$$F(\lambda) = -\int_0^\lambda \frac{\ln(1+x)}{1+x^2}dx + \frac{\ln 2}{2}\arctan\lambda + \frac{\pi}{8}\ln(1+\lambda^2)$$

and setting $\lambda = 1$ *we get to the equation*

$$2F(1) = \frac{\pi}{4}\ln 2.$$

That is,

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}dx = \frac{\pi}{8}\ln 2$$

Example 4.0.3. *We will now evaluate the following integral*

$$\int_0^\infty \left(\frac{1-e^{-x}}{x}\right)^2 dx$$

Consider the function

$$F(\lambda) = \int_0^\infty \left(\frac{1-e^{-\lambda x}}{x}\right)^2 dx = \lambda \ln 4 \qquad ; \lambda > 0$$

Now differentiate this function with respect to λ*. We find*

$$F'(\lambda) = 2\int_0^\infty \left(\frac{1 - e^{-\lambda x}}{x}\right) e^{-\lambda x} dx = 2\int_0^\infty \frac{e^{-\lambda x} - e^{-2\lambda x}}{x} dx.$$

By using Frullani's formula for the last equality (see below), we conclude that $F(\lambda)$ *is a linear function and since* $F(0) = 0$ *we can write* $F(\lambda) = \lambda \ln 4$*. With* $\lambda = 1$ *we find* $F(1) = \ln 4$*.*

Frullani's formula says that for appropriate functions $f(x)$ *we have*

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} dx = [f(0) - f(\infty)] \ln \frac{b}{a}$$

Now we will introduce more general examples

Example 4.0.4. *Consider the integral*

$$F(\lambda) = \int_0^\infty e^{-\beta x} \frac{1 - \cos \lambda x}{x} dx \qquad ; \beta > 0$$

Now differentiate this function we get

$$F'(\lambda) = \int_0^\infty e^{-\beta x} \sin \lambda x dx = \frac{\lambda}{\lambda^2 + \beta^2}$$

Integrating back we find

$$F(\lambda) = \frac{1}{2} \ln(\lambda^2 + \beta^2) + C(\beta)$$

To compute $C(\beta)$ we set $\lambda = 0$ and this gives $C(\beta) = -\frac{1}{2}\ln \beta^2$. Therefore,

$$\int_0^\infty e^{-\beta x}\frac{1-\cos\lambda x}{x}dx = \frac{1}{2}\ln\left(1+\frac{\lambda^2}{\beta^2}\right)$$

Example 4.0.5. *A symmetrical analog to the previous example is the integral*

$$F(\lambda) = \int_0^\infty \frac{1-e^{-\lambda x}}{x}\cos\beta x dx$$

defined for $\lambda \geq 0$ and $\beta \neq 0$. The integral is divergent at infinity when $\beta = 0$. Now differentiate this function we get

$$F'(\lambda) = \int_0^\infty e^{-\lambda x}\cos\beta x dx = \frac{\lambda}{\lambda^2+\beta^2}$$

Integrating back we find

$$F(\lambda) = \int_0^\infty \frac{1-e^{-\lambda x}}{x}\cos\beta x dx = \frac{1}{2}\ln\left(1+\frac{\lambda^2}{\beta^2}\right)$$

so that for any $\lambda \geq 0$, $\beta > 0$

$$\int_0^\infty \frac{1-e^{-\lambda x}}{x}\cos\beta x dx = \int_0^\infty e^{-\beta x}\frac{1-\cos\lambda x}{x}dx$$

Note that the integral

$$\int_0^\infty \frac{e^{-\lambda x}-e^{-\mu x}}{x}\cos\beta x dx$$

can be reduced to the above integral by writing

$e^{-\lambda x} - e^{-\mu x} = (e^{-\lambda x} - 1) + (1 - e^{-\mu x})$ *and splitting it in two integrals. Thus,*

$$\int_0^\infty \frac{e^{-\lambda x} - e^{-\mu x}}{x} \cos \beta x dx = \frac{1}{2} \ln \frac{\mu^2 + \beta^2}{\lambda^2 + \beta^2}$$

Example 4.0.6. *Using the well-known Gaussian integral, also known as the Euler-Poisson integral,*

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

we can evaluate for $\lambda \geq 0$ the integral

$$F(\lambda) = \int_0^\infty \frac{1 - e^{-\lambda x^2}}{x^2} dx.$$

Now differentiate this function we get

$$F'(\lambda) = \int_0^\infty e^{-\lambda x^2} dx = \frac{1}{\sqrt{\lambda}} \int_0^\infty e^{-(x\sqrt{\lambda})^2} dx\sqrt{\lambda} = \frac{\sqrt{\pi}}{2\sqrt{\lambda}}$$

Integrating back we find

$$F(\lambda) = \int_0^\infty \frac{1 - e^{-\lambda x^2}}{x^2} dx = \sqrt{\lambda \pi}$$

Example 4.0.7. *We can use partial derivatives as in the following integral. Consider the function*

$$F(\lambda, \mu) = \int_0^\infty \frac{e^{-px} \cos qx - e^{-\lambda x} \cos \mu x}{x} dx$$

Here $\lambda > 0$, μ will be variables and $p > 0$, q will be fixed.

Then the partial derivatives will be

$$\begin{aligned} F_\lambda(\lambda,\mu) &= \int_0^\infty e^{-\lambda x}\cos\mu x dx = \frac{\lambda}{\lambda^2+\mu^2} \\ F_\mu(\lambda,\mu) &= \int_0^\infty e^{-\lambda x}\sin\mu x dx = \frac{\mu}{\lambda^2+\mu^2} \end{aligned}$$

Integrate back we find

$$F(\lambda,\mu) = \frac{1}{2}\ln(\lambda^2+\mu^2) + C(p,q),$$

where $C(p,q)$ *is unknown. The integral will be* 0 *when* $\lambda = p$ *and* $\mu = q$*, so from the last equation we find* $C(p,q) = -\ln(p^2+q^2)/2$*. Therefore,*

$$\int_0^\infty \frac{e^{-px}\cos qx - e^{-\lambda x}\cos\mu x}{x}dx = \frac{1}{2}\ln\frac{\lambda^2+\mu^2}{p^2+q^2}$$

Example 4.0.8. *Now consider the function*

$$J(\lambda) = \int_0^\infty e^{-\lambda x}\frac{\sin(ax)\sin(bx)}{x}dx$$

Now differentiate we get

$$\begin{aligned} J'(\lambda) &= -\int_0^\infty e^{-\lambda x}\sin(ax)\sin(bx)dx \\ &= \frac{1}{2}\left\{\int_0^\infty e^{-\lambda x}\cos(a+b)xdx - \int_0^\infty e^{-\lambda x}\cos(a-b)xdx\right\} \\ &= \frac{1}{2}\left\{\frac{\lambda}{\lambda^2+(a+b)^2} - \frac{\lambda}{\lambda^2+(a-b)^2}\right\} \end{aligned}$$

Integrating with respect to λ we get

$$J(\lambda) = \frac{1}{4}\ln\frac{\lambda^2 + (a+b)^2}{\lambda^2 + (a-b)^2} + C$$

letting $\lambda \to \infty$ we get that $C = 0$. Hence,

$$J(\lambda) = \int_0^\infty e^{-\lambda x}\frac{\sin(ax)\sin(bx)}{x}dx = \frac{1}{4}\ln\frac{\lambda^2 + (a+b)^2}{\lambda^2 + (a-b)^2}$$

Example 4.0.9. *Using the previous example we can evaluate also*

$$G(\lambda) = \int_0^\infty e^{-\lambda x}\frac{\sin(ax)\sin(bx)}{x^2}dx$$

where again $a > b > 0$.

Differentiate now with respect to λ we get

$$G'(\lambda) = -\int_0^\infty e^{-\lambda x}\frac{\sin(ax)\sin(bx)}{x}dx = -J(\lambda)$$

and now integrating by parts,

$$G(\lambda) = \frac{\lambda}{4}\ln\frac{\lambda^2 + (a-b)^2}{\lambda^2 + (a+b)^2} - \frac{1}{4}\int\left(\frac{(a+b)^2}{\lambda^2 + (a-b)^2} - \frac{(a-b)^2}{\lambda^2 + (a-b)^2}\right)dx$$

and the integration becomes easy. The result is

$$G(\lambda) = \frac{\lambda}{4}\ln\frac{\lambda^2 + (a-b)^2}{\lambda^2 + (a+b)^2} + \frac{a-b}{2}\arctan\frac{\lambda}{a-b} - \frac{a+b}{2}\arctan\frac{\lambda}{a+b} + C$$

Letting $\lambda \to \infty$ *we get that,*

$$\begin{aligned} 0 &= 0 + \frac{(a-b)\pi}{4} - \frac{(a+b)\pi}{4} + C \\ 0 &= \frac{-\pi b}{2} + C \\ C &= \frac{\pi b}{2} \end{aligned}$$

Hence,

$$G(\lambda) = \frac{\lambda}{4} \ln \frac{\lambda^2 + (a-b)^2}{\lambda^2 + (a+b)^2} + \frac{a-b}{2} \arctan \frac{\lambda}{a-b} - \frac{a+b}{2} \arctan \frac{\lambda}{a+b} + \frac{\pi b}{2}$$

With $\lambda = 0$ *we get that*

$$\int_0^\infty \frac{\sin(ax)\sin(bx)}{x^2} dx = \frac{\pi b}{2}$$

where $(a \geq b > 0)$.

Example 4.0.10. *Consider the integral*

$$G(\lambda) = \int_0^\infty e^{-\lambda x} \frac{\sin(ax)\cos(bx)}{x} dx$$

Suppose $a > b > 0$. *Differentiate now with respect to* λ *we get*

$$\begin{aligned} G'(\lambda) &= -\int_0^\infty e^{-\lambda x} \sin(ax)\cos(bx) dx \\ &= \frac{-1}{2} \left\{ \int_0^\infty e^{-\lambda x} \sin(a+b)x dx + \int_0^\infty e^{-\lambda x} \sin(a-b)x dx \right\} \\ &= \frac{-1}{2} \left\{ \frac{a+b}{\lambda^2 + (a+b)^2} + \frac{a-b}{\lambda^2 + (a-b)^2} \right\} \end{aligned}$$

after integrating with respect to λ,

$$G(\lambda) = \frac{-1}{2}\left\{\arctan\frac{\lambda}{a+b} + \arctan\frac{\lambda}{a-b}\right\} + C$$

Letting $\lambda \to \infty$ we get that,

$$\begin{aligned} 0 &= \frac{-1}{2}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) + C \\ 0 &= \frac{-\pi}{2} + C \\ C &= \frac{\pi}{2} \end{aligned}$$

Hence,

$$G(\lambda) = \int_0^\infty e^{-\lambda x}\frac{\sin(ax)\cos(bx)}{x}dx = \frac{-1}{2}\left\{\arctan\frac{\lambda}{a+b} + \arctan\frac{\lambda}{a-b}\right\} + \frac{\pi}{2}$$

Setting $b \to a$ we find also

$$\int_0^\infty e^{-\lambda x}\frac{\sin(ax)\cos(bx)}{x}dx = \frac{\pi}{4} - \frac{1}{2}\arctan\frac{\lambda}{2a}.$$

Using the identity $2\sin(ax)\cos(ax) = \sin(2ax)$ the integral can be reduced to (4.0.1).

Example 4.0.11. *Consider the integral*

$$G(\lambda) = \int_0^\infty e^{-\lambda x}\frac{\sin^3(ax)}{x}dx$$

Differentiate now with respect to λ,

$$\begin{aligned}
G'(\lambda) &= -\int_0^\infty e^{-\lambda x}\sin^3(ax)dx \\
&= \frac{-1}{2}\int_0^\infty e^{-\lambda x}(1-\cos 2ax)\sin ax dx \\
&= \frac{-1}{2}\int_0^\infty e^{-\lambda x}\sin(ax)dx + \frac{1}{2}\int_0^\infty e^{-\lambda x}\cos(2ax)\sin(ax)dx \\
&= \frac{-1}{2}\int_0^\infty e^{-\lambda x}\sin(ax)dx + \frac{1}{4}\int_0^\infty e^{-\lambda x}\left(\sin(3ax)-\sin(ax)\right)dx \\
&= \frac{-1}{2}\int_0^\infty e^{-\lambda x}\sin(ax)dx + \frac{1}{4}\int_0^\infty e^{-\lambda x}\sin(3ax)dx - \frac{1}{4}\int_0^\infty \sin(ax)dx \\
&= \frac{-3}{4}\int_0^\infty e^{-\lambda x}\sin(ax)dx + \frac{1}{4}\int_0^\infty e^{-\lambda x}\sin(3ax)dx \\
G'(\lambda) &= \frac{-3}{4}\left(\frac{a}{\lambda^2+a^2}\right) + \frac{1}{4}\left(\frac{3a}{\lambda^2+9a^2}\right)
\end{aligned}$$

Integrate now with respect to λ *we get,*

$$G(\lambda) = \frac{-3}{4}\arctan\frac{\lambda}{a} + \frac{1}{4}\arctan\frac{\lambda}{3a} + C$$

Letting $\lambda \to \infty$ *we get that,*

$$\begin{aligned}
0 &= \frac{-3\pi}{8} + \frac{\pi}{8} + C \\
0 &= \frac{-\pi}{4} + C \\
C &= \frac{\pi}{4}
\end{aligned}$$

Hence,

$$\int_0^\infty e^{-\lambda x}\frac{\sin^3(ax)}{x}dx = \frac{-3}{4}\arctan\frac{\lambda}{a} + \frac{1}{4}\arctan\frac{\lambda}{3a} + \frac{\pi}{4}$$

With $\lambda = 0$ we get that,

$$\int_0^\infty \frac{\sin^3(ax)}{x}dx = \frac{\pi}{4}$$

Example 4.0.12. *Consider the integral*

$$J(\lambda) = \int_0^\infty e^{-\lambda x}\frac{\sin^2(ax)\sin(bx)\sin(cx)}{x}dx$$

Differentiate now with respect to λ we get,

$$\begin{aligned}
J'(\lambda) &= -\int_0^\infty e^{-\lambda x}\sin^2(ax)\sin(bx)\sin(cx)dx \\
&= -\int_0^\infty e^{-\lambda x}\left(\frac{1-\cos 2ax}{2}\right)\sin bx \sin cx dx \\
&= \frac{-1}{2}\int_0^\infty e^{-\lambda x}\sin bx \sin cx dx + \frac{1}{2}\int_0^\infty e^{-\lambda x}\cos 2ax \sin bx \sin cx dx
\end{aligned}$$

Let

$$\begin{aligned}
A &= \frac{-1}{2}\int_0^\infty e^{-\lambda x}\sin bx \sin cx dx \\
&= \frac{1}{4}\left\{\int_0^\infty e^{-\lambda x}\cos(b+c)x dx - \int_0^\infty e^{-\lambda x}\cos(b-c)x dx\right\} \\
&= \frac{1}{4}\left\{\frac{\lambda}{\lambda^2+(b+c)^2} - \frac{\lambda}{\lambda^2+(b-c)^2}\right\}
\end{aligned}$$

and let

$$\begin{aligned}
B &= \frac{1}{2}\int_0^\infty e^{-\lambda x}\cos 2ax\sin bx\sin cx dx \\
&= \frac{1}{4}\int_0^\infty e^{-\lambda x}\cos 2ax\left(\cos(b-c)x-\cos(b+c)x\right)dx \\
&= \frac{1}{4}\int_0^\infty e^{-\lambda x}\cos 2ax\cos(b-c)xdx-\frac{1}{4}\int_0^\infty e^{-\lambda x}\cos 2ax\cos(b+c)xdx \\
&= \frac{1}{8}\int_0^\infty e^{-\lambda x}\left(\cos(2a+b-c)x+\cos(2a-b+c)x\right)dx \\
&\quad -\frac{1}{8}\int_0^\infty e^{-\lambda x}\left(\cos(2a+b+c)x+\cos(2a-b-c)x\right)dx \\
&= \frac{1}{8}\int_0^\infty e^{-\lambda x}\cos(2a+b-c)xdx+\frac{1}{8}\int_0^\infty e^{-\lambda x}\cos(2a-b+c)xdx \\
&\quad -\frac{1}{8}\int_0^\infty e^{-\lambda x}\cos(2a+b+c)xdx-\frac{1}{8}\int_0^\infty e^{-\lambda x}\cos(2a-b-c)xdx \\
&= \frac{1}{8}\left(\frac{\lambda}{\lambda^2+(2a+b-c)^2}+\frac{\lambda}{\lambda^2+(2a-b+c)^2}-\frac{\lambda}{\lambda^2+(2a+b+c)^2}-\frac{\lambda}{\lambda^2+(2a-b-c)^2}\right)
\end{aligned}$$

We have

$$\begin{aligned}
J'(\lambda) &= A+B \\
&= \frac{1}{4}\left\{\frac{\lambda}{\lambda^2+(b+c)^2}-\frac{\lambda}{\lambda^2+(b-c)^2}\right\} \\
&\quad +\frac{1}{8}\left(\frac{\lambda}{\lambda^2+(2a+b-c)^2}+\frac{\lambda}{\lambda^2+(2a-b+c)^2}\right) \\
&\quad -\frac{1}{8}\left(\frac{\lambda}{\lambda^2+(2a+b+c)^2}+\frac{\lambda}{\lambda^2+(2a-b-c)^2}\right)
\end{aligned}$$

Integrating now with respect to λ we get

$$\begin{aligned} J(\lambda) &= \frac{1}{8}\ln|\lambda^2+(b+c)^2| - \frac{1}{8}\ln|\lambda^2+(b-c)^2| + \frac{1}{16}\ln|\lambda^2+(2a+b-c)^2| \\ &\quad + \frac{1}{16}\ln|\lambda^2+(2a-b+c)^2| - \frac{1}{16}\ln|\lambda^2+(2a+b+c)^2| - \frac{1}{16}\ln|\lambda^2+(2a-b-c)^2| \\ &= \frac{1}{8}\ln\left|\frac{\lambda^2+(b+c)^2}{\lambda^2+(b-c)^2}\right| + \frac{1}{16}\ln\left|\frac{\lambda^2+(2a+b-c)^2}{\lambda^2+(2a+b+c)^2}\right| + \frac{1}{16}\ln\left|\frac{\lambda^2+(2a-b+c)^2}{\lambda^2+(2a-b-c)^2}\right| + K \end{aligned}$$

Evaluating the constant of integration when $\lambda \to \infty$ we find that $K = 0$. Therefore,

$$J(\lambda) = \frac{1}{8}\ln\left|\frac{\lambda^2+(b+c)^2}{\lambda^2+(b-c)^2}\right| + \frac{1}{16}\ln\left|\frac{\lambda^2+(2a+b-c)^2}{\lambda^2+(2a+b+c)^2}\right| + \frac{1}{16}\ln\left|\frac{\lambda^2+(2a-b+c)^2}{\lambda^2+(2a-b-c)^2}\right|$$

Example 4.0.13. *Using the previous example we can evaluate also*

$$G(\lambda) = \int_0^\infty e^{-\lambda x}\frac{\sin^2(ax)\sin(bx)\sin(cx)}{x^2}dx,$$

where $a > 0$, $b > 0$ and $c > 0$. Differentiate with respect to λ we get

$$\begin{aligned} G'(\lambda) &= -\int_0^\infty e^{-\lambda x}\frac{\sin^2(ax)\sin(bx)\sin(cx)}{x}dx \\ G'(\lambda) &= -J(\lambda) = \frac{1}{8}\ln\left|\frac{\lambda^2+(b-c)^2}{\lambda^2+(b+c)^2}\right| + \frac{1}{16}\ln\left|\frac{\lambda^2+(2a+b+c)^2}{\lambda^2+(2a+b-c)^2}\right| + \frac{1}{16}\ln\left|\frac{\lambda^2+(2a-b-c)^2}{\lambda^2+(2a-b+c)^2}\right| \end{aligned}$$

Suppose

$$I_1 = \frac{1}{8}\ln\left|\frac{\lambda^2+(b-c)^2}{\lambda^2+(b+c)^2}\right|; I_2 = \frac{1}{16}\ln\left|\frac{\lambda^2+(2a+b+c)^2}{\lambda^2+(2a+b-c)^2}\right|; I_3 = \frac{1}{16}\ln\left|\frac{\lambda^2+(2a-b-c)^2}{\lambda^2+(2a-b+c)^2}\right|$$

Integrating by parts each function.

$$\int_0^\infty I_1 = \int_0^\infty \frac{1}{8} \ln \frac{\lambda^2 + (b-c)^2}{\lambda^2 + (b+c)^2} d\lambda$$

Let

$$\begin{aligned} u &= \ln \frac{\lambda^2 + (b-c)^2}{\lambda^2 + (b+c)^2} & dv = \frac{1}{8} d\lambda \\ du &= \frac{2\lambda(b+c)^2 - 2\lambda(b-c)^2}{(\lambda^2 + (b-c)^2)(\lambda^2 + (b+c)^2)} d\lambda & v = \frac{1}{8}\lambda \end{aligned}$$

Then, we have

$$\begin{aligned} \int_0^\infty I_1 &= uv - \int_0^\infty v du \\ &= \frac{\lambda}{8} \ln \frac{\lambda^2 + (b-c)^2}{\lambda^2 + (b+c)^2} - \frac{1}{4} \int_0^\infty \frac{\lambda^2 \left((b+c)^2 - (b-c)^2\right)}{(\lambda^2 + (b-c)^2)(\lambda^2 + (b+c)^2)} d\lambda \\ &= \frac{\lambda}{8} \ln \frac{\lambda^2 + (b-c)^2}{\lambda^2 + (b+c)^2} - \int_0^\infty \frac{bc\lambda^2}{(\lambda^2 + (b-c)^2)(\lambda^2 + (b+c)^2)} d\lambda \end{aligned}$$

Using partial fraction decomposition, we obtain

$$\begin{aligned} & \frac{bc\lambda^2}{(\lambda^2 + (b-c)^2)(\lambda^2 + (b+c)^2)} \\ = & \frac{A\lambda + B}{\lambda^2 + (b-c)^2} + \frac{C\lambda + D}{\lambda^2 + (b+c)^2} \\ = & \frac{(A\lambda + B)(\lambda^2 + (b+c)^2) + (C\lambda + D)(\lambda^2 + (b-c)^2)}{(\lambda^2 + (b-c)^2)(\lambda^2 + (b+c)^2)} \\ = & \frac{A\lambda^3 + A\lambda(b+c)^2 + B\lambda^2 + B(b+c)^2 + C\lambda^3 + C\lambda(b-c)^2 + D\lambda^2 + D(b-c)^2}{(\lambda^2 + (b-c)^2)(\lambda^2 + (b+c)^2)} \end{aligned}$$

We get

$$\begin{cases} A + C = 0 \\ B + D = bc \\ A(b+c)^2 + C(b-c)^2 = 0 \\ B(b+c)^2 + D(b-c)^2 = 0 \end{cases}$$

Then $A = -C = 0, D = \frac{1}{4}(b+c)^2$, *and* $B = \frac{-1}{4}(b-c)^2$. *We now have*

$$\begin{aligned} \int_0^\infty I_1 &= \frac{\lambda}{8}\ln\frac{\lambda^2+(b-c)^2}{\lambda^2+(b+c)^2} + \frac{1}{4}\int_0^\infty \frac{(b-c)^2}{\lambda^2+(b-c)^2}d\lambda - \frac{1}{4}\int_0^\infty \frac{(b+c)^2}{\lambda^2+(b+c)^2}d\lambda \\ &= \frac{\lambda}{8}\ln\frac{\lambda^2+(b-c)^2}{\lambda^2+(b+c)^2} + \frac{b-c}{4}\arctan\frac{\lambda}{b-c} - \frac{b+c}{4}\arctan\frac{\lambda}{b+c} + k_1 \\ &= \frac{\lambda}{8}\ln\frac{\lambda^2+(b-c)^2}{\lambda^2+(b+c)^2} + \frac{c-b}{4}\arctan\frac{\lambda}{c-b} - \frac{b+c}{4}\arctan\frac{\lambda}{b+c} + k_1 \end{aligned}$$

Letting $\lambda \to \infty$ *to find the constant of integration, we get*

$$\begin{aligned} 0 &= \left(\frac{(c-b)\pi}{8}\right) - \left(\frac{(b+c)\pi}{8}\right) + k_1 \\ 0 &= \frac{-\pi}{4}b + k_1 \\ k_1 &= \frac{\pi}{4}b \end{aligned}$$

Note that $c - b > 0$ *and* $c + b > 0$. *Hence,*

$$\int_0^\infty I_1 = \frac{\lambda}{8}\ln\frac{\lambda^2+(b-c)^2}{\lambda^2+(b+c)^2} + \frac{c-b}{4}\arctan\frac{\lambda}{c-b} - \frac{b+c}{4}\arctan\frac{\lambda}{b+c} + \frac{\pi}{4}b$$

Let's now integrate I_2.

$$\int_0^\infty I_2 = \int_0^\infty \frac{1}{16} \ln \frac{\lambda^2 + (2a+b+c)^2}{\lambda^2 + (2a+b-c)^2} d\lambda$$

Let

$$\begin{aligned} u &= \ln \frac{\lambda^2 + (2a+b+c)^2}{\lambda^2 + (2a+b-c)^2} & dv = \frac{1}{16} d\lambda \\ du &= \frac{2\lambda(2a+b-c)^2 - 2\lambda(2a+b+c)^2}{\left(\lambda^2 + (2a+b+c)^2\right)\left(\lambda^2 + (2a+b-c)^2\right)} & v = \frac{1}{16}\lambda \end{aligned}$$

We then have

$$\begin{aligned} \int_0^\infty I_2 &= \frac{\lambda}{16} \ln \frac{\lambda^2 + (2a+b+c)^2}{\lambda^2 + (2a+b-c)^2} - \frac{1}{8} \int_0^\infty \frac{\lambda^2 \left((2a+b-c)^2 - (2a+b+c)^2\right)}{\left(\lambda^2 + (2a+b+c)^2\right)\left(\lambda^2 + (2a+b-c)^2\right)} d\lambda \\ &= \frac{\lambda}{16} \ln \frac{\lambda^2 + (2a+b+c)^2}{\lambda^2 + (2a+b-c)^2} - \frac{1}{8} \int_0^\infty \frac{\lambda^2(-8ac - 4bc)}{\left(\lambda^2 + (2a+b+c)^2\right)\left(\lambda^2 + (2a+b-c)^2\right)} d\lambda \\ &= \frac{\lambda}{16} \ln \frac{\lambda^2 + (2a+b+c)^2}{\lambda^2 + (2a+b-c)^2} + \frac{1}{2} \int_0^\infty \frac{\lambda^2(2ac + bc)}{\left(\lambda^2 + (2a+b+c)^2\right)\left(\lambda^2 + (2a+b-c)^2\right)} d\lambda \\ &= \frac{\lambda}{16} \ln \frac{\lambda^2 + (2a+b+c)^2}{\lambda^2 + (2a+b-c)^2} + \int_0^\infty \frac{\lambda^2 ac}{\left(\lambda^2 + (2a+b+c)^2\right)\left(\lambda^2 + (2a+b-c)^2\right)} d\lambda \\ &\quad + \frac{1}{2} \int_0^\infty \frac{\lambda^2 bc}{\left(\lambda^2 + (2a+b+c)^2\right)\left(\lambda^2 + (2a+b-c)^2\right)} d\lambda \end{aligned}$$

Using partial fraction decomposition for the first integral, we obtain

$$\begin{aligned} & \frac{\lambda^2 ac}{\left(\lambda^2 + (2a+b+c)^2\right)\left(\lambda^2 + (2a+b-c)^2\right)} \\ = & \frac{A\lambda + B}{\lambda^2 + (2a+b+c)^2} + \frac{C\lambda + D}{\lambda^2 + (2a+b-c)^2} \\ = & \frac{(A\lambda + B)\left(\lambda^2 + (2a+b-c)^2\right) + (C\lambda + D)\left(\lambda^2 + (2a+b+c)^2\right)}{\left(\lambda^2 + (2a+b+c)^2\right)\left(\lambda^2 + (2a+b-c)^2\right)} \end{aligned}$$