A CRITERION FOR SETS OF LOCALLY FINITE PERIMETER

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Abstract of the Thesis

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Geometric measure theory could be described as differential geometry, generalized through measure theory to deal with maps and surfaces that are not necessarily smooth, and applied to the calculus of variations. Geometric measure theory is important because it studies sets, their variation and their boundaries (from the measure theoretic sense). In particular, a very interesting branch in Geometric measure theory, is the sets of locally finite perimeter. Just as their name actually shows, these sets are essentially sets

whose perimeter is (locally) finite. In this thesis we start by giving a formal definition for sets of locally finite perimeter. Moreover, we will use the Hausdorff measure (just like the surface measure) as a tool to give us the perimeter of the (measure theoretic) boundaries of these sets. Then we will prove a criteria for sets of locally finite perimeter, which states that a set is of locally finite perimeter, if and only if, (locally) the Hausdorff measure of its (measure theoretic) boundary is finite.

To my parents.

Contents

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> "I believe that education is all about being excited about something. Seeing passion and enthusiasm helps push an educational message." (Steve Irwin)

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> "God could not be everywhere, so he created parents."

Finally I thank God for giving me strength, and for his inspiration and blessings.

Notations

$\mathcal{L}^1$	the 1-dimensional Lebesgue measure
$\mathcal{L}^n$	$\mathcal{L}^{n-1} \times \mathcal{L}^1 = \mathcal{L}^1 \times \ldots \times \mathcal{L}^1$ (n times)
a.e.	almost everywhere
$\mathcal{H}^s$	s-dimensional Hausdorff measure
e_n	$(0, 0, \cdots, 1, 0, \cdots)$ with 1 in the nth slot
x	$(x_1, \cdots, x_n)$
$B(x, r)$	$\{y \in \mathbb{R}^n,\ \|x - y\| \leq r\}$ = closed ball with center x, radius $r\}$
α_s	$\dfrac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}+1\right)}$ $(0 \leq s < \infty)$
α_n	volume of the unit ball in $\mathbb{R}^n$
χ_A	indicator function of the set A
$\mathrm{Lip}\,(f)$	lipschitz constant of f
$\mathrm{div}\,\varphi$	$\left(\dfrac{\partial}{\partial x_1}, \ldots, \dfrac{\partial}{\partial x_n}\right) . (\varphi_1, \ldots, \varphi_n) = \dfrac{\partial \varphi_1}{\partial x_1} + \ldots + \dfrac{\partial \varphi_n}{\partial x_n}$
$\mathrm{spt}\,(f)$	support of f
Df	$\left(\dfrac{\partial f}{\partial x_1}, \ldots, \dfrac{\partial f}{\partial x_n}\right)$
U	open set in $\mathbb{R}^n$
∂U	boundary of U

Chapter 1

Introduction

The concept of a Caccioppoli set (locally finite perimeter) was firstly introduced by the Italian mathematician Renato Caccioppoli in 1927. He defined the measure or area of a plane set or a surface, as the total variation in the sense of their parametric equations, taking into consideration that this quantity should be bounded. The measure of the boundary of a set was defined on open sets, as a functional, precisely a set function. Moreover, it can be defined on all Borel sets and its value can be approximated by the values it takes on an increasing net of subsets.

Lamberto Cesari introduced the generalization of functions of bounded variation to the case of several variables only in 1936. In 1951, Caccioppoli improved the version of his theory in the talk at the IV UMI Congress followed by five notes published in the Rendiconti of the Accademia Nazionale dei Lincei.

In 1952 Ennio de Giorgi presented his first results, developing the ideas of

Caccioppoli, on the definition of the measure of boundaries of sets at the Salzburg Congress of the Austrian Mathematical Society. He used a smoothing operator, similar to a mollifier, constructed from the Gaussian function, in proving some results of Caccioppoli. In 1953, he published his first paper on the topic. But his approach to sets of finite perimeter became widely known, after he finished the paper in 1954 and it was reviewed by Laurence Chisholm Young in the Mathematical Reviews. The last paper of De Giorgi on the theory of perimeters was published in 1958. In 1959, after the death of Caccioppoli, he started to call sets of finite perimeter "Caccioppoli sets".

In 1960, Herbert Federer and Wendell Fleming published their paper, changing the approach to the theory. They defined currents. Federer showed that Caccioppoli sets are normal currents of dimension n in n- dimensional Euclidean spaces.

Sets of locally finite perimeter, those whose characteristic functions have locally bounded variation, are extremely important in studying many problems involving interfaces, in areas such as material science, fluid mechanics, surface physics, image processing, oncology, and computer vision. Moreover, Sets of finite perimeter play an important role in the theory of minimal surfaces, capillarity problems, phase transitions, and optimal partitions. They are general enough to adequately model complex physical phenomena with singularities, they have useful local approximation properties, and they satisfy vital compactness results which do not hold for classes of sets having smooth boundaries (see [1] and [2]).

Green's theorem is one of the four fundamental theorems of vector calculus and one of the basic results in analysis, it transforms the line integral around a closed curve C into a double integral over the region inside C . This result emphasizes the importance of sets of finite perimeter (see [3] and [4]).

Since sets of locally finite perimeter are of great importance, it is an interesting question to find a criterion for them. We give first the formal definition of sets of locally finite perimeter.

Definition 1.0.1. *An $\mathcal{L}^n$-measurable subset $E \subset \mathbb{R}^n$ has locally finite perimeter in U if*

$$\sup \left\{ \int_E div\varphi \, \mathrm{d}x | \varphi \in C_c^1 \left(U; \mathbb{R}^n \right), |\varphi| \leq 1 \right\} < \infty$$

where U is an open set subset to $\mathbb{R}^n$.

To be able to introduce the very well known criterion theorem for sets of locally finite perimeter, we first need to recall the definition of the Lebesgue and the Hausdorff measure, and mainly focus on the latter one. The Hausdorff measure is basically the surface measure. In other words, the Hausdorff measure can measure length, area, or in general the n-volume of an n-dimensional object that lives in a higher dimensional space. For example if we have a 2-dimensional surface living in $\mathbb{R}^5$, we need a measure that gives us the area of this surface even if it is not living in $\mathbb{R}^2$. Moreover, if we have an n-dimensional set E and we want to work on its boundary, then we cannot use the lebesgue measure $\mathcal{L}^n$, because the boundary is of a smaller dimension that means of dimension $n-1$ or smaller. Since the n-dimensional Lebesgue measure can only work on n-dimensional sets and measures only the n-dimensional volume, we need the surface measure to measure the volume of the boundary. For this reason the mathematician Felix Hausdorff introduced the Hausdorff measure in 1918.

Now we give the formal definition of the Hausdorff measure.

Definition 1.0.2. *1. Let $A \subset \mathbb{R}^n,\ 0 \leq \delta < \infty$.*

Let us define

$$\mathcal{H}^n_\delta(A) = \inf\left\{\sum_{j=1}^{\infty} \alpha(n)\left(\frac{diam\ C_j}{2}\right)^n;\ A \subset \bigcup_{j=1}^{\infty} C_j;\ diam\ C_j \leq \delta\right\}$$

and where $\alpha(n) = \dfrac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$.

2. For $A \subset \mathbb{R}^n$, let us define

$$\mathcal{H}^n(A) = \lim_{\delta \to 0} \mathcal{H}^n_\delta(A) = \sup_{\delta > 0} \mathcal{H}_\delta(A).$$

We call $\mathcal{H}^n$ an n-dimensional Hausdorff measure on $\mathbb{R}^n$.

Since the set E that I work with lives in a higher dimensional space than its *own dimension, the boundary of E cannot be defined in the standard way. In fact I introduce a definition of the boundary from the measure theoretical point of view, called the measure theoretic boundary.*

Definition 1.0.3. *(see section 5.8 p.208 in [5])*
Let $x \in \mathbb{R}^n$. We say $x \in \partial_* E$, the measure theoretic boundary of E if

$$\limsup_{r \to 0} \frac{\mathcal{L}^n \left(B \left(x, r \right) \cap E \right)}{r^n} > 0$$

and

$$\limsup_{r \to 0} \frac{\mathcal{L}^n \left(B \left(x, r \right) \setminus E \right)}{r^n} > 0.$$

In this thesis, my aim is to prove in extreme rigour and detail the very well known criterion theorem for sets of locally finite perimeter. This theorem states that a set is of locally finite perimeter if and only if locally, the hausdorff measure of the measure theoretic boundary is finite. In fact,

Theorem 1.0.4. *Criterion for Finite Perimeter*
Let $E \subset \mathbb{R}^n$ be $\mathcal{L}^n$-measurable. Then, E has locally finite perimeter. if and only if ,

$$\mathcal{H}^{n-1} \left(K \cap \partial_* E \right) < \infty \tag{1.0.1}$$

for each compact set $K \subset \mathbb{R}^n$.

In Chapter 2, I give a review on the part of the measure theory needed in the thesis. In particular I put the definition of Borel, Radon, Lebesgue and Hausdorff measure, Lipchitz functions, and state some theorems and corollaries related to them. In chapter 3, I state the definition for the mollifier and a theorem that shows the importance of a mollifier in relating a non smooth function to a smooth one. In chapter 4, I give the definition for bounded variation, locally bounded variation, and state a very important theorem which relates functions of bounded varations to Radon measures. More precisely this theorem states

Theorem 1.0.5. *Structure Theorem for BV_{loc} functions (see section 5.1 p.167 in [5])*
Let $U \subset \mathbb{R}^n$ and $f \in BV_{loc}(U)$. Then there exists a Radon measure μ on U and a μ-measurable function $\sigma : U \to \mathbb{R}^n$ such that

1. $|\sigma(x)| = 1$ *μ a.e, and*

2. $\int_U f \, div\varphi \, \mathrm{d}x = -\int_U \varphi.\sigma \, \mathrm{d}\mu$

for all $\sigma \in C_c^1(U; \mathbb{R}^n)$.

When I take E to be any set and $f = \chi_E$, I will denote μ by $||\partial E||$, which is called the boundary E measure. Furthermore, from this measure we can now *define the measure theoretic boundary. Next, I give the definition of essential variation on* $\mathbb{R}$*, which will be important in the next chapter to be able to relate it to functions of bounded variation. In chapter 5, I extend the definition of essential variation to* $\mathbb{R}^n$*, and prove a theorem that relates functions of bounded variation to the essential variation of a function. Finally, in chapter 6, I state and prove two lemmas which will be helpful in proving my main theorem. We finish chapter 6 by proving our main theorem.*

Chapter 2

Preliminaries : Measure, Hausdorff Measure and Lipschitz Functions

We begin by defining measures,Borel measures, Radon measures and measurable sets and functions.

Definition 2.0.1. *Measure*

Let X denote a set, and 2^X the power set of X, that is the set of all subsets of X.

A mapping $\mu : 2^X \to [0, \infty]$ is called a measure on X if

1. $\mu(\phi) = 0$
2. $\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k)$ *whenever* $A \subset \bigcup_{k=1}^{\infty} A_k$.

Definition 2.0.2. *Let* μ *be a measure on* X *and* $A \subset X$*.Then* μ *restricted to* A*,*

written

$$\mu \llcorner A,$$

is the measure defined by

$$\mu \llcorner A\,(B) = \mu\,(A \cap B)\,\textit{for all } B \subset X.$$

Definition 2.0.3. *A set* $A \subset X$ *is* μ*-measurable if for each set* $B \subset X$*,*

$$\mu\,(B) = \mu\,(B \cap A) + \mu\,(B - A)\,.$$

Definition 2.0.4. *A borel set is a set that is made of unions and intersections of open and closed sets.*

Definition 2.0.5. *A measure* μ *on* $\mathbb{R}^n$ *is called a Borel measure if every Borel set is* μ*-measurable.*

Definition 2.0.6. *A measure* μ *on* $\mathbb{R}^n$ *is Borel regular if* μ *is Borel and for each* $A \subset \mathbb{R}^n$ *there exists a Borel set* B *such that* $A \subset B$ *and* $\mu\,(A) = \mu\,(B)\,.$

Definition 2.0.7. *A measure* μ *on* $\mathbb{R}^n$ *is a Radon measure if* μ *is Borel regular and* $\mu\,(K) < \infty$ *for each compact set* $K \subset \mathbb{R}^n$*.*

Definition 2.0.8. *A function* $f : X \to Y$ *is called* μ*-measurable if for each open* $U \subset Y$*,* $f^{-1}\,(U)$ *is* μ*-measurable.*

Definition 2.0.9. 1. *A μ-measurable function f is μ-integrable if*

$$\int f \, \mathrm{d}\mu < \infty$$

and we say $f \in L^1(\mu)$.

2. *A μ-measurable function f is locally μ-integrable if*

$$\int_K f \, \mathrm{d}\mu < \infty \text{ for every compact set } K$$

and we say $f \in L^1_{loc}(\mu)$.

Definition 2.0.10. *A function $f \in C^\infty(\mathbb{R}^n)$ is smooth, if it is infinitely differentiable.*

Next, we state some standard measure theory theorems that will be useful later in this thesis.

Definition 2.0.11. *$f : \mathbb{R}^n \to \mathbb{R}^m$ is approximetly continuous at $x \in \mathbb{R}^n$ if*

$$ap \lim_{y \to x} f(y) = f(x),$$

such an x is called a point of approximate continuity.

Definition 2.0.12. *$spt(f) = \overline{\{f \neq 0\}}$ $x \in X$ such that $f(x) \neq 0$. And if the spt is compact, then it is compactly supported.*

Theorem 2.0.13. *Fatou's Lemma (see section 1.3 p.19 in [5])*

Let $f_k : X \to [0, \infty]$ *be* μ*-measurable (k=1,...).Then*

$$\int \lim_{k\to\infty} \inf f_k \, \mathrm{d}\mu \leq \lim_{k\to\infty} \inf \int f_k \, \mathrm{d}\mu.$$

Theorem 2.0.14. *Beppo-Levi (see section 2.2 p.51 in [6])*

Let (X, m, μ) *be a measure space.*

Let $\{f_n\}$ *be a sequence of positive integrable functions, then,*

$$\int \sum_{n=1}^{\infty} f_n \, \mathrm{d}\mu = \sum_{n=1}^{\infty} \int f_n \, \mathrm{d}\mu.$$

Now, let us recall the definition of lebesgue measure, some properties of the lebesgue measure, and some useful theorems concerning it.

Definition 2.0.15. *One-dimensional Lebesgue measure* $\mathcal{L}^1$ *on* $\mathbb{R}^1$ *is defined by*

$$\mathcal{L}^1(A) \equiv \inf \left\{ \sum_{i=1}^{\infty} diam\, C_i \,|\, A \subset \bigcup_{i=1}^{\infty} C_i,\, C_i \subset \mathbb{R} \right\}$$

for all $A \subset \mathbb{R}$.

Definition 2.0.16. *Let* $B(x, r)$ *be a closed ball of center* x *and radius* r

$$\mathcal{L}^n\Big(B(x, r)\Big) = \alpha_n r^n.$$

Definition 2.0.17. *The average of* f *over the set* E *with respect to* μ *by*

$$⨍_E f \, \mathrm{d}\mu \equiv \frac{1}{\mu(E)} \int_E f \, \mathrm{d}\mu,$$

provided $0 < \mu(E) < \infty$ *and the integral on the right is defined.*

Definition 2.0.18. *Let* μ *be a radon measure on* $\mathbb{R}^n$, $1 \leq p < \infty$, *and* $f \in L^p_{loc}(\mathbb{R}^n, \mu)$. *A point* x *for which*

$$\lim_{r \to 0} \fint_{B(x,r)} |f - f(x)|^p \, \mathrm{d}\mu = 0$$

is called a lebesgue point of f *with respect to* μ.

Corollary 2.0.19. *Density Theorem (see section 1.7.1 p.45 in [5])*

Let $E \subset \mathbb{R}^n$ *be* $\mathcal{L}^n$*-measurable .Then*

$$\lim_{r \to 0} \frac{\mathcal{L}^n(B(x,r) \cap E)}{\mathcal{L}^n(B(x,r))} = 1 \text{ for } \mathcal{L}^n \text{ a.e } x \in E$$

and

$$\lim_{r \to 0} \frac{\mathcal{L}^n(B(x,r) \cap E)}{\mathcal{L}^n(B(x,r))} = 0 \text{ for } \mathcal{L}^n \text{ a.e } x \in \mathbb{R}^n - E.$$

Theorem 2.0.20. *Isodiametric Inequality (see section 2.2 p.69 in [5])*

For all sets $A \subset \mathbb{R}^n$,

$$\mathcal{L}^n(A) \leq \alpha_n \left(\frac{diam\ A}{2} \right)^n.$$

We proceed with the definition and some properties of the Hausdorff measure.

Definition 2.0.21. *Let* $A \subset \mathbb{R}^n, 0 \leq s < \infty, 0 < \delta \leq \infty$. *Define*

$$\mathcal{H}^s_\delta(A) \equiv \inf \left\{ \sum_{j=1}^{\infty} \alpha_s \left(\frac{diam C_j}{2} \right)^s, A \subset \bigcup_{j=1}^{\infty} C_j, diam\ C_j \leq \delta \right\}$$

where

$$\alpha_s \equiv \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2}+1)}.$$

Here $\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}\,\mathrm{d}x$, *(*$0 < s < \infty$*), is the usual gamma function.*

Definition 2.0.22. *s-dimensional Hausdorff measure on* $\mathbb{R}^n$

Let $A \subset \mathbb{R}^n, 0 \leq s < \infty, 0 < \delta \leq \infty$*.Define*

$$\mathcal{H}^s(A) \equiv \lim_{\delta \to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A).$$

Theorem 2.0.23. *(see section 2.1 p. 63 in [5])*

$\mathcal{H}^0$ *is the counting measure.*

Theorem 2.0.24. *(see on p.70 in [5])*

$$\mathcal{H}^n = \mathcal{L}^n \text{ on } \mathbb{R}^n.$$

Finally let us finish this chapter by defining lipschitz functions.

Definition 2.0.25. 1. *A function* $f : \mathbb{R}^n \to \mathbb{R}^m$ *is called Lipschitz if there exists a constant* C *such that*

$$|f(x) - f(y)| \leq C|x - y| \quad \forall\ x, y \in \mathbb{R}^n.$$

2. $Lip(f) \equiv \sup\left\{\frac{|f(x) - f(y)|}{|x - y|}; x, y \in \mathbb{R}^n, x \neq y\right\}$

Theorem 2.0.26. *Let* $P : \mathbb{R}^n \to \mathbb{R}^m$ *be the projection map, and let* $A \subset \mathbb{R}^n$*, where* $m < n$*. Then* P *is a lipschitz function such that*

$$Lip\,(P) = 1 \tag{2.0.1}$$

and

$$diam\Big(P\,(A)\Big) \leq diam\,(A) \tag{2.0.2}$$

Proof. By Definition 2.0.25, we have

$$|P\,(x) - P\,(y)\,| \leq |x - y| \tag{2.0.3}$$

and

$$Lip\,(P) \leq 1.$$

Let us prove (2.0.1) by contradiction. Suppose that $\mathrm{Lip}\,(P) < 1$, thus

$$|P\,(x) - P\,(y)\,| < |x - y| \quad \forall\ x, y \in \mathbb{R}^n. \tag{2.0.4}$$

But, in particular if $x, y \in \mathbb{R}^m$, then $P\,(x) = x$ and $P\,(y) = y$. Replacing in (2.0.4) , we get $|P\,(x) - P\,(y)\,| = |x - y| < |x - y|$, which is a contradiction.

To prove (2.0.2) , first recall that

$$\mathrm{diam}\,(A) = \sup |x - y| \quad x, y \in A.$$

and

$$\mathrm{diam}\Big(P\,(A)\Big) = \sup |P\,(x) - P\,(y)\,| \quad x, y \in A.$$

But by Definition 2.0.25, we have

$$|P(x) - P(y)| \leq \mathrm{Lip}(P)\,|x - y|$$

Putting sup on both sides, we get

$$\sup|P(x) - P(y)| \leq \mathrm{Lip}(P)\sup|x - y|$$

that is

$$\mathrm{diam}\Big(P(A)\Big) \leq Lip(P)\,\mathrm{diam}(A) \tag{2.0.5}$$

However by (2.0.1), $\mathrm{Lip}(P) = 1$, thus (2.0.5) becomes

$$\mathrm{diam}\Big(P(A)\Big) \leq \mathrm{diam}(A).$$

□

Chapter 3

Preliminaries: Mollifiers and Their Importance

This chapter concerns itself with the definitions needed to state our main theorem and the prerequisites needed to prove it. We start by introducing mollifiers.

We will use a mollifier to smooth out our non smooth functions.

Definition 3.0.1. *1. If* $\epsilon > 0$ *and* $U \subset \mathbb{R}^n$, *we write* $U_\epsilon \equiv \{x \in U; dist(x, \partial U) > \epsilon\}$

2. Define the $C^\infty - function$ $\eta : \mathbb{R}^n \to \mathbb{R}$ *as follows:*

$$\eta(x) \equiv \begin{cases} c\ exp\left(\dfrac{1}{|x|^2 - 1}\right) & |x| < 1 \\ 0 & |x| \geq 1, \end{cases}$$

the constant c adjusted so

$$\int_{\mathcal{R}^n} \eta(x)\ \mathrm{d}x = 1. \tag{3.0.1}$$

Next define

$$\eta_\epsilon(x) \equiv \frac{1}{\epsilon^n}\eta\left(\frac{x}{\epsilon}\right) \quad (\epsilon > 0, x \in \mathbb{R}^n);$$

η_ϵ *is the standard mollifier.*

3. *If* $f \in L^1_{loc}(U)$,*define*

$$f^\epsilon \equiv \eta_\epsilon * f;$$

that is,

$$f^\epsilon(x) \equiv \int_U \eta_\epsilon(x-y)\, f(y)\ \mathrm{d}y \quad (x \in U_\epsilon).$$

To see the importance of mollifiers, note the following theorem.

Theorem 3.0.2. *(see section 4.2.1 p.123 in [5])*

1. *For each* $\epsilon > 0$, $f^\epsilon \in C^\infty(U_\epsilon)$.

2. *If* $f \in C(U)$, *then*

$$f^\epsilon \to f$$

uniformly on compact subsets of U.

3. *If* $f \in L^p_{loc}(U)$ *for some* $1 \le p < \infty$, *then*

$$f^\epsilon \to f \ \text{ in } L^p_{loc}(U).$$

4. *Furthermore,* $f^{\epsilon}(x) \to f(x)$ *if* x *is a Lebesgue point of* f*; in particular*

$$f^{\epsilon} \to f \quad \mathcal{L}^n \text{ a.e.}$$

5. *For some* $1 \leq p < \infty$ *,then*

$$\frac{\partial f^{\epsilon}}{\partial x_i} = \eta_{\epsilon} * \frac{\partial f}{\partial x_i} \quad (i = 1, \ldots, n)$$

on U_{ϵ}.

Chapter 4

Bounded Variation, Reduced and Measure Theoretic Boundary

We are mainly interested in sets of locally finite perimeter. To be able to define these sets, we start by definition of bounded variation.

Definition 4.0.1. *Bounded Variation in U*

A function $f \in L^1(U)$ has bounded variation in U if

$$\sup \left\{ \int_U f \, div\varphi \ \mathrm{d}x \ |\varphi \in C_c^1 (U; \mathbb{R}^n), |\varphi| \leq 1 \right\} < \infty$$

We write

$$BV(U)$$

to denote the space of functions of bounded variation.

Definition 4.0.2. *Locally Bounded Variation in U*

A function $f \in L^1_{loc}(U)$ *has locally bounded variation in U if for each open set* $V \subset\subset U$.

$$\sup\left\{\int_V f\, div\varphi \, \mathrm{d}x \; |\varphi \in C^1_c(V;\mathbb{R}^n), |\varphi| \leq 1\right\} < \infty$$

We write

$$BV_{loc}(U)$$

to denote the space of such functions.

Definition 4.0.3. *An* $\mathcal{L}^n$*-measurable subset* $E \subset \mathbb{R}^n$ *has locally finite perimeter in* U *if*

$$\chi_E \in BV_{loc}(U).$$

The next theorem is of great importance since it relates $f \in BV$ *to a Radon measure.*

Theorem 4.0.4. *Structure Theorem for* BV_{loc} *functions (see section 5.1 p.167 in [5])*

Let $f \in BV_{loc}(U)$. *Then there exists a Radon measure* μ *on* U *and a* μ*-measurable function* $\sigma : U \to \mathbb{R}^n$ *such that*

1. $|\sigma(x)| = 1$ μ *a.e, and*

2. $\int_U f\, div\varphi \, \mathrm{d}x = -\int_U \varphi.\sigma \, \mathrm{d}\mu$

for all $\sigma \in C^1_c(U;\mathbb{R}^n)$.

Notations *In view of Theorem 4.0.4, we denote* μ *by* $||Df||$*. In case* $f = \chi_E$*, we denote* μ *by* $||\partial E||$*, and* $\sigma = -\nu_E$*.*

Let us remark here a couple of things:

Remark

$$||Df||\,(U) = \sup\left\{ \int_U f\, div\varphi \ \mathrm{d}x | \varphi \in C_c^1\,(U;\mathbb{R}^n)\,, |\varphi| \leq 1 \right\}$$

Remark *Note that from the Theorem 4.0.4 and definition of* $||Df||$*, we can define generalized partial derivatives and generalized derivative of function* f *as long as* $f \in BV_{loc}\,(U)$ *. (see p.169 in [5]).*

For the sake of the thesis here, we will denote this partial generalized derivative by

$$\frac{\partial f}{\partial x_i} \text{ where } i = 1, \ldots, n. \tag{4.0.1}$$

And the generalized derivative by

$$Df = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) \tag{4.0.2}$$

Remark *Throughout this thesis, when* $f \in BV_{loc}$*, and we write* $\dfrac{\partial f}{\partial x_i}$ *and* Df*, we mean by them the generalized partial derivative and generalized derivative, respectively.*

However, when we have functions that are smooth, like f^ϵ *in the lemma and the theorem of chapter 5,* $\dfrac{\partial f^\epsilon}{\partial x_i}$ *and* Df^ϵ *will denote the usual partial derivative and usual derivative of smooth functions, respectively.*

We continue by defining the reduced and measure theoretic boundary of sets of locally finite perimeter. These are basically a generalization of the boundary of smooth sets.

Definition 4.0.5. *Let $x \in \mathbb{R}^n$ and E is a set of locally finite perimeter in $\mathbb{R}^n$. We say $x \in \partial^* E$, the reduced boundary of E, if*

1. $||\partial E|| \left(B\left(x, r\right)\right) > 0 \;\; \forall \; r > 0,$
2. $\lim_{r \to 0} \fint_{B(x,r)} \nu_E \; \mathrm{d}||\partial E|| = \nu_E\left(x\right),$ *and*
3. $|\nu_E\left(x\right)| = 1.$

The reduced boundary is of extreme importance since

Theorem 4.0.6. *(see remark in section 5.7.1 p.194 in [5])*
Let E be a set of locally finite perimeter in $\mathbb{R}^n$. Then

$$\|\partial E\| \left(\mathbb{R}^n \setminus \partial^* E\right) = 0.$$

Theorem 4.0.7. *(see section 5.7.3 p.205 in [5])*
Assume E has locally finite perimeter in $\mathbb{R}^n$. Then

$$\|\partial E\| = \mathcal{H}^{n-1} \llcorner \partial^* E.$$

Definition 4.0.8. *Let $x \in \mathbb{R}^n$. We say $x \in \partial_* E$, the measure theoretic boundary of E if*

$$\limsup_{r\to 0} \frac{\mathcal{L}^n\left(B\left(x,r\right)\cap E\right)}{r^n} > 0$$

and

$$\limsup_{r\to 0} \frac{\mathcal{L}^n\left(B\left(x,r\right)\setminus E\right)}{r^n} > 0.$$

Lemma 4.0.9. *(see section 5.11 p.222 in [5])*

Let $E \subset \mathbb{R}^n$ be $\mathcal{L}^n$-measurable.

$\partial_* E$ is a Borel measurable set.

The measure theoretic boundary and reduced boundary are very related, that is

Lemma 4.0.10. *(see section 5.8 p.208 in [5])*

Assume E is a set of locally finite perimeter in $\mathbb{R}^n$.

1. *$\partial^* E \subset \partial_* E$.*

2. *$\mathcal{H}^{n-1}\left(\partial_* E \setminus \partial^* E\right) = 0$.*

We recall the definition of variation for smooth and essential variation for non smooth functions $f : \mathbb{R} \to \mathbb{R}$.

Definition 4.0.11. *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. Then the variation is*

$$V_a^b f = \sup\left\{ \sum_{j=1}^{m} \left| f\left(t_{j+1}\right) - f\left(t_j\right) \right| \right\}.$$

Theorem 4.0.12. *(see section 3.4.1 p. 134 in [7])*

Let f be a smooth function then

$$V_a^b f = \int_a^b |f'(x)| \, \mathrm{d}x.$$

Definition 4.0.13. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{L}^1$-measurable, $-\infty \leq a < b \leq \infty$. The essential variation of f on the interval (a, b) is*

$$ess\ V_a^b f = V_a^b f \equiv \sup \left\{ \sum_{j=1}^{m} |f(t_{j+1}) - f(t_j)| \right\}$$

the supremum taken over all finite partitions $a < t_1 < \ldots < t_{m+1} < b$ such that each t_j is a point of approximate continuity of f.

It turns out $||Df||$ and $ess\, V_a^b f$ are equal. This will be a key point while proving our main theorem.

Theorem 4.0.14. *(see section 5.10.1 p.217 in [5])*

Suppose $f \in L^1(a, b)$. Then $||Df||(a, b) = ess\ V_a^b f$. Thus $f \in BV(a, b) \iff ess\ V_a^b f < \infty$.

Chapter 5

Extending The Essential Variation to Functions on $\mathbb{R}^n$

To be able to prove our main theorem, we need to extend the definition of the essential variation $ess\,V_a^b$ *to function* f *on* $\mathbb{R}^n$.

Let us begin by some notations : Write $x' \in \mathbb{R}^{n-1}$ *as* $x' = (x_1, \ldots x_{k-1}, x_{k+1}, \ldots x_n)$ *for* $k = 1, \ldots, n$.
Assume $f : \mathbb{R}^n \to \mathbb{R}$. *For a fixed* $x' \in \mathbb{R}^{n-1}$, *define* f_k *as a function of* $t \in]a, b[$ *as*

$$f_k(x', t) \equiv f(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_n)$$

where $k = 1, \ldots, n$.
We now define the essential variation of f_k *as follows*

$$ess\ V_a^b f_k = \sup \sum_{j=1}^{n} |f_k(x', t_{j+1}) - f_k(x', t_j)|$$

for each fixed x', and $a = t_1 < t_2 \ldots < t_n = b$ forming a partition of $]a, b[$.

The next theorem relates bounded variation to essential variation. To be able to do that, we need the following lemma.

Lemma 5.0.1. *Let $f \in BV_{loc}(\mathbb{R}^n)$. Let $f^\epsilon = \eta_\epsilon * f$ where η_ϵ is the standard mollifier defined in Definition 3.0.1. Then we have*

1. *$f^\epsilon \to f$ a.e $x \in \mathbb{R}^n$.*
2. *$\limsup_{\epsilon \to 0} \int_C |Df^\epsilon| \, \mathrm{d}x < \infty$. (where $C \subseteq \mathbb{R}^n$ is a compact set)*

Proof. Notice that (1) is true by Theorem 3.0.2 part (4).

To see (2), notice first that $\forall\, i = 1, 2, \ldots, n$

$$\Big\| \left(\frac{\partial f}{\partial x_i} \right)^\epsilon \Big\|_{L^1(C)} \leq \Big\| \frac{\partial f}{\partial x_i} \Big\|_{L^1(C)} \tag{5.0.1}$$

(see p.124 Theorem 1 part (3) in [5])

However, by Definition 3.0.1 part (3) and Theorem 3.0.2 part (5), we have

$$\left(\frac{\partial f}{\partial x_i} \right)^\epsilon = \eta_\epsilon * \frac{\partial f}{\partial x_i} = \frac{\partial f^\epsilon}{\partial x_i} \tag{5.0.2}$$

Replacing (5.0.2) in (5.0.1), we get

$$\Big\| \frac{\partial f^\epsilon}{\partial x_i} \Big\|_{L^1(C)} \leq \Big\| \frac{\partial f}{\partial x_i} \Big\|_{L^1(C)} \tag{5.0.3}$$

Notice that (5.0.3) translates to

$$\int_C \left|\frac{\partial f^\epsilon}{\partial x_i}\right| \,\mathrm{d}x \leq \int_C \left|\frac{\partial f}{\partial x_i}\right| \,\mathrm{d}x := M_i \tag{5.0.4}$$

where M_i is a constant depending on $i = 1, \ldots, n$.

By summing both sides of (5.0.4) , we get

$$\sum_{i=1}^{n} \int_C \left|\frac{\partial f^\epsilon}{\partial x_i}\right| \,\mathrm{d}x \leq M_1 + \ldots + M_n := M \tag{5.0.5}$$

Moreover,

$$|Df^\epsilon| = \sqrt{\sum_{i=1}^{n} \left|\frac{\partial f^\epsilon}{\partial x_i}\right|^2} \leq \sum_{i=1}^{n} \sqrt{\left|\frac{\partial f^\epsilon}{\partial x_i}\right|^2} = \sum_{i=1}^{n} \left|\frac{\partial f^\epsilon}{\partial x_i}\right| \tag{5.0.6}$$

Integrating both sides of (5.0.6), and using (5.0.5) , we get

$$\int_C |Df^\epsilon| \,\mathrm{d}x \leq \int_C \sum_{i=1}^{n} \left|\frac{\partial f^\epsilon}{\partial x_i}\right| \,\mathrm{d}x = \sum_{i=1}^{n} \int_C \left|\frac{\partial f^\epsilon}{\partial x_i}\right| \,\mathrm{d}x \leq M \tag{5.0.7}$$

Finally, taking lim sup on both sides of (5.0.7) , we get

$$\limsup_{\epsilon \to 0} \int_C |Df^\epsilon| \,\mathrm{d}x \leq M < \infty,$$

which finishes the proof of (2) . □

Theorem 5.0.2. *Assume* $f \in L^1_{loc}(\mathbb{R}^n)$. *Thus*

$$f \in BV_{loc}(\mathbb{R}^n) \iff \int_K ess\ V_a^b f_k \, \mathrm{d}x' < \infty$$

for each $k = 1, \ldots, n$, $a, b \in \mathbb{R}$, $a < b$, *and compact set* $K \subset \mathbb{R}^{n-1}$.

Proof. We begin by proving the first direction of the theorem. Suppose $f \in BV_{\text{loc}}(\mathbb{R}^n)$. Set

$$C = \Big\{ x \in \mathbb{R}^n | a \leq x_k \leq b, (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \in K \Big\}$$

such that k, a, b, K are defined as above.

Let $f^\epsilon \equiv \eta_\epsilon * f$ $(for\ \epsilon > 0$ small enough).Thus by Lemma 5.0.1 we have :

1. $f^\epsilon \to f$ a.e $x \in \mathbb{R}^n$.
2. $\limsup_{\epsilon \to 0} \int_C |Df^\epsilon| \, \mathrm{d}x < \infty$.

Notice that (1) translates to $f_k^\epsilon(x', t) \to f_k(x', t)$ for $\mathcal{L}^{n-1}$ a.e $x' \in \mathbb{R}^{n-1}$, and $\mathcal{L}^1$ a.e $t \in \mathbb{R}$ where,

$$f_k^\epsilon(x', t) \equiv f^\epsilon(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_n).$$

Which means that for a.e fixed $x' \in \mathbb{R}^{n-1}$, we have

$$f_k^\epsilon \to f_k \quad \text{a.e} \ \ t \in \mathbb{R}. \tag{5.0.8}$$

Let $a = t_1 < t_2 < \ldots < t_n = b$ be a partition of $]a, b[$. Then, by using (5.0.8), and the definition of ess $V_a^b f_k$, we have,

$$\begin{aligned}
\sum_{j=1}^{n} |f_k(x', t_{j+1}) - f_k(x', t_j)| &= \sum_{j=1}^{n} \lim_{\epsilon\to 0} |f_k^\epsilon(x', t_{j+1}) - f_k^\epsilon(x', t_j)| \\
&= \lim_{\epsilon\to 0} \sum_{j=1}^{n} |f_k^\epsilon(x', t_{j+1}) - f_k^\epsilon(x', t_j)| \\
&= \liminf_{\epsilon\to 0} \sum_{j=1}^{n} |f_k^\epsilon(x', t_{j+1}) - f_k^\epsilon(x', t_j)| \\
&\leq \liminf_{\epsilon\to 0} \text{ess } V_a^b f_k^\epsilon
\end{aligned}$$

This is true for all partitions of $]a, b[$, thus, taking the sup on both sides

$$\text{ess } V_a^b f_k \leq \liminf_{\epsilon\to 0} \text{ess } V_a^b f_k^\epsilon. \tag{5.0.9}$$

Thus by (5.0.9) and Theorem 2.0.13, Theorem 4.0.12, the fact that $\left|\frac{\partial f^\epsilon}{\partial x_i}\right| \leq |Df^\epsilon|$, and by (2)

$$\begin{aligned}
\int_K \text{ess } V_a^b f_k \, \mathrm{d}x' &\leq \int_K \liminf_{\epsilon\to 0} \text{ess } V_a^b f_k^\epsilon \, \mathrm{d}x' \\
&\leq \liminf_{\epsilon\to 0} \int_K \text{ess } V_a^b f_k^\epsilon \, \mathrm{d}x' \\
&= \liminf_{\epsilon\to 0} \int_K \int_a^b |f_k^\epsilon(x_k)|' \, \mathrm{d}x_k \, \mathrm{d}x' \\
&= \liminf_{\epsilon\to 0} \int_C |f^\epsilon(x_k)|' \, \mathrm{d}x \\
&= \liminf_{\epsilon\to 0} \int_C \left|\frac{\partial f^\epsilon}{\partial x_k}\right| \mathrm{d}x \\
&\leq \liminf_{\epsilon\to 0} \int_C |Df^\epsilon| \, \mathrm{d}x \\
&\leq \limsup_{\epsilon\to 0} \int_C |Df^\epsilon| \, \mathrm{d}x \\
&< \infty.
\end{aligned}$$

This finishes the proof of the first direction of the theorem.

We end by proving the second direction of the theorem. Suppose $f \in L^1_{loc}(\mathbb{R}^n)$ and

$$\int_K \text{ess } V_a^b f_k \, \mathrm{d}x' < \infty,$$

for any compact $K \subseteq \mathbb{R}^n$, $\forall a, b \in \mathbb{R}$, $a < b$, and any $k = 1, \ldots, n$.

Now, fix $k = 1, \ldots, n$. Let $\varphi_k \in C_c^\infty(\mathbb{R}^n)$, $|\varphi_k| \leq 1$. Then choose a, b, and K such that

$$\text{spt}(\varphi_k) \subseteq \Big\{x \in \mathbb{R}^n, a \leq x_k \leq b, (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \in K\Big\}$$

By Remark 4 , and Theorem 4.0.14 we have

$$\int_a^b f_k \frac{\partial \varphi_k}{\partial x_k} \, \mathrm{d}x_k \leq \sup \int_a^b f_k \frac{\partial \varphi_k}{\partial x_k} \, \mathrm{d}x_k = ||Df_k|| = \text{ess } V_a^b f_k. \tag{5.0.10}$$

So, by (5.0.10) we get

$$\int_{\mathbb{R}^n} f \frac{\partial \varphi_k}{\partial x_k} \, \mathrm{d}x = \int_{\text{spt}(\varphi_k)} f \frac{\partial \varphi_k}{\partial x_k} \, \mathrm{d}x \leq \int_K \int_a^b f_k \frac{\partial \varphi_k}{\partial x_k} \, \mathrm{d}x_k \mathrm{d}x' \leq \int_K \text{ess } V_a^b f_k \, \mathrm{d}x' < \infty \tag{5.0.11}$$

Finally, let $\varphi = (\varphi_1, \ldots, \varphi_n)$, $\varphi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Then by (5.0.11) we get

$$\int_{\mathbb{R}^n} f \text{div}\varphi \, \mathrm{d}x = \int_{\mathbb{R}^n} f \sum_{k=1}^n \frac{\partial \varphi_k}{\partial x_k} \, \mathrm{d}x = \sum_{k=1}^n \int_{\mathbb{R}^n} f \frac{\partial \varphi_k}{\partial x_k} \, \mathrm{d}x < \infty$$

which means that $f \in BV_{loc}(\mathbb{R}^n)$. $\square$

Chapter 6

Criterion for Sets of Locally Finite Perimeter

To be able to prove our main theorem, we need the following two lemmas.

We start by some notations: $x = (x', t) \in \mathbb{R}^n$, *for* $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, $t = x_n \in \mathbb{R}$.

The projection $P : \mathbb{R}^n \to \mathbb{R}^{n-1}$ *is defined by :*

$$P(x) = x'.$$

Finally, Set

$$N(P|A, x') = \mathcal{H}^0 \left(A \cap P^{-1}\{x'\}\right) \tag{6.0.1}$$

for Borel sets $A \subset \mathbb{R}^n$ *and* $x' \in \mathbb{R}^{n-1}$.

Lemma 6.0.1. $\int_{\mathbb{R}^{n-1}} N(P|A, x')\ \mathrm{d}x' \leq \mathcal{H}^{n-1}(A)$.

Proof. Fix $j \in \mathbb{N}$. By Definition 2.0.21, $\exists \{B_i^j\}_{i=1}^{\infty}$ such that $A \subset \bigcup_{i=1}^{\infty} B_i^j$ where diam $\left(B_i^j\right) \leq \dfrac{1}{j}$ for every i, and

$$\sum_{i=1}^{\infty} \alpha_{n-1} \left(\frac{\operatorname{diam}\left(B_i^j\right)}{2}\right)^{n-1} \leq \mathcal{H}^{n-1}(A) + \frac{1}{j}. \tag{6.0.2}$$

Let

$$g_i^j = \chi_{P\left(B_i^j\right)}. \tag{6.0.3}$$

Then,

$$\mathcal{H}^0_{\frac{1}{j}}\left(A \cap P^{-1}\left(\{x'\}\right)\right) \leq \sum_{i=1}^{\infty} g_i^j\left(x'\right). \tag{6.0.4}$$

To see (6.0.4), notice that by Theorem 2.0.23, $\mathcal{H}^0_{\frac{1}{j}}\left(A \cap P^{-1}\left(\{x'\}\right)\right)$ counts the points in $A \cap P^{-1}\{x'\}$.Then, if

$$\begin{aligned} x \in A \cap P^{-1}\{x'\} \quad &\longrightarrow \quad \exists \text{ at least one } i \text{ such that } x \in B_i^j \text{ and } P(x) = x' \in P\left(B_i^j\right) \\ &\longrightarrow \quad g_i^j\left(x'\right) = 1 \end{aligned}$$

which implies that, $\sum_{i=1}^{\infty} g_i^j\left(x'\right)$ counts the same points each at least once (for each x, and each i, where $x \in B_i^j$, $g_i^j\left(x'\right)$ gives us a 1), thus leading to (6.0.4).

By (6.0.1), Definition 2.0.22, Theorem 2.0.13, (6.0.4), Theorem 2.0.14, (6.0.3), Theorems 2.0.24, 2.0.20 and 2.0.26, and finally by (6.0.2), we get,

$$\begin{aligned}
\int_{\mathbb{R}^{n-1}} N(P|A,x')\ \mathrm{d}x' &= \int_{\mathbb{R}^{n-1}} \mathcal{H}^0\Big(A\cap P^{-1}(\{x'\})\Big)\ \mathrm{d}\mathcal{H}^{n-1} \\
&= \int_{\mathbb{R}^{n-1}} \lim_{j\to\infty} \mathcal{H}^0_{\frac{1}{j}}\Big(A\cap P^{-1}(\{x'\})\Big)\ \mathrm{d}\mathcal{H}^{n-1} \\
&= \int_{\mathbb{R}^{n-1}} \liminf_{j\to\infty} \mathcal{H}^0_{\frac{1}{j}}\Big(A\cap P^{-1}(\{x'\})\Big)\ \mathrm{d}\mathcal{H}^{n-1} \\
&\le \liminf_{j\to\infty} \int_{\mathbb{R}^{n-1}} \mathcal{H}^0_{\frac{1}{j}}\Big(A\cap P^{-1}(\{x'\})\Big)\ \mathrm{d}\mathcal{H}^{n-1} \\
&\le \liminf_{j\to\infty} \int_{\mathbb{R}^{n-1}} \sum_{i=1}^{\infty} g_i^j(x')\ \mathrm{d}\mathcal{H}^{n-1} \\
&= \liminf_{j\to\infty} \sum_{i=1}^{\infty} \int_{\mathbb{R}^{n-1}} g_i^j(x')\ \mathrm{d}\mathcal{H}^{n-1} \\
&= \liminf_{j\to\infty} \sum_{i=1}^{\infty} \int_{\mathbb{R}^{n-1}} \chi_{P\left(B_i^j\right)}\ \mathrm{d}\mathcal{H}^{n-1} \\
&= \liminf_{j\to\infty} \sum_{i=1}^{\infty} \mathcal{H}^{n-1}\Big(P\left(B_i^j\right)\Big) \\
&= \liminf_{j\to\infty} \sum_{i=1}^{\infty} \mathcal{L}^{n-1}\Big(P\left(B_i^j\right)\Big) \\
&\le \liminf_{j\to\infty} \sum_{i=1}^{\infty} \alpha_{n-1}\left(\frac{\operatorname{diam}\Big(P\left(B_i^j\right)\Big)}{2}\right)^{n-1} \\
&\le \liminf_{j\to\infty} \sum_{i=1}^{\infty} \alpha_{n-1}\left(\frac{\operatorname{diam}\left(B_i^j\right)}{2}\right)^{n-1} \\
&\le \liminf_{j\to\infty} \left(\mathcal{H}^{n-1}(A)+\frac{1}{j}\right) \\
&= \mathcal{H}^{n-1}(A)+\liminf_{j\to\infty}\frac{1}{j} \\
&= \mathcal{H}^{n-1}(A).
\end{aligned}$$

This finishes the proof of the lemma. □

Lemma 6.0.2. *Let $E \subset \mathbb{R}^n$ be $\mathcal{L}^n$-measurable. We define*

$$I := \left\{ x \in \mathbb{R}^n | \lim_{r \to 0} \frac{\mathcal{L}^n\left(B\left(x,r\right) \setminus E\right)}{r^n} = 0 \right\} \tag{6.0.5}$$

and

$$O := \left\{ x \in \mathbb{R}^n | \lim_{r \to 0} \frac{\mathcal{L}^n\left(B\left(x,r\right) \cap E\right)}{r^n} = 0 \right\} \tag{6.0.6}$$

to be the measure theoretic interior and exterior of E, respectively. Then

$$\mathcal{L}^n\Big(\left(I \setminus E\right) \cup \left(E \setminus I\right) \Big) = 0 \tag{6.0.7}$$

and

$$\mathcal{L}^n\Big(\left(O \setminus E^c\right) \cup \left(E^c \setminus O\right) \Big) = 0 \tag{6.0.8}$$

Proof. We will only prove (6.0.7) , since the proof of (6.0.8) is similar. To do this we will show that $\mathcal{L}^n\left(I \setminus E\right) = 0$ and $\mathcal{L}^n\left(E \setminus I\right) = 0$. First, by Corollary 2.0.19 on E^c we have

$$\lim_{r \to 0} \frac{\mathcal{L}^n\left(B\left(x,r\right) \cap E^c\right)}{\mathcal{L}^n\left(B\left(x,r\right)\right)} = 1 \quad \text{a.e } x \in E^c. \tag{6.0.9}$$

This means that $E^c = A \cup F$ with

$$\mathcal{L}^n\left(F\right) = 0 \tag{6.0.10}$$

and

$$\lim_{r \to 0} \frac{\mathcal{L}^n\left(B\left(x,r\right) \setminus E\right)}{\alpha_n r^n} = 1 \quad \forall\ x \in A. \tag{6.0.11}$$

Similarly by Corollary 2.0.19 on E, we can write $E = B \cup G$ with

$$\mathcal{L}^n(G) = 0 \tag{6.0.12}$$

and

$$\lim_{r \to 0} \frac{\mathcal{L}^n(B(x,r) \cap E)}{\alpha_n r^n} = 1 \ \ \forall\ x\ \in B. \tag{6.0.13}$$

Now, let us prove that $I \setminus E \subset F$. Notice that

$$I \setminus E = I \cap E^c = I \cap (A \cup F) = (I \cap A) \cup (I \cap F). \tag{6.0.14}$$

But, $I \cap A = \phi$ because, if $x \in A$, then by (6.0.11) $\lim_{r \to 0} \frac{\mathcal{L}^n(B(x,r) \setminus E)}{\alpha_n r^n} = 1$, while if $x \in I$, then by (6.0.5) $\lim_{r \to 0} \frac{\mathcal{L}^n(B(x,r) \setminus E)}{r^n} = 0$.

Thus (6.0.14) becomes

$$I \setminus E = I \cap F \subset F \tag{6.0.15}$$

which is exactly what we want. Since by (6.0.10) we have $\mathcal{L}^n(F) = 0$, (6.0.15) implies that $\mathcal{L}^n(I \setminus E) = 0$.

Next, let us show that

$$E \setminus I \subset G \tag{6.0.16}$$

To see this, take

$$x \in E \setminus I = (B \cup G) \cap I^c \subset B \cup G$$

Notice that if $x \in B$,then by Definition 2.0.16 and (6.0.13) we have :

$$\begin{aligned} 1 &= \lim_{r \to 0} \frac{\mathcal{L}^n (B(x,r))}{\alpha_n r^n} \\ &= \lim_{r \to 0} \frac{\mathcal{L}^n (B(x,r) \cap (E \cup E^c))}{\alpha_n r^n} \\ &= \lim_{r \to 0} \frac{\mathcal{L}^n (B(x,r) \cap E)}{\alpha_n r^n} + \lim_{r \to 0} \frac{\mathcal{L}^n (B(x,r) \cap E^c)}{\alpha_n r^n} \\ &= 1 + \lim_{r \to 0} \frac{\mathcal{L}^n (B(x,r) \cap E^c)}{\alpha_n r^n} \end{aligned}$$

which implies that , $\lim_{r \to 0} \frac{\mathcal{L}^n (B(x,r) \cap E^c)}{\alpha_n r^n} = 0$. In other words $x \in I$, which contradicts the fact that we took $x \in E \setminus I$. Thus, $x \notin B$ and therefore we have (6.0.16) .

However, recall that by (6.0.12) , we have, $\mathcal{L}^n (G) = 0$ so by (6.0.16), we get

$$\mathcal{L}^n (E \setminus I) = 0.$$

Finally, by countable subadditivity:

$$\mathcal{L}^n \Big((I \setminus E) \cup (E \setminus I) \Big) \leq \mathcal{L}^n (I \setminus E) + \mathcal{L}^n (E \setminus I) = 0.$$

Thus,

$$\mathcal{L}^n \Big((I \setminus E) \cup (E \setminus I) \Big) = 0.$$

□

We are now ready to discuss the criterion for sets of locally finite perimeter.

Theorem 6.0.3. *Criterion for Finite Perimeter*

Let $E \subset \mathbb{R}^n$ be $\mathcal{L}^n$-measurable. Then, E has locally finite perimeter if and only if ,

$$\mathcal{H}^{n-1}\left(K \cap \partial_* E\right) < \infty \tag{6.0.17}$$

for each compact set $K \subset \mathbb{R}^n$.

Proof. We start by proving the first direction of the theorem.We know by Theorem 4.0.4 that if E is a set of locally finite perimeter, then $||\partial E||$ is a radon measure. Furthermore, by Theorem 4.0.7 and Definition 2.0.2 , we have

$$||\partial E||\left(K\right) = \mathcal{H}^{n-1} \llcorner \partial^* E\left(K\right) = \mathcal{H}^{n-1}\left(\partial^* E \cap K\right). \tag{6.0.18}$$

Hence, by Lemma 4.0.10 and (6.0.18) we have :

$$\begin{aligned}
\mathcal{H}^{n-1}\left(K \cap \partial_* E\right) &= \mathcal{H}^{n-1}\Big(K \cap \Big(\partial^* E \cup \left(\partial_* E \setminus \partial^* E\right)\Big)\Big) \\
&= \mathcal{H}^{n-1}\Big(\left(K \cap \partial^* E\right) \cup \Big(K \cap \left(\partial_* E \setminus \partial^* E\right)\Big)\Big) \\
&\leq \mathcal{H}^{n-1}\left(K \cap \partial^* E\right) + \mathcal{H}^{n-1}\Big(K \cap \left(\partial_* E \setminus \partial^* E\right)\Big) \\
&= \mathcal{H}^{n-1}\left(K \cap \partial^* E\right) \\
&= ||\partial E||\left(K\right) < \infty
\end{aligned}$$

where the last step comes from Lemma 4.0.9 and Definition 2.0.7.

We proceed by proving the second direction of the theorem. We have $\mathcal{H}^{n-1}(K \cap \partial_* E) < \infty$ for any compact $K \subset \mathbb{R}^n$ and want to prove that E is of locally finite perimeter. Now recall that by Lemma 6.0.2, $\mathcal{L}^n(E \setminus I) = 0$ and $\mathcal{L}^n(I \setminus E) = 0$. But any two sets that differ in at most a set of $\mathcal{L}^n$-measure zero, define the same boundary measure. Thus it is enough to prove I of locally finite perimeter. Hence, by Definition 4.0.3 and Theorem 5.0.2 we need to prove that

$$\int_V ess\ V_{-a}^{a}(\chi_I)_l \ \mathrm{d}x' < \infty \tag{6.0.19}$$

for $V =]-a, a[^{n-1} \subset \mathbb{R}^{n-1}$ for any fixed $a > 0$, $l = 1, \ldots, n$, and $x' = (x_1, \ldots x_{l-1}, x_{l+1}, \ldots, x_n) \in \mathbb{R}^{n-1}$.

We will do this proof for $l = n$ only, as the rest of the cases follow accordingly. For $x \in \mathbb{R}^n$, we will denote $x = (x', x_n)$ where $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$ and $x' = P(x)$ the projection of x on $\mathbb{R}^{n-1}$. We will denote by z a general point in $\mathbb{R}^{n-1}$, and by t a general point in $\mathbb{R}$.

We begin by defining the following sets, for positive integers k and m :

$$G(k) = \left\{ x \in \mathbb{R}^n, \mathcal{L}^n(B(x,r) \cap O) \leq \frac{\alpha_{n-1}}{3^{n+1}} r^n \text{ for } 0 < r < \frac{3}{k} \right\} \tag{6.0.20}$$

$$H(k) = \left\{ x \in \mathbb{R}^n, \mathcal{L}^n(B(x,r) \cap I) \leq \frac{\alpha_{n-1}}{3^{n+1}} r^n \text{ for } 0 < r < \frac{3}{k} \right\} \tag{6.0.21}$$

and,

$$G^{+}(k,m) = \left\{ x \in G(k), x + se_n \in O \text{ for } 0 < s < \frac{3}{m} \right\} \quad (6.0.22)$$

$$G^{-}(k,m) = \left\{ x \in G(k), x - se_n \in O \text{ for } 0 < s < \frac{3}{m} \right\} \quad (6.0.23)$$

$$H^{+}(k,m) = \left\{ x \in H(k), x + se_n \in I \text{ for } 0 < s < \frac{3}{m} \right\}$$

$$H^{-}(k,m) = \left\{ x \in H(k), x - se_n \in I \text{ for } 0 < s < \frac{3}{m} \right\} \quad (6.0.24)$$

Claim # 1: $\mathcal{L}^{n-1}\Big(P\big(G^{+}(k,m)\big)\Big) = \mathcal{L}^{n-1}\Big(P\big(G^{-}(k,m)\big)\Big) = \mathcal{L}^{n-1}\Big(P\big(H^{+}(k,m)\big)\Big)$
$= \mathcal{L}^{n-1}\Big(P\big(H^{-}(k,m)\big)\Big) = 0 \ \forall\ k,m \in \mathbb{N}$.

Proof of Claim # 1:We will only prove $\mathcal{L}^{n-1}\Big(P\big(G^{+}(k,m)\big)\Big) = 0$, because all other cases are treated similarly. Let $G^{+}(k,m) = \bigcup_{j=-\infty}^{\infty} G_j$, where

$$G_j = G^{+}(k,m) \cap \left\{ x \in \mathbb{R}^n,\ \frac{j-1}{m} \le x_n < \frac{j}{m} \right\} \quad (6.0.25)$$

Assume that

$$z \in \mathbb{R}^{n-1}, 0 < r < \min\left\{\frac{1}{k}, \frac{1}{m}\right\}, \text{and } B(z,r) \cap P(G_j) \neq \phi \quad (6.0.26)$$

Thus, $P^{-1}\big(B(z,r)\big) \cap G_j \neq \phi$. So, $\sup\left\{ x_n \in \mathbb{R},\ x \in P^{-1}\big(B(z,r)\big) \cap G_j \right\}$ is defined.

Therefore, there exists $b \in P^{-1}\big(B(z,r)\big) \cap G_j$ such that

$$b_n + \frac{r}{2} > \sup\left\{x_n \in \mathbb{R},\ x \in P^{-1}\Big(B(z,r)\Big) \cap Gj\right\}. \tag{6.0.27}$$

But, notice that

$$\left\{y \in \mathbb{R}^n,\ b_n + \frac{r}{2} \le y_n \le b_n + r\right\} \cap P^{-1}\Big(P(G_j) \cap B(z,r)\Big) \subset O \cap B(b,3r) \tag{6.0.28}$$

To show this, let $y \in$ left hand side of $(6.0.28)$. We prove first that $y \in B(b,3r)$.

Take $y' = (y_1, \ldots, y_{n-1})$ and $b' = (b_1, \ldots, b_{n-1})$. Then,

$$\begin{aligned}
|y-b| &= \sqrt{\sum_{i=1}^{n} |y_i - b_i|^2} \\
&= \sqrt{\sum_{i=1}^{n-1} |y_i - b_i|^2 + |y_n - b_n|^2} \\
&\le \sqrt{\sum_{i=1}^{n-1} |y_i - b_i|^2} + \sqrt{|y_n - b_n|^2} \\
&= |y' - b'| + |y_n - b_n| \\
&\le |y' - z| + |z - b'| + |y_n - b_n| \\
&\le 3r
\end{aligned}$$

where the last inequality is due to the following statements respectively:

1. by (6.0.28) $y \in P^{-1}\Big(P(G_j) \cap B(z,r)\Big) \longrightarrow y' \in P(G_j) \cap B(z,r) \longrightarrow y' \in B(z,r)$.

2. by (6.0.27) $b \in P^{-1}\Big(B(z,r)\Big) \cap G_j \longrightarrow b' \in B(z,r) \cap P(G_j) \longrightarrow b' \in B(z,r)$.

3. by (6.0.28) $\frac{r}{2} \leq y_n - b_n \leq r \longrightarrow y_n \in B(b_n, r)$.

Next we prove that the left hand side of (6.0.28) is in O.
Since $y \in P^{-1}\Big(P(G_j) \cap B(z,r)\Big)$ then , $y' \in P(G_j) \cap B(z,r)$ that is, there exists $x \in G_j$ such that $x' = y'$. Thus, we get $x \in P^{-1}\Big(B(z,r)\Big)$, that is $x \in G_j \cap P^{-1}\Big(B(z,r)\Big)$. So by (6.0.27) and (6.0.28) , we get

$$x_n < b_n + \frac{r}{2} \leq y_n \tag{6.0.29}$$

So by (6.0.29) , $\exists\ s > 0$ such that

$$y_n = x_n + s \leq b_n + r. \tag{6.0.30}$$

Thus, we get

$$s \leq b_n + r - x_n.$$

Moreover, by (6.0.25), we have $x_n \geq \frac{j-1}{m}$, and thus

$$s \leq b_n + r - \frac{j}{m} + \frac{1}{m}. \tag{6.0.31}$$

But, $b \in P^{-1}\Big(B(z,r)\Big) \cap G_j$, and so by (6.0.25) , we get

$$b_n < \frac{j}{m}. \tag{6.0.32}$$

Finally, by replacing (6.0.32) in (6.0.31) and using (6.0.26) we get

$$0 < s \leq \frac{2}{m} < \frac{3}{m}. \tag{6.0.33}$$

To see how this leads us to $y \in O$, recall that we have $x \in G_j$. Thus by (6.0.25) $x \in G^+ (k, m)$. Hence, by (6.0.33) $x + se_n \in O$.

Now, with the fact that $x' = y'$ and by (6.0.30), we have

$$x + se_n = (x', x_n + s) = (y', x_n + s) = (y', y_n) = y.$$

So $y \in O$ and we have (6.0.28). Taking the $\mathcal{L}^n$-measure of each side of (6.0.28) and by (6.0.20) used on b and $3r$, we get

$$\mathcal{L}^n \left(\left\{ y \in \mathbb{R}^n,\ b_n + \frac{r}{2} \leq y_n \leq b_n + r \right\} \cap P^{-1} \Big(P\left(G_j\right) \cap B\left(z, r\right) \Big) \right) \leq \mathcal{L}^n \Big(O \cap B\left(b, 3r\right) \Big) \leq \frac{\alpha_{n-1}}{3^{n+1}} \left(3r\right)^n . \tag{6.0.34}$$

However, notice that

$$\Big(P\left(G_j\right) \cap B\left(z, r\right) \Big) \times \left[b_n + \frac{r}{2}, b_n + r \right] \subseteq \left\{ y \in \mathbb{R}^n,\ b_n + \frac{r}{2} \leq y_n \leq b_n + r \right\} \cap P^{-1} \Big(P\left(G_j\right) \cap B\left(z, r\right) \Big). \tag{6.0.35}$$

To see this take $y = (y', y_n) \in P\left(G_j\right) \cap B\left(z, r\right) \times \left[b_n + \frac{r}{2}, b_n + r \right]$. This implies that

1. $y' \in P\left(G_j\right) \cap B\left(z, r\right) \longrightarrow y \in P^{-1} \Big(P\left(G_j\right) \cap B\left(z, r\right) \Big)$
2. $y_n \in \left[b_n + \frac{r}{2}, b_n + r \right] \longrightarrow y \in \left\{ y \in \mathbb{R}^n,\ b_n + \frac{r}{2} \leq y_n \leq b_n + r \right\}.$

Therefore, replacing (6.0.35) in (6.0.34), we get

$$\mathcal{L}^n\left(\left(P\left(G_j\right)\cap B\left(z,r\right)\right)\times[b_n+\frac{r}{2},b_n+r]\right)\leq\frac{\alpha_{n-1}}{3^{n+1}}\left(3r\right)^n. \tag{6.0.36}$$

Now,by applying Definition 2.0.15 on (6.0.36), we get

$$\frac{r}{2}\mathcal{L}^{n-1}\left(P\left(G_j\right)\cap B\left(z,r\right)\right)\leq\frac{\alpha_{n-1}}{3^{n+1}}\left(3r\right)^n. \tag{6.0.37}$$

Doing some arithmetic operations on (6.0.37) and taking $\limsup_{r\to 0}$ on both sides, we get

$$\limsup_{r\to 0}\frac{\mathcal{L}^{n-1}\left(P\left(G_j\right)\cap B\left(z,r\right)\right)}{\alpha_{n-1}r^{n-1}}\leq\frac{2}{3}\quad\forall\, z\in\mathbb{R}^{n-1}. \tag{6.0.38}$$

But, if we apply Corollary 2.0.19 on $P\left(G_j\right)$, then we know that

$$\limsup_{r\to 0}\frac{\mathcal{L}^{n-1}\left(P\left(G_j\right)\cap B\left(z,r\right)\right)}{\alpha_{n-1}r^{n-1}}=1\quad\text{a.e } z\in P\left(G_j\right),$$

which means that, $\exists\ N\subset P\left(G_j\right)$ such that

$$\mathcal{L}^{n-1}\left(N\right)=0,$$

whereas on $P\left(G_j\right)\setminus N$

$$\limsup_{r\to 0}\frac{\mathcal{L}^{n-1}\left(P\left(G_j\right)\cap B\left(z,r\right)\right)}{\alpha_{n-1}r^{n-1}}=1\ \ \forall\ z\in P\left(G_j\right)\setminus N.$$

However, by (6.0.38) we have $\limsup_{r\to 0} \frac{\mathcal{L}^{n-1}\Big(P\left(G_j\right)\cap B\left(z,r\right)\Big)}{\alpha_{n-1}r^{n-1}} \leq \frac{2}{3} < 1$, which implies that $P\left(G_j\right)\setminus N = \phi$, and $P\left(G_j\right) = N$. Thus, we get

$$\mathcal{L}^{n-1}\Big(P\left(G_j\right)\Big) = \mathcal{L}^{n-1}\left(N\right) = 0. \tag{6.0.39}$$

Finally, by (6.0.39) we have :

$$\begin{aligned}
\mathcal{L}^{n-1}\left(P\Big(G^{+}\left(k,m\right)\Big)\right) &= \mathcal{L}^{n-1}\left(P\left(\bigcup_{j=-\infty}^{\infty} G_j\right)\right) \\
&= \mathcal{L}^{n-1}\left(\bigcup_{j=-\infty}^{\infty} P\left(G_j\right)\right) \\
&\leq \sum_{j=-\infty}^{\infty} \mathcal{L}^{n-1}\Big(P\left(G_j\right)\Big) = 0.
\end{aligned}$$

And the claim is proved.

Define $U =]-a,a[^n \subset \mathbb{R}^n$.

We want to apply Lemma 6.0.1 on the set $U \cap \partial_* E$. To be able to do so, notice that U is an open set, and thus (by Definition 2.0.4) a borel set, and by Lemma 4.0.9, $\partial_* E$ is a borel set. Hence, $U \cap \partial_* E$ is a borel set. So by applying Lemma 6.0.1 on the set $U \cap \partial_* E$, and by (6.0.17) , we have

$$\int_{\mathbb{R}^{n-1}} N\left(P|U \cap \partial_* E, z\right)\ \mathrm{d}z \leq \mathcal{H}^{n-1}\left(U \cap \partial_* E\right) < \infty. \tag{6.0.40}$$

Since an integrable function is finite almost everywhere, from (6.0.40) we get that

$$N\left(P|U \cap \partial_* E, z\right) < \infty \ \ \mathcal{L}^{n-1} - \text{a.e } z \in \mathbb{R}^{n-1}, \tag{6.0.41}$$

and therefore by (6.0.1) we have

$$\mathcal{H}^0\Big(U \cap \partial_* E \cap P^{-1}(\{z\})\Big) < \infty \quad \mathcal{L}^{n-1} - \text{a.e } z \in \mathbb{R}^{n-1}. \tag{6.0.42}$$

Now let $z \in V \setminus \bigcup_{k,m=1}^{\infty} P\left[G^+(k,m) \cup G^-(k,m) \cup H^+(k,m) \cup H^-(k,m)\right]$, such that (6.0.42) holds.

Then, by Theorem 2.0.23 and by (6.0.42), we can write

$$U \cap \partial_* E \cap P^{-1}(\{z\}) = \{\tau_i\}_{i=1}^{M}. \tag{6.0.43}$$

Now, assume $-a < t_1 < \ldots < t_{m+1} < a$ are points of approximate continuity of $f^z(t) := \chi_I(z,t)$. Notice that $\forall\ j \in \{1, \ldots, m\}$ we have

$$|f^z(t_{j+1}) - f^z(t_j)| = 0 \quad \text{or} \quad |f^z(t_{j+1}) - f^z(t_j)| = 1.$$

Suppose that $|f^z(t_{j+1}) - f^z(t_j)| = 1$, and assume without any loss of generality that,

$$(z, t_j) \in I \text{ and } (z, t_{j+1}) \notin I. \tag{6.0.44}$$

We would like to show that every neighbourhood of t_{j+1} that is inside the partition $]-a, a[$ must contain points s such that $(z,s) \in O$ and f^z is approximetly continuous at s $(\bigstar)$

To prove $(\bigstar)$, consider two cases :

Case 1: If $(z, t_{j+1}) \in O$, then $(\bigstar)$ is satisfied for $s = t_{j+1}$.

Case 2: If $(z, t_{j+1}) \notin O$, then by Definition 4.0.8 and by (6.0.5) and (6.0.6) we can write $\mathbb{R}^n = I \cup O \cup \partial_* E$. By (6.0.44), we get $(z, t_{j+1}) \notin I \cup O$, hence

$(z, t_{j+1}) \in \partial_* E.$

We now have the following information about (z, t_{j+1})

1. $(z, t_{j+1}) \in \partial_* E$

2. $(z, t_{j+1}) \in P^{-1}(\{z\})$

3. $(z, t_{j+1}) = (z_1, \dots z_{n-1}, t_{j+1}) \in U =]-a, a[^n$,where $z_1, \dots z_{n-1}$ $\in]-a, a[^{n-1}$ and $t_{j+1} \in]-a, a[$.

Thus by (6.0.43), and without any loss of generality, we have $(z, t_{j+1}) = \tau_1$. To prove ($\bigstar$), we go by contradiction. Assume ($\bigstar$) is not true.Then, there exists a neighborhood W of t_{j+1} such that for any $s \in W$, $(z, s) \notin O$. Thus there exists $r_0 > 0$ such that

$$|\tau_i - \tau_l| > 2r_0 \quad \forall\, i, l \in \{1, \dots M\}$$

and

$$(z, s) \in I \quad \forall\, s \in (t_{j+1} - r_0, t_{j+1} + r_0) \setminus \{t_{j+1}\}.$$

Now, for sufficiently large integer N, we have

$$B\left((z, t_{j+1}), \frac{r}{2}\right) \subset B\left(\left(z, t_{j+1} + \frac{r}{N}\right), r\right) \tag{6.0.45}$$

where $(z, t_{j+1}) \in \partial_* E$ and $\left(z, t_{j+1} + \dfrac{r}{N}\right) \in I$.

Next, putting $\mathcal{L}^n$ measure, intersecting by E^c, dividing by r^n and taking

$\limsup_{r\to 0}$ on both sides of (6.0.45) we have, by (6.0.5) that

$$\begin{aligned}\limsup_{r\to 0}\frac{\mathcal{L}^n\left(B\left((z,t_{j+1}),\frac{r}{2}\right)\cap E^c\right)}{r^n} &\leq \limsup_{r\to 0}\frac{\mathcal{L}^n\left(B\left(\left(z,t_{j+1}+\frac{r}{N}\right),r\right)\cap E^c\right)}{r^n}\\ &= \lim_{r\to 0}\frac{\mathcal{L}^n\left(B\left(\left(z,t_{j+1}+\frac{r}{N}\right),r\right)\cap E^c\right)}{r^n}\end{aligned} \tag{6.0.46}$$

with $(z,t_{j+1})\in\partial_*E$ and $\left(z,t_{j+1}+\frac{r}{N}\right)\in I$.

But, by Definition 4.0.8, the left hand side of (6.0.46) is strictly positive, while by (6.0.5), the right hand side of (6.0.46) is zero. This leads to a contradiction, and ($\bigstar$) is proved.

Consequently, for $z\in V\setminus\bigcup_{k,m=1}^{\infty}P\left[G^+(k,m)\cup G^-(k,m)\cup H^+(k,m)\cup H^-(k,m)\right]$ such that (6.0.42) holds, we have

$$\text{ess } V_{-a}^{a}f^z=\sup\left\{\sum_{j=1}^{m}|f^z(t_{j+1})-f^z(t_j)|\right\}$$

the sup taken over all points $-a<t_1<\ldots<t_m<a$ such that $(z,t_i)\in O\cup I$ and f^z is approximetly continuous at each t_i.

Claim # 2: Let $z\in V\setminus\bigcup_{k,m=1}^{\infty}P\left[G^+(k,m)\cup G^-(k,m)\cup H^+(k,m)\cup H^-(k,m)\right]$. If $(z,u)\in I$ and $(z,v)\in O$, with $u<v$, then $\exists\ u<t<v$ such that $(z,t)\in\partial_*E$.

Proof of Claim # 2: Fix $z \in V \setminus \bigcup_{k,m=1}^{\infty} P\left[G^{+}(k,m) \cup G^{-}(k,m) \cup H^{+}(k,m) \cup H^{-}(k,m)\right]$, and let $(z,u) \in I$ and $(z,v) \in O$. We will prove ***Claim # 2*** by contradiction. Assume $(z,t) \notin \partial_{*}E \ \forall\ u < t < v$. Then $(z,t) \in O \cup I$ $\forall\ u < t < v$.

Subclaim # 1: $\forall\ k \in \mathbb{N}$, $G(k)$ and $H(k)$ are both closed and increasing as a sequence of k.

Proof of Subclaim # 1: For simplifying notation, let

$$f_r(x) = \frac{\mathcal{L}^n(B(x,r) \cap O)}{\frac{\alpha_{n-1}}{3^{n+1}}}.$$

Notice that $f_r(x)$ is a continuous function of x.

Now, recall that by the definition of $G(k)$ (see 6.0.20), we have

$$G(k) = \left\{x \in \mathbb{R}^n, f_r(x) \leq r^n \text{ for } 0 < r < \frac{3}{k}\right\} \tag{6.0.47}$$

Let us show that

$$G(k) = \bigcap_{0 < r < \frac{3}{k}} f_r^{-1}([0, r^n]) \tag{6.0.48}$$

To see that, take $x \in G(k)$, then by (6.0.47) we have

$$
\begin{aligned}
f_r(x) \le r^n \ \forall\ 0 < r < \frac{3}{k} \quad &\Longleftrightarrow \quad 0 \le f_r(x) \le r^n \ \forall\ 0 < r < \frac{3}{k} \\
&\Longleftrightarrow \quad f_r(x) \in [0, r^n] \ \forall\ 0 < r < \frac{3}{k} \\
&\Longleftrightarrow \quad x \in f_r^{-1}([0, r^n]) \ \forall\ 0 < r < \frac{3}{k} \\
&\Longleftrightarrow \quad x \in \bigcap_{0<r<\frac{3}{k}} f_r^{-1}([0, r^n]) \ \forall\ 0 < r < \frac{3}{k}
\end{aligned}
$$

and (6.0.48) is proved.

Thus, since f_r continuous and $[0, r^n]$ is a closed interval, then (6.0.48) implies that $G(k)$ is closed.

Furthermore, $G(k)$ is an increasing sequence in k, since by (6.0.47) if $x \in G(k)$ then $f_r(x) \le r^n \ \forall\ 0 < r < \frac{3}{k}$.

Thus $f_r(x) \le r^n \ \forall\ 0 < r < \frac{3}{k+1}$ which implies that $x \in G(k+1)$. Hence, $G(k) \subset G(k+1)$.

By a similar argument $H(k)$ is also closed and increasing as a sequence of k.

Subclaim # 2:

$$I \subset \bigcup_{k=1}^{\infty} G(k) \tag{6.0.49}$$

and

$$O \subset \bigcup_{k=1}^{\infty} H(k) \tag{6.0.50}$$

Proof of Subclaim # 2: To show (6.0.49), take $x \in I$. Then by (6.0.5)

$$\lim_{r \to 0} \frac{\mathcal{L}^n(B(x, r) \cap E^c)}{r^n} = 0. \tag{6.0.51}$$

On one hand we have

$$\begin{aligned} B(x,r)\cap E^c &= \Big(B(x,r)\cap\big(E^c\cap(O\cup O^c)\big)\Big) \\ &= \big(B(x,r)\cap(E^c\cap O)\big)\cup\big(B(x,r)\cap(E^c\cap O^c)\big) \end{aligned} \tag{6.0.52}$$

Putting $\mathcal{L}^n$-measure on both sides of (6.0.52), we get

$$\begin{aligned} \mathcal{L}^n\big(B(x,r)\cap E^c\big) &= \mathcal{L}^n\big(B(x,r)\cap(E^c\cap O)\big)+\mathcal{L}^n\big(B(x,r)\cap(E^c\cap O^c)\big) \\ &= \mathcal{L}^n\big(B(x,r)\cap(E^c\cap O)\big), \end{aligned} \tag{6.0.53}$$

we got the last step since by Lemma 6.0.2, $\mathcal{L}^n\big(B(x,r)\cap(E^c\cap O^c)\big)\le\mathcal{L}^n(E^c\setminus O)=0$.

On the other hand we have

$$\begin{aligned} B(x,r)\cap O &= \Big(B(x,r)\cap\big(O\cap(E\cup E^c)\big)\Big) \\ &= \big(B(x,r)\cap(O\cap E)\big)\cup\big(B(x,r)\cap(O\cap E^c)\big) \end{aligned} \tag{6.0.54}$$

Putting $\mathcal{L}^n$-measure on both sides of (6.0.54), we get

$$\begin{aligned} \mathcal{L}^n\big(B(x,r)\cap O\big) &= \mathcal{L}^n\big(B(x,r)\cap(O\cap E)\big)+\mathcal{L}^n\big(B(x,r)\cap(O\cap E^c)\big) \\ &= \mathcal{L}^n\big(B(x,r)\cap(O\cap E^c)\big), \end{aligned} \tag{6.0.55}$$

we got the last step since by Lemma 6.0.2, $\mathcal{L}^n\big(B(x,r)\cap(O\cap E)\big)\le\mathcal{L}^n(O\setminus E^c)$ $=0$.

Therofore, by (6.0.53) and (6.0.55) , (6.0.51) becomes

$$\lim_{r \to 0} \frac{\mathcal{L}^n\left(B\left(x,r\right)\cap O\right)}{r^n} = 0.$$

Now, fix $\epsilon = \dfrac{\alpha_{n-1}}{3^{n+1}}$, hence there exists $k \in \mathbb{N}$ such that for all $0 < r < \dfrac{3}{k}$ we have

$$\frac{\mathcal{L}^n\left(B\left(x,r\right)\cap O\right)}{r^n} < \frac{\alpha_{n-1}}{3^{n+1}}.$$

Therefore, by (6.0.20) $x \in G\left(k\right)$, so $x \in \bigcup_{k=1}^{\infty} G\left(k\right)$.

We can argue similarly to prove (6.0.50).

Subclaim # 3: $\exists\ k_0 \in \mathbb{N}$ such that

$$\left(z,u\right) \in G\left(k_0\right), \tag{6.0.56}$$

and

$$\left(z,v\right) \in H\left(k_0\right), \tag{6.0.57}$$

and

$$G\left(k_0\right) \cap H\left(k_0\right) = \phi. \tag{6.0.58}$$

Proof of Subclaim # 3: Recall (by the first line in proof of **Claim # 2**) that $\left(z,u\right) \in I$ and $\left(z,v\right) \in O$.

Using (6.0.49) and (6.0.50) $\exists\ k$ such that $\left(z,u\right) \in G\left(k\right)$ and $\exists\ k'$ such that $\left(z,v\right) \in H\left(k'\right)$.

Let $k_0 = \max\{k, k'\}$, and using the fact that $G\left(k\right)$ and $H\left(k\right)$ are increasing as functions of k, we get $(z, u) \in G(k_0)$ and $(z, v) \in H(k_0)$.

Let us now show that $G(k_0) \cap H(k_0) = \phi$. To prove it, we go by contradiction. Assume there exists $x \in G(k_0) \cap H(k_0)$. Then for every $0 < r < \dfrac{3}{k_0}$, we have

$$\frac{\mathcal{L}^n(B(x,r) \cap O)}{r^n} = \frac{\mathcal{L}^n(B(x,r) \cap E^c)}{r^n} \leq \frac{\alpha_{n-1}}{3^{n+1}}, \tag{6.0.59}$$

and

$$\frac{\mathcal{L}^n(B(x,r) \cap I)}{r^n} = \frac{\mathcal{L}^n(B(x,r) \cap E)}{r^n} \leq \frac{\alpha_{n-1}}{3^{n+1}}. \tag{6.0.60}$$

By adding (6.0.60) and (6.0.59), we get

$$\lim_{r \to 0} \frac{\mathcal{L}^n\big(B(x,r)\big)}{r^n} = \lim_{r \to 0} \frac{\mathcal{L}^n(B(x,r) \cap E^c) + \mathcal{L}^n(B(x,r) \cap E)}{r^n} \leq \frac{2\alpha_{n-1}}{3^{n+1}} < \alpha_n, \tag{6.0.61}$$

where the last inequality comes from the fact that α_n is increasing and thus $\dfrac{2}{3^{n+1}} < 1 < \dfrac{\alpha_n}{\alpha_{n-1}}$.

However, by Definition 2.0.16 we have

$$\lim_{r \to 0} \frac{\mathcal{L}^n\big(B(x,r)\big)}{r^n} = \lim_{r \to 0} \frac{\alpha_n r^n}{r^n} = \alpha_n. \tag{6.0.62}$$

Thus by (6.0.61) and (6.0.62), $\alpha_n < \alpha_n$, which is a contradiction. Hence, $G(k_0) \cap H(k_0) = \phi$.

Subclaim # 4: There exists two numbers u_0 and v_0, such that

$$u < u_0 < v_0 < v \tag{6.0.63}$$

and

$$\{(z,t)\,,u_0<t<v_0\}\cap\Big(G\,(k_0)\cup H\,(k_0)\Big)=\phi, \tag{6.0.64}$$

Proof of Subclaim # 4: Define

$$u_0:=\sup\{t|\,(z,t)\in G\,(k_0)\,,t<v\} \tag{6.0.65}$$

We would like to show that $u_0<v$, and to do that we go by contradiction. Suppose that $u_0=v$. Then, by the definition of sup, there exists a sequence $\{t_i\}$ in $\mathbb{R}$ such that $t_i<v$, $(z,t_i)\in G\,(k_0)$ and $\lim\limits_{i\to\infty}t_i=v$. Thus $\lim\limits_{i\to\infty}(z,t_i)=(z,v)\,.$

Moreover, $G\,(k_0)$ is closed, and thus, $(z,v)\in G\,(k_0)$ which is a contradiction, since by (6.0.57) $(z,v)\in H\,(k_0)$ and by (6.0.58) $G\,(k_0)\cap H\,(k_0)=\phi$.

Notice that $(z,u_0)\in G\,(k_0)$ since by the definition of sup, we got the sequence t_i that converges to u_0 such that $(z,t_i)\in G\,(k_0)\,.$ Then by the fact that $G\,(k_0)$ is closed, we get $\lim\limits_{i\to\infty}(z,t_i)=(z,u_0)\in G\,(k_0)\,.$

Now, let

$$v_0:=\inf\{t|\,(z,t)\in H\,(k_0)\,,t>u_0\} \tag{6.0.66}$$

We would like to show that $u_0<v_0$, and to do that we go by contradiction. Suppose that $u_0=v_0$. Then, by the definition of inf, there exists a sequence $\{t_i\}$, such that $t_i>u_0$, $(z,t_i)\in H\,(k_0)$ and $\lim\limits_{i\to\infty}t_i=u_0$. Thus, $\lim\limits_{i\to\infty}(z,t_i)=(z,u_0)\,.$

Moreover, $H(k_0)$ is closed and thus, $(z, u_0) \in H(k_0)$ which is a contradiction since $(z, u_0) \in G(k_0)$ and by (6.0.58) $G(k_0) \cap H(k_0) = \phi$.

Summarizing our work above and recallng (6.0.56) and (6.0.57) we get (6.0.63). We are now ready to show (6.0.64). Notice that (6.0.64) can be rewritten as

$$\Big(\{(z,t), u_0 < t < v_0\} \cap G(k_0)\Big) \cup \Big(\{(z,t), u_0 < t < v_0\} \cap H(k_0)\Big) = \phi. \tag{6.0.67}$$

To prove (6.0.67), we prove the first union to be empty. Let's proceed by contradiction. Assume $(z,t) \in \{(z,t), u_0 < t < v_0\} \cap G(k_0)$. Then $(z,t) \in G(k_0)$, $u_0 < t$ and by (6.0.63) $t < v_0 < v$. But by (6.0.65) $t \leq u_0$. Thus leading to a contradiction.

Similarly, we can show that the second union is also empty.
This concludes the proof of (6.0.67) and hence **Subclaim # 4**.

Subclaim # 5: Since,

$$z \in V \setminus \bigcup_{k,m=1}^{\infty} P\left[G^{+}(k,m) \cup G^{-}(k,m) \cup H^{+}(k,m) \cup H^{-}(k,m)\right] \tag{6.0.68}$$

then there exists

$$u_0 < s_1 < t_1 < v_0,$$

such that

$$(z, s_1) \in I \text{ and } (z, t_1) \in O. \tag{6.0.69}$$

Proof of Subclaim # 5: We also prove (6.0.69) by contradiction. Assume $(z,t) \in I \ \ \forall\, t \in]u_0, v_0[$ (we can argue similarly if $(z,t) \in O \ \ \forall\, t \in]u_0, v_0[$). Let $m \in \mathbb{N}$ such that $v_0 - \dfrac{3}{m} > u_0$. Thus we have

1. $(z, v_0) \in H\,(k_0)$ (by **Subclaim # 2**)

2. $(z, v_0) - se_n = (z, v_0 - s)$,where $v_0 - s$ is one of those t's, such that $(z,t) \in I\ \forall\, t \in]u_0, v_0[$. Thus, $(z, v_0) - se_n = (z, v_0 - s) \in I$ for $0 < s < \dfrac{3}{m}$.

Hence by (6.0.24) , $(z, v_0) \in H^-\,(k_0, m)$ and thus $z \in P\big(H^-\,(k_0, m)\big)$. This contradicts (6.0.68) .

Repeating **Subclaim # 3** till the end of **Subclaim # 4** we get $k_1 > k_0$ and numbers u_1, v_1 such that

$$u_0 < u_1 < v_1 < v_0, \quad (z, u_1) \in G\,(k_1)\,, \text{ and } (z, v_1) \in H\,(k_1)\,,$$

and by (6.0.64) , $(z,t) \notin G\,(k_1) \cup H\,(k_1)$ for $u_1 < t < v_1$.

Continuing like this, $\exists$ sequences $\{k_j\}_{j=1}^{\infty}, \{u_j\}_{j=1}^{\infty}$ and $\{v_j\}_{j=1}^{\infty}$ such that

$$\left\{ \begin{array}{c} u_0 < u_1 < \ldots, v_0 > v_1 > v_2 \ldots \\ u_j < v_j \ \forall\ j = 1, 2, \ldots \\ (z, u_j) \in G\,(k_j)\,, (z, v_j) \in H\,(k_j) \\ (z,t) \notin G\,(k_j) \cup H\,(k_j) \quad \text{if } u_j < t < v_j \end{array} \right\}$$

Take

$$\lim_{j\to\infty} u_j \le t \le \lim_{j\to\infty} v_j,$$

Note that $\forall\ j \in \mathbb{N}, u_j < t < v_j$. Thus we have

$$y := (z,t) \notin \bigcup_{j=1}^{\infty} \Big[G(k_j) \cup H(k_j) \Big]. \tag{6.0.70}$$

Let us show that $y \in \partial_* E$. Notice that by $(6.0.70)$, $y \notin G(k_j)$, thus by $(6.0.21)$

$$\frac{\mathcal{L}^n (B(y,r) \cap O)}{r^n} \ge \frac{\alpha_{n-1}}{3^{n+1}}, \tag{6.0.71}$$

so by $(6.0.53)$, $(6.0.55)$ and by applying $\limsup$ on both sides of $(6.0.71)$, we get

$$\limsup_{r\to 0} \frac{\mathcal{L}^n (B(y,r) \cap E^c)}{r^n} > \frac{\alpha_{n-1}}{3^{n+1}}. \tag{6.0.72}$$

Similarly, we can get

$$\limsup_{r\to 0} \frac{\mathcal{L}^n (B(y,r) \cap E)}{r^n} > \frac{\alpha_{n-1}}{3^{n+1}}. \tag{6.0.73}$$

Then, by Definition 4.0.8, $y = (z,t) \in \partial_* E$, hence, we get a contradiction and the claim is proven.

After proving ***claim # 1*** we got that $ess\ V_{-a}^{a} f^z$ is taken over all the points such that $(z, t_i) \in O \cup I$. Furthermore, by ***claim # 2***, if $(z,u) \in I$ and $(z,v) \in O$ then, $(z,t) \in \partial_* E$. So the essential variation actually counts the points in $\partial_* E$. Thus, we get

$$\begin{aligned}
\text{ess } V_{-a}^{a} f^z &= \sup \left\{ \sum_{j=1}^{m} \left| f^z\left(t_{j+1}\right) - f^z\left(t_j\right) \right| \right\} \\
&\leq \text{Card}\big\{ t \big| -a < t < a, (z,t) \in \partial_* E \big\} \\
&= N\left(P|U \cap \partial_* E, z\right),
\end{aligned}$$

We got the last inequality since, by (6.0.1), $N\left(P|U \cap \partial_* E, z\right) = \mathcal{H}^0\left(U \cap \partial_* E \cap P^{-1}\{z\}\right)$ which is the counting measure by Theorem 2.0.23.

Thus by Lemma 6.0.1 and (6.0.17) , we have

$$\begin{aligned}
\int_V \text{ess } V_{-a}^{a} f^z \, \mathrm{d}z &\leq \int_V N\left(P|U \cap \partial_* E, z\right) \, \mathrm{d}z \\
&\leq \mathcal{H}^{n-1}\left(U \cap \partial_* E\right) \\
&< \infty
\end{aligned}$$

Hence, I is of locally finite perimeter, as we wanted. $\square$

Bibliography

[1] *David G.Caraballo. Local Simplicity, Topology and Sets of Finite Perimeter. 2010.*

[2] *Luigi Ambrosio. Characterizations of Sets of Finite Perimeter: Old and Recent Results. 2014.*

[3] *William P. Ziemer.* Weakly Differentiable Functions*. Springer Science and Business Media New York, 1989.*

[4] *Nykamp DQ. The Idea Behind Green's Theorem. URL* `http://mathinsight.org/greens_theorem_idea`.

[5] *Lawrence C. Evans and Ronald F. Gariepy.* Measure Theory and Fine Properties of Functions*. Studies in Advanced Mathematics, CRC Press, 1992.*

[6] *Gerald B. Folland.* Real Analysis, Modern Techniques and Their Applications*. John Wiley and sons INC., 1999.*

[7] *Elias M. Stein and Rami Shakarchi.* Real Analysis, Measure Theory, Integration and Hilbert Spaces*. Princeton University Press, Princeton and Oxford, 2005.*