A Criterion For N- Rectifiable Sets

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Abstract of the Thesis

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Geometric measure theory was developed in the second half of the 20^{th} century to manipulate the structure and regularity questions in the calculus of variations. The main goal of this thesis is to introduce the theory of " rectifiability of sets". Rectifiable sets are considered smooth in a certain measure theoretic sense. Rectifiable sets are basic concepts in geometric measure theory. Their theory began with the study and determination of length, area or volume of sets in Euclidean space. Rectifiable sets have many of the desirable properties that smooth sets have. In this thesis, we will

discuss one of their most important features which is the existence of what we call approximate tangent planes. In fact, we will show that a set that has an n-dimensional approximate tangent plane at almost every point is n-rectifiable.

To my family.

Contents

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Notations

$\mathcal{L}^1$	The 1-dimensional Lebesgue measure
$\mathcal{L}^n$	The n-dimensional Lebesgue measure
$f\restriction_E$	f restricted to the set E
a.e.	almost everywhere
$\mathcal{H}^s$	s-dimensional Hausdorff measure
$(X, \mathcal{M}, \mu)$	Measure Space : X is the set, $\mathcal{M}$ is measurable set on X , μ is the measure on X
C_c^1	Compactly supported
L_{loc}^1	Locally bounded variation
L^1	Set of all μ-summable functions
$\{\theta > a\}$	$\{x \in \mathbb{R}^{n+k}, \theta(x) > a\}$
$B\left(x, r\right)$	$\{y \in \mathbb{R}^n,\ \lvert x-y\rvert \leq r\}$ = closed ball with center x, radius r}
$\alpha\left(s\right)$	$\dfrac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}+1\right)} \quad (0 \leq s < \infty)$
$\alpha\left(n\right)$	volume of the unit ball in $\mathbb{R}^n$
χ_A	characteristic function of the set A
$\bar{f}$	an extension of f
$Lip\left(f\right)$	Lipschitz constant of f

Chapter 1

Introduction

In measure theory, the notion of n-rectifiable sets provides a measure theoretic notion of smoothness for surfaces which are not smooth in the usual sense. In fact, a set is rectifiable if it is basically a subset of a union of Lipschitz graphs. By definition Lipschitz functions are functions that do not vary very much. More formally,

Definition 1.0.1. $|f(x) - f(y)| \leq C|x - y|$ *for all* x *and* y *in* $\mathbb{R}^n$ *where* C *is a constant*

Rademacher (see book [1]) proved that Lipschitz functions are differentiable almost everywhere. For this reason it is known that Lipschitz functions are a measure theoretic generalization of smooth functions, making rectifiable sets a generalization of smooth surfaces. A standard example of 1-rectifiable set in the plane is the graph of the function $f(x) = |x|$.

To be able to give the formal definition of rectifiable sets, we need the notion of the Hausdorff measure.

The Hausdorff measure is essentially the surface measure, in the sense that it measures the n-dimensional volumes of an n- dimensional set that lives in a higher dimensional space $\mathbb{R}^{n+k}$. For example, if we have a 2-dimensional surface living in $\mathbb{R}^5$, we need a measure that gives us the area of this surface even if it is living in a very high dimensional space $\mathbb{R}^5$. Thus, the 1-dimensional Hausdorff measure of a simple curve in $\mathbb{R}^{n+k}$ is equal to the length of the curve, the 2- dimensional measure of a plane living in $\mathbb{R}^{n+k}$ is its area, and the 3- dimensional measure of a solid living in $\mathbb{R}^{n+k}$ is its volume and so on. This new measure is known as the Hausdorff measure and it was introduced in 1918 by the mathematician Felix Hausdorff. Since rectifiable sets will be an n- dimensional sets that live in the higher space $\mathbb{R}^{n+k}$, the Hausdorff measure is the correct tool to measure the volume of these sets.
We are now ready to give the formal definition of Hausdorff measure and then rectifiable sets

Definition 1.0.2. *1. Let $A \subset \mathbb{R}^{n+k}$*

Let us define

$$\mathcal{H}_{\delta}^{n}(A) = \inf\left\{ \sum_{j=1}^{\infty} \alpha(n) \left(\frac{diam\ C_j}{2} \right)^n \ ; A \subset \bigcup_{j=1}^{\infty} C_j \ ; diam\ C_j \leq \delta \right\}$$

and where $\alpha(n) = \dfrac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$.

2. For $A \subset \mathbb{R}^{n+k}$, let us define

$$\mathcal{H}^n(A) = \lim_{\delta \to 0} \mathcal{H}^n_\delta(A) = \sup_{\delta > 0} \mathcal{H}^n_\delta(A).$$

We call $\mathcal{H}^n$ an n-dimensional Hausdorff measure on $\mathbb{R}^{n+k}$.

Definition 1.0.3. *We say that $M \subset \mathbb{R}^{n+k}$ is countably n- rectifiable if*

$$M \subset M_o \bigcup \left(\bigcup_{i=1}^{\infty} f_i(A_i) \right)$$

where $\mathcal{H}^n(M_o) = 0$ and $f_i : A_i \longrightarrow \mathbb{R}^{n+k}$ is Lipschitz and $A_i \subset \mathbb{R}^n$ for $i = 1, 2 \cdots$.

Rectifiability was first introduced by Bescovitch for 1-dimensional sets in the plane. His work was extended by Federer to n- sets of $\mathbb{R}^{n+k}$, with n an integer. Finally, rectifiability was generalized by Mastrand to fractal sets (which is defined by Mandelbrot as a shape made of parts similar to the whole in some way) in the plane whose Hausdorff dimension is any positive real number (see book [2]).

From the definition of rectifiable sets, we notice that they are a natural and convenient generalization of smooth n-dimensional surfaces. In fact, smooth surfaces are described (locally) by smooth functions, whereas rectifiable sets are described by Lipschitz functions, which as we said earlier are a generalization of smooth functions. Thus, rectifiable sets are sets of extreme importance and *one comes to a very important and interesting question concerning them which is how we can characterize rectifiable sets?*

There is a wide variety of known geometric characterizations of rectifiability. We shall state three of them which are given in terms of densities, projections and approximate tangent planes (see book [3]).

One way of characterizing rectifiable sets is through density: Preiss' Theorem is one of the great landmarks of geometric measure theory. In fact, his theorem states that a set is rectifiable if and only if the variation in its density (relative to the Hausdorff measure) is controlled. More precisely,

Theorem 1.0.4. *(see book [4])*
Let $E \subset \mathbb{R}^{n+k}$ is a borel set with $\mathcal{H}^n(E) < \infty$ such that $\theta^n_*(x, E) = \liminf_{r\to 0} \frac{\mu(B(x,r)\cap E)}{2^n} > 0$ for $\mathcal{H}^n$a.e $x \in E$. Then the following are equivalent:

$$(i) E \;\; is \;\; n - rectifiable.$$

$$(ii) \int_0^1 \left| \frac{\mathcal{H}^n\left(B(x,r)\cap E\right)}{r^n} - \frac{\mathcal{H}^n\left(B(x,2r)\cap E\right)}{(2r)^n} \right|^2 \frac{dr}{r} < \infty \quad for\; \mathcal{H}^n - a.e\; x \in E.$$

$$(iii) \lim_{r\to 0} \left(\frac{\mathcal{H}^n\left(B(x,r)\cap E\right)}{r^n} - \frac{\mathcal{H}^n\left(B(x,2r)\cap E\right)}{(2r)^n} \right) = 0 \quad for\; \mathcal{H}^n - a.e\; x \in E$$

Another characterization of rectifiable sets is through projection: In fact, it is shown (see book [5]) a set A is n-rectifiable if and only if the image of every subset $B \subset A$ of positive $\mathcal{H}^n$- measure under a projection has a positive $\mathcal{H}^n$-measure. i.e $\mathcal{H}^n(P_v(B)) > 0$.

In this thesis, we will focus on the third known characterization of rectifiability in terms of approximate tangent planes. The main importance of the class of rectifiable sets is that it posesses many of the nice properties of the smooth surfaces which one is seeking to generalize. Although in general, tangent planes may not exist for rectifiable sets, they do admit (at $\mathcal{H}^n$ a.e of their points) what we call an approximate tangent plane. Let us formally here define an approximate tangent plane:

Definition 1.0.5. *We say that n-dimensional subspace $P(x)$ is the approximate tangent space of μ at x if there exists $\theta(x) \in (0, \infty)$ such that*

$$\lim_{\lambda \to 0} \int f(y)\, d\mu_{x,\lambda}(y) = \theta(x) \int_{P(x)} f(y)\, d\mathcal{H}^n(y) \quad \forall f \in C_c^1(\mathbb{R}^{n+k}, \mathbb{R}) \tag{1.0.1}$$

where $\mu_{x,\lambda}$ be the measure given by

$$\mu_{x,\lambda}(A) = \frac{\mu(x + \lambda A)}{\lambda^n}$$

In this thesis, we will show that having an approximate tangent plane at *almost every point is a sufficient criterion for a set to be n- rectifiable. More precisely, we will prove the following theorem:*

Theorem 1.0.6. *Let μ be a radon measure on $\mathbb{R}^{n+k}$, $x \in \mathbb{R}^{n+k}$, and $\lambda > 0$. Let $\mu_{x,\lambda}$ be the measure given by $\mu_{x,\lambda}(A) = \dfrac{\mu(x+\lambda A)}{\lambda^n}$. Suppose for μ - a.e $x \in \mathbb{R}^{n+k}$, there exists $\theta(x) \in (0, \infty)$ and there exists an n - dimensional space $P(x) \subset \mathbb{R}^{n+k}$ such that (3.0.1) holds. Let*

$$M = \{x \in \mathbb{R}^{n+k} \ \ such \ \ that \ \ (3.0.1) \ \ holds \ \ for \ \ some \ \ P \ \ and \ \ some \ \ \theta\}$$

Let $\theta = 0$ on $\mathbb{R}^{n+k} \setminus M$. Then θ is $\mathcal{H}^n$ - measurable, and M is countably n-rectifiable.

In order to establish the proof of this theorem, we need the following chapters:
In chapter 1, we state some preliminary definitions and theorems: definition of measures, Beppo-Levi, Dominated Convergence Theorem and Egoroff Theorem.
In chapter 2, we define the notion of Hausdorff measure and state some of its properties then we define Lipschitz functions and introduce their extension theorem.
In chapter 3, we introduce the definition of countable n-rectifiable sets. Then, we state two lemmas that will help us in proving our main theorem. We finish the thesis by the proof of theorem 3.0.53.

Chapter 2

Preliminaries

Let us begin by defining measures, Borel measures, and Radon Measures.

Definition 2.0.1. *Let X be any set and 2^X be the set of all subsets of X. A mapping $\mu : 2^X \longrightarrow [0, \infty]$ is called a measure on X. If:*

1. $\mu(\phi) = 0$
2. $\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k)$ *whenever* $A \subset \bigcup_{k=1}^{\infty} A_k$.

Definition 2.0.2. *Let X be any set, μ is a measure on X and $A \subset X$. We say A is μ - measurable if $\mu(E) = \mu(E \cap A) + \mu(E \cap A^c) \quad \forall E \in 2^X$*

Definition 2.0.3. *The Borel σ - algebra of $\mathbb{R}^n$ is the smallest σ- algebra of $\mathbb{R}^n$ containing the open sets of $\mathbb{R}^n$. The sets that belong to the σ- algebra are called Borel sets.*

Definition 2.0.4. *A measure μ on $\mathbb{R}^n$ is called Borel measure if every borel set is μ - measurable.*

Definition 2.0.5. *Let μ be a measure on X. We say μ is regular if for each set $A \subset X$, there exists a μ- measurable set B, such that $A \subset B$ and $\mu(A) = \mu(B)$.*

Definition 2.0.6. *Let μ be a measure on $\mathbb{R}^n$. We say μ is Borel regular if μ is a Borel measure and for each $A \subset \mathbb{R}^n$, there exists a Borel set B such that $A \subset B$ and $\mu(A) = \mu(B)$.*

Definition 2.0.7. *Let μ be a measure on $\mathbb{R}^n$. We say that μ is a radon measure if μ is a borel regular measure and $\mu(K) < \infty$ for each compact $K \subset \mathbb{R}^n$.*

Definition 2.0.8. *A function $f : X \longrightarrow \mathbb{R}$ is said to be upper semi-continous at x , if for each $\epsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies that $f(y) < f(x) + \epsilon$.*

Next, we recall some theorems from measure theory that we will use in this thesis.

Theorem 2.0.9. *Let μ be a radon measure. Then, the function $f : x \longrightarrow \mu(B(x, r))$ is upper semi- continues.*

Theorem 2.0.10. *Beppo-Levi (see book [6])*
Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{f_n\}$ be a sequence of positive measurable functions then,

$$\int \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} f_n \, d\mu.$$

Theorem 2.0.11. *Dominated Convergence Theorem (see book [6])*
Let $(X, \mathcal{M}, \mu)$ be a measure space, $f, \{f_n\}$ be measurable functions and ϕ be a positive function. If :

1. $\lim_{n\to\infty} f_n = f$, *pointwise.*

2. $|f_n| \leq \phi$ *for all* n.

3. $\int \phi \, d\mu < \infty$, *that is* $\phi \in L^1(\mu)$. *Then,*

$$\lim_{n\to\infty} \int |f_n - f| \, d\mu = 0$$

and

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu.$$

Theorem 2.0.12. *(see Theorem 7 p: 13 in book [1])*

Assume $f : X \longrightarrow [0, \infty]$ *be* μ *- measurable.Then there exists* μ *-measurable sets* $\{A_k\}_{k=1}^{\infty}$ *in* X *such that* $f = \sum_{k=1}^{\infty} (\frac{1}{k})\chi_{A_k}$.

Theorem 2.0.13. *(see Theorem 5 p: 5 in book [1])*

Let μ *be a regular measure on* X. *If* $A_1 \subset A_2 \subset A_3 \cdots \subset A_k \cdots$ *then*

$$\lim_{k\to\infty} \mu(A_k) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right)$$

Theorem 2.0.14. *Let* $(X, \mathcal{M}, \mu)$ *be a measure space. Let f, g be positive functions. If:* $f \leq g$ *then* $\int f \, d\mu \leq \int g \, d\mu$

Theorem 2.0.15. *Egoroff Theorem (see Theorem 3 p:16 in book [1])*

Let μ *be a measure on* $\mathbb{R}^n$ *and suppose* $f_k : \mathbb{R}^n \longrightarrow \mathbb{R}^m (k = 1, 2 \cdots)$ *are* μ *- measurable. Assume* $A \subset \mathbb{R}^n$ *is* μ *- measurable with* $\mu(A) < \infty$ *and*

$f_k \longrightarrow g\,\mu$ - a.e on A. Then for each $\epsilon > 0$ there exists a μ - measurable set $B \subset A$ such that :

1. *$\mu(A - B) < \epsilon$*

2. *$f_k \longrightarrow g$ uniformly on B as $k \longrightarrow \infty$*

Next, we define the Hausdorff Measure and state some of its properties.

Definition 2.0.16. 1. *Let $A \subset \mathbb{R}^n$, $0 \leq s < \infty$, $0 \leq \delta < \infty$.*

Let us define

$$\mathcal{H}^s_\delta (A) = \inf \left\{ \sum_{j=1}^{\infty} \alpha (s) \left(\frac{diam\ C_j}{2} \right)^s \ ; A \subset \bigcup_{j=1}^{\infty} C_j \ ; diam\ C_j \leq \delta \right\}$$

and where $\alpha (s) = \dfrac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}+1\right)}$.

2. *For $A \subset \mathbb{R}^n$ and $0 \leq s < \infty$, let us define*

$$\mathcal{H}^s (A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta (A) = \sup_{\delta > 0} \mathcal{H}^s_\delta (A) .$$

We call $\mathcal{H}^s$ an s-dimensional Hausdorff measure on $\mathbb{R}^n$.

Theorem 2.0.17. *(see Theorem 1 p:61 in book [1])*
$\mathcal{H}^s$ is a borel regular measure.$(0 \leq s < \infty)$.

Finally,we define Lipschitz functions and introduce their extension theorem.

Definition 2.0.18. *A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is called a Lipschitz function if there exists a constant C such that $|f(x) - f(y)| \leq C|x - y|$ for all x and y in $\mathbb{R}^n$.*

Definition 2.0.19. *Let f be a Lipschitz function. Define*

$$Lip\,(f) = \sup\left\{\frac{|f\,(x) - f\,(y)|}{|x - y|}\ ; x, y \in \mathbb{R}^n\ , x \neq y\right\}$$

We call $Lip\,(f)$ the Lipschitz constant of the function f.

Theorem 2.0.20. *Extension of Lipschitz functions (see Theorem 1 p:80 in book [1])*

Suppose $f : A \longrightarrow \mathbb{R}^m$ is a Lipschitz function where $A \subset \mathbb{R}^n$, then there exists a Lipschitz function $\bar{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that :

1. *$\bar{f} = f$ on A.*
2. *$Lip\left(\bar{f}\right) \leq \sqrt{m}\ Lip\,(f)$.*

Lemma 2.0.21. *(See Theorem 6.9 p:95 in [7])*

Let $A \subset \mathbb{R}^{n+k}$. If

$$\lim_{r\to 0} \frac{\mu\,(B(x,r))}{\alpha_n r^n} \geq t\ \ \forall\, x \in A,$$

then, $c\,t\,\mathcal{H}^n(A) \leq \mu(A)$

where c is a constant depending only on n.

Chapter 3

Rectifiable Sets

Recall that our main theorem gives a criterion for a set to be n-rectifiable. So let us begin this section by introducing the definition of countably n-rectifiable sets.

Definition 3.0.1. *$M \subset \mathbb{R}^{n+k}$ is said to be countably n- rectifiable if $M \subset M_o \bigcup \left(\bigcup_{i=1}^{\infty} f_i(\mathbb{R}^n) \right)$ where $\mathcal{H}^n(M_o) = 0$ and $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+k}$ is Lipschitz for $i = 1, 2, \cdots$ Notice that by the extension theorem of Lipschitz Functions, it is enough to have $M \subset M_o \bigcup \left(\bigcup_{i=1}^{\infty} f_i(A_i) \right)$ where $\mathcal{H}^n(M_o) = 0$ and $f_i : A_i \longrightarrow \mathbb{R}^{n+k}$ is Lipschitz, and $A_i \subset \mathbb{R}^n$, for $i = 1, 2, \cdots$*

Definition 3.0.2. *Let μ be a Radon measure on $\mathbb{R}^{n+k}$, and fix $x \in \mathbb{R}^{n+k}$. We say that n-dimensional subspace $P(x)$ is the approximate tangent space of μ at x if there exists $\theta(x) \in (0, \infty)$ such that*

$$\lim_{\lambda \to 0} \int f(y)\, d\mu_{x,\lambda}(y) = \theta(x) \int_{P(x)} f(y)\, d\mathcal{H}^n(y) \quad \forall f \in C_c^1(\mathbb{R}^{n+k}, \mathbb{R}) \tag{3.0.1}$$

where $\mu_{x,\lambda}$ *be the measure given by*

$$\mu_{x,\lambda}(A) = \frac{\mu(x+\lambda A)}{\lambda^n}$$

To prove our main theorem, we need the following two Lemmas.

Lemma 3.0.3. *Let* $S \subset \mathbb{R}^{n+k}, \epsilon \in (0,1)$, *and* $\delta \in (0,1)$. *Let* $0 \in S$. *Assume that there exists an* n- *plane* L *containing the origin, such that for every* $\rho \in [0, \delta]$ *and for each* $x \in S \cap B(0, \delta)$, *we have*

$$S \cap B(x,\rho) \subset \epsilon\rho - neighborhood\ \ of\ \ (L+x) \cap B(x,\rho) \tag{3.0.2}$$

Then $S \cap B(0, \frac{\delta}{2})$ *is contained in the graph of a Lipschitz function defined on* L , *and is thus contained in a Lipschitz image of* $\mathbb{R}^n$.

Proof. Let P_L denote the projection onto the plane L. Fix $x, y \in S \cap B(0, \frac{\delta}{2})$. Let $|y-x| = \rho < \delta$. So, $y \in S \cap B(x,\rho)$, which by (3.0.2) ensures

$$\begin{aligned} |P_{(L+x)^\perp}(y) - x| &\leq \epsilon\rho & (3.0.3) \\ &= \epsilon|y-x| & (3.0.4) \end{aligned}$$

Now, if we translate by x and use the linearity of the projection map, (3.0.3) becomes

$$|P_{L^\perp}(y-x)| \leq \epsilon|y-x|$$

which becomes

$$|P_{L^\perp}(y) - P_{L^\perp}(x)| \leq \epsilon |y - x| \tag{3.0.5}$$

However, notice that

$$|y - x| = |P_L(y) + P_{L^\perp}(y) - (P_L(x) + P_{L^\perp}(x))| \tag{3.0.6}$$

Thus, using the triangle inequality on (3.0.6) and recalling (3.0.5), we get

$$\begin{aligned} |y - x| &\leq |P_L(y) - P_L(x)| + |P_{L^\perp}(y) - P_{L^\perp}(x)| \\ &\leq |P_L(y) - P_L(x)| + \epsilon |y - x| \end{aligned}$$

So,

$$|P_L(y) - P_L(x)| \geq (1 - \epsilon)|y - x| \tag{3.0.7}$$

Notice that (3.0.7) shows that P_L is injective on $S \cap B(0, \frac{\delta}{2})$. So,

$$P_L \restriction_{S \cap B(0, \frac{\delta}{2})} : S \cap B(0, \frac{\delta}{2}) \longrightarrow P_L(S \cap B(0, \frac{\delta}{2}))$$

is bijective.

Now, recall $\forall x, y \in S \cap B(0, \frac{\delta}{2})$, we have

$$x = (P_L(x), P_{L^\perp}(x)) = P_L(x) + P_{L^\perp}(x)$$

$$y = (P_L(y), P_{L^\perp}(y)) = P_L(y) + P_{L^\perp}(y)$$

Define function

$$f : P_L\left(S \cap B(0, \frac{\delta}{2})\right) \longrightarrow L^\perp \textit{ such that } \forall\, x \in S \cap B\left(0, \frac{\delta}{2}\right)$$

$$f(P_L(x)) = P_L^\perp(x)$$

Now, by (3.0.7), we have

$$|P_L(y) - P_L(x)| \geq (1-\epsilon)|\,(P_L(y), P_{L^\perp}(y)) - (P_L(x), P_{L^\perp}(x))\,|$$

Squaring both sides, we get

$$\begin{aligned}
|P_L(y) - P_L(x)|^2 &\geq (1-\epsilon)^2|\,(P_L(y), P_{L^\perp}(y)) - (P_L(x), P_{L^\perp}(x))\,|^2 \\
&= (1-\epsilon)^2\left(|P_L(y) - P_L(x)|^2 + |P_{L^\perp}(y) - P_{L^\perp}(x)|^2\right) \\
&= (1-\epsilon)^2\left(|P_L(y) - P_L(x)|^2 + |f(P_L(y)) - f(P_L(x))|^2\right) \\
&= (1-\epsilon)^2|P_L(y) - P_L(x)|^2 + (1-\epsilon)^2|f(P_L(y)) - f(P_L(x))|^2
\end{aligned}$$

So,

$$\left(1 - (1-\epsilon)^2\right)|P_L(y) - P_L(x)|^2 \geq (1-\epsilon)^2|f(P_L(y)) - f(P_L(x))|^2$$

which makes

$$|f(P_L(y)) - f(P_L(x))| \leq \sqrt{\frac{1-(1-\epsilon)^2}{(1-\epsilon)^2}}|P_L(y) - P_L(x)|$$

So, f is Lipschitz function and $S \cap B(0, \frac{\delta}{2}) \subset$ graph(f).

By Extension theorem for Lipschitz functions , we can extend f to a Lipschitz function on L such that $S \cap B(0, \frac{\delta}{2}) \subset$ graph(f).

Next , consider the rotation r that takes $\mathbb{R}^n$ to L. Set

$$h := (Id \times f) \circ r$$

Thus,

$$h : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+k}$$

and

$$S \cap B\left(0, \frac{\delta}{2}\right) \subset h(\mathbb{R}^n)$$

We finish the proof by showing h is a Lipschitz function. Since

$$\begin{aligned}
|h(x) - h(y)| &= |\,(r(x), f(r(x)) - (r(y), f(r(y))\,| \\
&\leq |r(x) - r(y)| + |f(r(x)) - f(r(y))| \\
&\leq |r(x) - r(y)| + Lip(f)|r(x) - r(y)| \\
&\leq (Lip(f) + 1)|r(x) - r(y)|
\end{aligned}$$

Thus,

$$|h(x) - h(y)| \leq (Lip(f) + 1)|x - y|$$

$\square$

Lemma 3.0.4. *Let μ be a Radon measure. Then the function $f : x \longrightarrow \mu\left(B(x,r)\right)$ is a borel function.*

Proof. By Theorem 2.0.9, we know $f : x \longrightarrow \mu\left(B(x,r)\right)$ is upper semi-continous . To prove f is a borel function , we show

$$U = \{x : f(x) < t\}$$

is a borel set. However, we will show that U is an open set, and hence borel. Take $x_o \in U$ and $\epsilon = t - f(x_o)$. Since f is upper semi- continous, there exists δ such that if $|y - x_o| < \delta$, then

$$f(y) < f(x_o) + \epsilon$$

that is,

$$f(y) < f(x_o) + t - f(x_o)$$

so,

$$f(y) < t.$$

Hence, $y \in U$ and hence $B(x_o, \delta) \subset U$.

This shows that U is open , and as mentioned borel. This finishes the proof. $\square$

We are now ready to prove our main theorem.

Theorem 3.0.5. *Let μ be a Radon measure on $\mathbb{R}^{n+k}$, $x \in \mathbb{R}^{n+k}$, and $\lambda > 0$. Let $\mu_{x,\lambda}$ be the measure given by $\mu_{x,\lambda}(A) = \dfrac{\mu(x+\lambda A)}{\lambda^n}$. Suppose for μ - a.e $x \in \mathbb{R}^{n+k}$, there exists $\theta(x) \in (0,\infty)$ and there exists an n - dimensional space $P(x) \subset \mathbb{R}^{n+k}$ such that (3.0.1) holds. Let*

$$M = \{x \in \mathbb{R}^{n+k} \ such \ that \ (3.0.1) \ holds \ for \ some \ P \ and \ some \ \theta\}$$

Let $\theta = 0$ on $\mathbb{R}^{n+k} \setminus M$. Then θ is $\mathcal{H}^n$ - measurable, and M is countably n-rectifiable.

Proof. In this proof, $B(0,1)$ will denote the closed unit ball and $U(0,1)$ will denote the open unit ball. We begin by assuming $\mu(\mathbb{R}^{n+k}) < \infty$ since otherwise, we just replace it by $\mu \lceil_{B(0,R)}$.

We start the proof by showing that $\theta(x)$ is $\mathcal{H}^n$ - measurable.

Claim # 1:

$$\lim_{\rho \to 0} \frac{\mu(B(x,\rho))}{\alpha_n \rho^n} = \theta(x), \qquad \mu - a.e \ x \in \mathbb{R}^{n+k}$$

Proof of Claim # 1: First for $f \in C_c^1(\mathbb{R}^{n+k}, \mathbb{R})$, let us show that

$$\frac{1}{r^n} \int f\left(\frac{y-x}{r}\right) d\mu = \int f(y) d\mu_{x,r} \tag{3.0.8}$$

(3.0.8) is clear for $f = \chi_A$ since in this case the right hand side of (3.0.8) gives us

$$\int \chi_A \, d\mu_{x,r} = \mu_{x,r}(A) \tag{3.0.9}$$

and left hand side gives us

$$\begin{aligned} \frac{1}{r^n} \int \chi_A \left(\frac{y-x}{r} \right) d\mu &= \frac{1}{r^n} \int \chi_{x+rA}(y) \, d\mu \\ &= \frac{1}{r^n} \int_{x+rA} d\mu \\ &= \frac{\mu(x+rA)}{r^n} \end{aligned} \tag{3.0.10}$$

and (3.0.9) and (3.0.10) are equal by the definition of $\mu_{x,r}$.

Now, for any positive function f, by Theorem 2.0.12, f can be written as

$$f = \sum_{i=1}^{\infty} \frac{1}{i} \chi_{A_i}$$

and thus we have,

$$\frac{1}{r^n} \int \chi_{A_i} \left(\frac{y-x}{r} \right) d\mu = \int \chi_{A_i}(y) \, d\mu_{x,r}$$

So,

$$\frac{1}{i} \frac{1}{r^n} \int \chi_{A_i} \left(\frac{y-x}{r} \right) d\mu = \frac{1}{i} \int \chi_{A_i}(y) \, d\mu_{x,r}$$

that is,

$$\sum_{i=1}^{\infty} \frac{1}{i} \frac{1}{r^n} \int \chi_{A_i}\left(\frac{y-x}{r}\right) d\mu = \sum_{i=1}^{\infty} \frac{1}{i} \int \chi_{A_i}(y)\, d\mu_{x,r}$$

Using Beppo-Levi (see Theorem 2.0.10) we get,

$$\frac{1}{r^n} \int \sum_{i=1}^{\infty} \frac{1}{i} \chi_{A_i}\left(\frac{y-x}{r}\right) d\mu = \int \sum_{i=1}^{\infty} \frac{1}{i} \chi_{A_i}(y)\, d\mu_{x,r}$$

which is exactly what we want. So we have (3.0.8) for positive functions. Finally, for $f \in C_c^1(\mathbb{R}^{n+k}, \mathbb{R})$ write $f = f^+ - f^-$ where f^+, f^- are positive functions. Thus, we have,

$$\frac{1}{r^n} \int f^+\left(\frac{y-x}{r}\right) d\mu = \int f^+(y)\, d\mu_{x,r}$$

and

$$\frac{1}{r^n} \int f^-\left(\frac{y-x}{r}\right) d\mu = \int f^-(y)\, d\mu_{x,r}$$

Thus,

$$\frac{1}{r^n} \int f\left(\frac{y-x}{r}\right) d\mu = \int f(y)\, d\mu_{x,r},$$

which finishes the proof of (3.0.8).

Now, for every $0 < \delta < 1$, let $g_\delta \in C_c^\infty(\mathbb{R}^{n+k}, \mathbb{R})$, where $0 \leq g_\delta \leq 1$ and $g_\delta = 1$ on $B(0,1)$ and 0 outside $B(0, 1+\delta)$ and $\lim_{\delta \to 0} g_\delta = \chi_{B(0,1)}$ pointwise on $\mathbb{R}^{n+k}$.

Then, for all $0 < \delta < 1$, we have

$$\chi_{B(0,1)} \leq g_\delta$$

Thus,by Theorem 2.0.14, we have

$$\int \chi_{B(0,1)} \left(\frac{y-x}{\rho} \right) d\mu \quad \leq \quad \int g_\delta \left(\frac{y-x}{\rho} \right) d\mu.$$

Multiplying by $\dfrac{1}{\rho^n}$ and taking lim sup on both sides,we get

$$\limsup_{\rho \to 0} \frac{1}{\rho^n} \int \chi_{B(0,1)} \left(\frac{y-x}{\rho} \right) d\mu \leq \limsup_{\rho \to 0} \frac{1}{\rho^n} \int g_\delta \left(\frac{y-x}{\rho} \right) d\mu. \quad (3.0.11)$$

For the left side of (3.0.11), notice that

$$\frac{1}{\rho^n} \int \chi_{B(0,1)} \left(\frac{y-x}{\rho} \right) d\mu = \frac{1}{\rho^n} \int \chi_{B(x,\rho)}(y)\, d\mu \qquad (3.0.12)$$

For the right hand side of (3.0.11), notice that by (3.0.1) and (3.0.8), we have

$$\frac{1}{\rho^n} \int g_\delta \left(\frac{y-x}{\rho} \right) d\mu = \theta(x) \int_{P(x)} g_\delta(y)\, d\mathcal{H}^n(y) \qquad (3.0.13)$$

Replacing (3.0.12) and (3.0.13) in (3.0.11) , we get

$$\limsup_{\rho \to 0} \frac{1}{\rho^n} \int \chi_{B(x,\rho)}(y)\, d\mu \leq \theta(x) \int_{P(x)} g_\delta(y)\, d\mathcal{H}^n(y) \qquad (3.0.14)$$

Notice that (3.0.14) is true $\forall\, 0 < \delta < 1$. Now,

$$\lim_{\delta\to 0} g_\delta \chi_{P(x)} = \chi_{B(0,1)}\chi_{P(x)} \text{ pointwise,}$$

and

$$g_\delta \chi_{P(x)} \leq \chi_{B(0,2)\cap P(x)},$$

and

$$\int_{B(0,2)\cap P(x)} d\mathcal{H}^n(y) < \infty.$$

Thus, by Dominated Convergence Theorem (see Theorem 2.0.11) , we have

$$\lim_{\delta\to 0} \int_{P(x)} g_\delta(y) d\mathcal{H}^n(y) = \int_{P(x)} \chi_{B(0,1)}(y)\, d\mathcal{H}^n(y) \tag{3.0.15}$$

Thus, taking the limit as $\delta \longrightarrow 0$ in (3.0.14) and using (3.0.15), we get

$$\limsup_{\rho\to 0} \frac{1}{\rho^n} \int \chi_{B(x,\rho)}(y)\, d\mu \leq \theta(x) \int_{P(x)} \chi_{B(0,1)}(y)\, d\mathcal{H}^n(y)$$

that is,

$$\begin{aligned} \limsup_{\rho\to 0} \frac{\mu\left(B(x,\rho)\right)}{\rho^n} &\leq \theta(x) \int_{P(x)} \chi_{B(0,1)}(y)\, d\mathcal{H}^n(y) \\ &= \theta(x) \int_{P(x)\cap B(0,1)} d\mathcal{H}^n(y) \\ &= \theta(x)\alpha_n \end{aligned} \tag{3.0.16}$$

Now, let $g_{1-\delta} \in C_c^\infty(\mathbb{R}^{n+k}, \mathbb{R})$, where $0 \leq g_{1-\delta} \leq 1$ and $g_{1-\delta} = 1$ on $B(0, 1-\delta)$ and 0 outside $U(0,1)$ and $\lim_{\delta \to 0} g_{1-\delta} = \chi_{U(0,1)}$ pointwise on $\mathbb{R}^{n+k}$.

Then, for all $0 < \delta < 1$, we have

$$\chi_{U(0,1)} \geq g_{1-\delta}$$

Thus, by Theorem 2.0.14, we have

$$\int \chi_{U(0,1)} \left(\frac{y-x}{\rho} \right) d\mu \geq \int g_{1-\delta} \left(\frac{y-x}{\rho} \right) d\mu$$

Multiplying by $\frac{1}{\rho^n}$ and taking $\liminf$ on both sides, we get

$$\liminf_{\rho \to 0} \frac{1}{\rho^n} \int \chi_{U(0,1)} \left(\frac{y-x}{\rho} \right) d\mu \geq \liminf_{\rho \to 0} \frac{1}{\rho^n} \int g_{1-\delta} \left(\frac{y-x}{\rho} \right) d\mu \quad (3.0.17)$$

For the left hand side of (3.0.17), notice that

$$\frac{1}{\rho^n} \int \chi_{U(0,1)} \left(\frac{y-x}{\rho} \right) d\mu = \frac{1}{\rho^n} \int \chi_{U(x,\rho)}(y) d\mu \quad (3.0.18)$$

For the right hand side of (3.0.17), notice that by (3.0.1) and (3.0.8), we have

$$\frac{1}{\rho^n} \int g_{1-\delta} \left(\frac{y-x}{\rho} \right) d\mu = \theta(x) \int_{P(x)} g_{1-\delta}(y) \, d\mathcal{H}^n(y) \quad (3.0.19)$$

Replacing (3.0.18) and (3.0.19) in (3.0.17), we get

$$\liminf_{\rho \to 0} \frac{1}{\rho^n} \int \chi_{U(x,\rho)}(y) d\mu \geq \theta(x) \int_{P(x)} g_{1-\delta}(y) \, d\mathcal{H}^n(y) \quad (3.0.20)$$

Notice that (3.0.20) is true $\forall\, 0 < \delta < 1$. Now,

$$\lim_{\delta \to 0} g_{1-\delta}\, \chi_{P(x)} = \chi_{U(0,1)}\, \chi_{P(x)} \text{ pointwise,}$$

and

$$g_{1-\delta}\, \chi_{P(x)} \leq \chi_{U(0,2)\cap P(x)},$$

and

$$\int_{U(0,2)\cap P(x)} d\mathcal{H}^n(y) < \infty.$$

Thus, by Dominated Convergence Theorem (see Theorem 2.0.11), we have

$$\lim_{\delta \to 0} \int_{P(x)} g_{1-\delta}(y)\, d\mathcal{H}^n(y) = \int_{P(x)} \chi_{U(0,1)} d\mathcal{H}^n\,(y) \tag{3.0.21}$$

Thus, taking the limit as $\delta \longrightarrow 0$ in (3.0.20) and using (3.0.21), we get

$$\liminf_{\rho \to 0} \frac{1}{\rho^n} \int \chi_{U(x,\rho)}(y)\, d\mu \geq \theta(x) \int_{P(x)} \chi_{U(0,1)}(y)\, d\mathcal{H}^n\,(y)$$

that is ,

$$\begin{aligned} \liminf_{\rho \to 0} \frac{\mu(U(x,\rho))}{\rho^n} \;&\geq\; \theta(x) \int_{P(x)} \chi_{U(0,1)}(y)\, d\mathcal{H}^n(y) \\ &=\; \theta(x) \int_{P(x)\cap U(0,1)} d\mathcal{H}^n(y) \\ &=\; \theta(x)\alpha_n \end{aligned} \tag{3.0.22}$$

Finally, note that

$$U(x,\rho) \subset B(x,\rho)$$

So,

$$\mu\left(U(x,\rho)\right) \leq \mu\left(B(x,\rho)\right)$$

which implies

$$\frac{\mu\left(U(x,\rho)\right)}{\rho^n} \leq \frac{\mu\left(B(x,\rho)\right)}{\rho^n}$$

that is,

$$\liminf_{\rho\to 0} \frac{\mu\left(U(x,\rho)\right)}{\rho^n} \leq \liminf_{\rho\to 0} \frac{\mu\left(B(x,\rho)\right)}{\rho^n} \tag{3.0.23}$$

So, by (3.0.16), (3.0.23) and (3.0.22), we get

$$\theta(x)\alpha_n \geq \limsup_{\rho\to 0} \frac{\mu\left(B(x,\rho)\right)}{\rho^n} \geq \liminf_{\rho\to 0} \frac{\mu\left(B(x,\rho)\right)}{\rho^n} \geq \liminf_{\rho\to 0} \frac{\mu\left(U(x,\rho)\right)}{\rho^n} \geq \theta(x)\alpha_n.$$

Thus, $\lim_{\rho\to 0} \frac{\mu\left(B(x,\rho)\right)}{\rho^n}$ exists and

$$\lim_{\rho\to 0} \frac{\mu\left(B(x,\rho)\right)}{\alpha_n \rho^n} = \theta(x)$$

which finishes the proof of claim 1 .

Now, recall by Lemma 3.0.4, that the function $x \longrightarrow \mu\left(B(x,\rho)\right)$ is borel. Thus, by claim 1, we have

$$\lim_{\rho \to 0} \frac{\mu\left(B(x,\rho)\right)}{\alpha_n \rho^n} = \lim_{m \to \infty} \frac{\mu\left(B(x,\frac{1}{m})\right)}{\alpha_n \left(\frac{1}{m}\right)^n} \tag{3.0.24}$$

making θ the limit of a sequence of borel functions, that is $\theta(x)$ is borel. But recall by Theorem 2.0.17 , $\mathcal{H}^n$ is a borel regular measure, which means that every borel function is $\mathcal{H}^n$-measurable. So, θ is $\mathcal{H}^n$- measurable.

We are left to prove that M is countably n-rectifiable. To be able to do this, we first recall G(n,n+k) the metric space we call the Grassmanian, whose elements are k-dimensional spaces of $\mathbb{R}^{n+k}$. The distance between any two k-dimensional subspaces is

$$d(\pi, \pi^{'}) = \sup_{|x|=1} |P_\pi(x) - P_{\pi'}(x)|$$

where P_π denotes the orthogonal projection of $\mathbb{R}^{n+k}$ onto π.
Now, recalling Theorem 2.0.13 and the fact that $\theta(x) > 0$ μ -a.e, we get

$$\mu\left(\mathbb{R}^{n+k}\right) = \mu\left(\bigcup_{m=1}^{\infty} \{\theta > \frac{1}{m}\}\right) = \lim_{m\to\infty} \mu\left(\{\theta > \frac{1}{m}\}\right).$$

So, let $\epsilon = \frac{1}{2}\mu(\mathbb{R}^{n+k})$. Then, there exists N_o such that $\forall m > N_o$, we have

$$\left|\mu(\mathbb{R}^{n+k}) - \mu\left(\{\theta > \frac{1}{m}\}\right)\right| \leq \frac{1}{2}\mu(\mathbb{R}^{n+k})$$

that is,

$$\mu(\mathbb{R}^{n+k}) - \mu\left(\{\theta > \frac{1}{m}\}\right) \leq \frac{1}{2}\mu(\mathbb{R}^{n+k}) \tag{3.0.25}$$

Now, take $m_o > N_o$ and denote by

$$\theta_o = \frac{1}{m_o}$$

and

$$F = \{x \in \mathbb{R}^{n+k};\ \theta(x) > \theta_o\} \tag{3.0.26}$$

Moreover, (3.0.25), becomes

$$\mu(\mathbb{R}^{n+k}) - \mu(F) \leq \frac{1}{2}\mu(\mathbb{R}^{n+k})$$

Since $\theta(x)$ is $\mathcal{H}^n$ measurable, then F is $\mathcal{H}^n$ measurable. Finally, notice that by claim 1, we have

$$\theta(x) = \lim_{\rho \to 0} \frac{\mu\left(B(x, \rho)\right)}{\alpha_n \rho^n}, \qquad \mu - a.e\, x \in \mathbb{R}^{n+k}$$

Hence, by definition of F (see (3.0.26)), we get that the

$$\lim_{\rho \to 0} \frac{\mu(B(x, \rho))}{\alpha_n \rho^n} > \theta_o \qquad x \in F \tag{3.0.27}$$

Now, to be able to construct the sets eligible to be our Lipschitz images, we need to introduce cones. For $x \in \mathbb{R}^{n+k}$, and π a k-dimensional space, let $X_\alpha(\pi, x)$ denote the following cone

$$X_\alpha(\pi, x) = \{y \in \mathbb{R}^{n+k}, |P_\pi(y-x)| \geq \alpha|y-x|\}.$$

Claim # 2: $\lim_{\rho \to 0} \dfrac{\mu\left(X_{\frac{1}{2}}(\pi_x, x) \cap (B(x,\rho)\right)}{\rho^n} = 0$ where $x \in F$ and $\pi_x = P^\perp(x)$

Proof of Claim # 2: For $0 < \delta < 1$, consider the cone

$$C := X_{\frac{1}{2}}(\pi_x, 0) \cap B(0,1)$$

Let N_δ be the δ - neighborhood of C .Let $g_\delta \in C_c^\infty(\mathbb{R}^{n+k}, \mathbb{R})$, where $0 \leq g_\delta \leq 1$ and $g_\delta = 1$ on C and 0 on N_δ^c, such that $\lim_{\delta \to 0} g_\delta = \chi_C$ pointwise on $\mathbb{R}^{n+k}$. Then, for all $0 < \delta < 1$, we have

$$\chi_C \leq g_\delta$$

By theorem 2.0.14, we have

$$\int \chi_C\left(\frac{y-x}{\rho}\right) d\mu \leq \int g_\delta\left(\frac{y-x}{\rho}\right) d\mu$$

Multiplying by $\dfrac{1}{\rho^n}$ and taking $\limsup$ on both sides, we get

$$\limsup_{\rho\to 0}\frac{1}{\rho^n}\int \chi_C\left(\frac{y-x}{\rho}\right)d\mu \;\le\; \limsup_{\rho\to 0}\frac{1}{\rho^n}\int g_\delta\left(\frac{y-x}{\rho}\right)d\mu$$
$$= \lim_{\rho\to 0}\frac{1}{\rho^n}\int g_\delta\left(\frac{y-x}{\rho}\right)d\mu \qquad (3.0.28)$$

For the left hand side of (3.0.28), notice that

$$\frac{1}{\rho^n}\int \chi_C\left(\frac{y-x}{\rho}\right)d\mu \;=\; \frac{1}{\rho^n}\int \chi_{X_{\frac{1}{2}}(\pi_x,x)\cap B(x,\rho)}(y)\,d\mu \qquad (3.0.29)$$

For the right hand side of (3.0.28), notice that by (3.0.1) and (3.0.8), we have

$$\lim_{\rho\to 0}\frac{1}{\rho^n}\int g_\delta\left(\frac{y-x}{\rho}\right)d\mu = \theta(x)\int_{P(x)} g_\delta(y)\,d\mathcal{H}^n(y) \qquad (3.0.30)$$

Replacing (3.0.29) and (3.0.30) in (3.0.28), we get

$$\limsup_{\rho\to 0}\frac{1}{\rho^n}\int \chi_{X_{\frac{1}{2}}(\pi_x,x)\cap B(x,\rho)}(y)\,d\mu \le \theta(x)\int_{P(x)} g_\delta(y)\,d\mathcal{H}^n(y) \qquad (3.0.31)$$

Notice that (3.0.31) is true $\forall\, 0<\delta<1$. Now

$$\lim_{\delta\to 0} g_\delta \chi_{P(x)} = \chi_C\,\chi_{P(x)} \text{ pointwise,}$$

$$g_\delta\,\chi_{P(x)} \le \chi_{sptg_\delta \cap P(x)}$$

and

$$\int_{sptg_\delta \cap P(x)} d\mathcal{H}^n(y) < \infty.$$

Thus , by Dominated Convergence theorem (see Theorem 2.0.11) , we get

$$\lim_{\delta \to 0} \int_{P(x)} g_\delta(y)\, d\mathcal{H}^n(y) = \int_{P(x)} \chi_C(y)\, d\mathcal{H}^n(y) \tag{3.0.32}$$

Thus, taking the limit as $\delta \longrightarrow 0$ in (3.0.31) and using (3.0.32), we get

$$\limsup_{\rho \to 0} \frac{\mu\left(X_{\frac{1}{2}}(\pi_x, x) \cap B(x,\rho)\right)}{\rho^n} \leq \theta(x) \int_{P(x)} \chi_C(y)\, d\mathcal{H}^n(y) \tag{3.0.33}$$

We finish the proof of claim 2 by showing that right hand side of (3.0.33) is zero, that is

$$\int_{P(x)} \chi_C(y)\, d\mathcal{H}^n(y) = \int_{P(x)\cap C} d\mathcal{H}^n = \mathcal{H}^n(P(x) \cap C) = 0.$$

To do that, it is enough to show that $P(x) \cap C = \{0\}$. Since $y \in P(x)$ then $P_{P(x)}(y) = y$ and $P_{P(x)^\perp}(y) = 0$ where $P_{P(x)}$ denotes the orthogonal projection of $\mathbb{R}^{n+k}$ onto $P(x)$. But $y \in C$, so $|P_{P^\perp(x)}(y)| \geq \dfrac{|y|}{2}$. Hence, we get $0 \geq \dfrac{|y|}{2}$ which means $y = 0$. and thus $P(x)\, \cap C = \{0\}$.

$$\limsup_{\rho \to 0} \frac{\mu\left(X_{\frac{1}{2}}(\pi_x, x) \cap B(x,\rho)\right)}{\rho^n} = 0,$$

and claim 2 is proved.

Now, we are ready to construct the sets eligible for being Lipschitz images : For $k = 1, 2, \cdots$ and $x \in F$ (as constructed in (3.0.26)), let

$$f_k(x) = \inf_{(0<\rho<\frac{1}{k})} \frac{\mu\left(B(x,\rho)\right)}{\alpha_n \rho^n} \tag{3.0.34}$$

and

$$q_k(x) = \sup_{(0<\rho<\frac{1}{k})} \frac{\mu\left(X_{\frac{1}{2}}(\pi_x, x) \cap B(x,\rho)\right)}{\alpha_n \rho^n} \tag{3.0.35}$$

For every $x \in F$, by claim 1 and (3.0.27), we have

$$\theta(x) = \lim_{k\to\infty} f_k(x) \geq \theta_o \text{ and } \lim_{k\to\infty} q_k(x) = 0 \tag{3.0.36}$$

Now, by Egoroff's Theorem (see Theorem 2.0.15), there exists $A_1 \subset F$ such that

$$\mu(F \setminus A_1) < \frac{1}{4}\mu(\mathbb{R}^{n+k}) \textit{ and } \lim_{k\to\infty} f_k(x) = \theta(x) \text{ uniformly on } A_1$$

By applying Egoroff's Theorem again, there exists a set $A_2 \subset A_1$ such that

$$\mu(A_1 \setminus A_2) < \frac{1}{4}\mu(\mathbb{R}^{n+k}) \text{ and } \lim_{k\to\infty} q_k(x) = 0 \text{ uniformly on } A_2$$

Set $E := A_2$. Then, $E \subset F$ and

$$\begin{aligned}\mu(F \setminus E) &\leq \mu(F \setminus A_1) + \mu(A_1 \setminus E)\\ &= \mu(F \setminus A_1) + \mu(A_1 \setminus A_2)\\ &< \frac{1}{2}\mu(\mathbb{R}^{n+k})\end{aligned} \tag{3.0.37}$$

Moreover,

$$\lim_{k\to\infty} f_k(x) \geq \theta_o, \text{ and } \lim_{k\to\infty} q_k(x) = 0 \text{ uniformly on } E \tag{3.0.38}$$

Now, fix $0 < \epsilon < 1$ and using uniform convergence, then there exists k_o such that for all $k \geq k_o$, and for all $x \in E$, and using (3.0.36), we have

$$|f_k(x) - \theta(x)| \leq \theta_o\epsilon \text{ and } |q_k(x)| \leq \theta_o\epsilon$$

So, $\forall\, k \geq k_o$, we have

$$f_k(x) \geq \theta_o(1-\epsilon) \text{ and } q_k(x) \leq \theta_o\epsilon \tag{3.0.39}$$

Then,$\forall\, k \geq k_o$, substituting (3.0.39) in (3.0.34) and (3.0.35) respectively ,we get

$$\inf_{(0<\rho<\frac{1}{k})} \frac{\mu\left(B(x,\rho)\right)}{\alpha_n\rho^n} \geq \theta_o(1-\epsilon) \text{ and } \sup_{(0<\rho<\frac{1}{k})} \frac{\mu\left(X_{\frac{1}{2}}(\pi_x,x)\cap(B(x,\rho)\right)}{\alpha_n\rho^n} \leq \theta_o\epsilon$$

Thus, for $\rho < \frac{1}{k_o}$, we get

$$\frac{\mu\left(B(x,\rho)\right)}{\alpha_n \rho^n} \geq \theta_o(1-\epsilon) \;\; \text{and} \;\; \frac{\mu\left(X_{\frac{1}{2}}(\pi_x, x) \cap B(x,\rho)\right)}{\alpha_n \rho^n} \leq \theta_o \epsilon. \quad (3.0.40)$$

Let $\delta = \dfrac{1}{k_o}$. Then (3.0.40) holds for all $\rho \leq \delta$.

We know that the Grassmanian G(n+k , k) is compact in the d- metric space, so for $\epsilon = \dfrac{1}{16}$, we can choose k- dimensional subspaces $\pi_1, \cdots \pi_N$ of $\mathbb{R}^{n+k}$ such that for any k- dimensional π of $\mathbb{R}^{n+k}$, there exists a $j \in \{1, \cdots, N\}$ such that $d(\pi, \pi_j) \leq \dfrac{1}{16}$.

Let $E_1, \cdots, E_n$ be subsets of E defined by :

$$E_j = \left\{ x \in E, d(\pi_j, \pi_x) < \frac{1}{16} \right\}$$

Its is clear that $E = \bigcup_{j=1}^{N} E_j$.

Now, to be able to prove that Ej is inside a countable union of Lipschitz images, we need to prove the following claim.

Claim # 3: For $\epsilon = \dfrac{1}{2^n 16^n + 1}$, and $\delta = \dfrac{1}{k_o}$ as above (see line below (3.0.40)), we have

$$X_{\frac{3}{4}}(\pi_j, x) \cap E_j \cap B\left(x, \frac{\delta}{2}\right) = \{x\} \, \forall \; x \in E_j \;\; j = (1, \cdots, N)$$

Proof of Claim # 3: Fix ϵ and δ as in statement of the claim. We proceed by contradiction. Let $x \in E_j$ and suppose there exist $y \neq x$ such that $y \in X_{\frac{3}{4}}(\pi_j, x) \cap E_j \cap B(x, \frac{\delta}{2})$. So , there exists $\rho < \frac{\delta}{2}$ such that $y \in \partial B(x, \rho)$

that is $|y-x| = \rho < \frac{\delta}{2}$.

Since $x \in E_j \subset E$ and $2\rho < \delta$, then by (3.0.40), we have

$$\mu\left(X_{\frac{1}{2}}(\pi_x, x) \cap B(x, 2\rho)\right) \leq \theta_o\, \epsilon\, \alpha_n (2\rho)^n \tag{3.0.41}$$

In order to reach a contradiction, we need to show that

$$B\left(y, \frac{\rho}{16}\right) \subset X_{\frac{1}{2}}(\pi_x, x) \cap B(x, 2\rho)$$

Take $z \in B(y, \frac{\rho}{16})$. Then

$$|z-y| \leq \frac{\rho}{16}. \tag{3.0.42}$$

But, by the Triangle Inequality, we have

$$\begin{aligned} |z-x| &\leq |z-y| + |y-x| \\ &\leq \frac{\rho}{16} + \rho \\ &= \frac{17\rho}{16} < 2\rho. \end{aligned} \tag{3.0.43}$$

Thus, $z \in B(x, 2\rho)$.

We still need to show that $z \in X_{\frac{1}{2}}(\pi_x, x)$, that is we need to show that

$$|P_{\pi_x}(z-x)| \geq \frac{|z-x|}{2}$$

Notice that $x, y \in E_j$, so we have

$$d(\pi_j, \pi_x) \leq \frac{1}{16} \text{ and } d(\pi_j, \pi_y) \leq \frac{1}{16}. \tag{3.0.44}$$

Now,

$$\begin{aligned} |P_{\pi_x}(z-x)| &= |P_{\pi_x}(z-x) + P_{\pi_j}(z-x) - P_{\pi_j}(z-x)| \\ &\geq |P_{\pi_j}(z-x)| - |P_{\pi_x}(z-x) - P_{\pi_j}(z-x)| \end{aligned} \tag{3.0.45}$$

To bound the second summand of the right hand side of (3.0.45), note that by (3.0.44) and the the definition of distance, we have

$$\left| P_{\pi_x}\left(\frac{z-x}{|z-x|}\right) - P_{\pi_j}\left(\frac{z-x}{|z-x|}\right) \right| \leq d(\pi_x, \pi_j) \leq \frac{1}{16}$$

Thus, by linearity of the projection function, we get

$$\left| \frac{1}{|z-x|} \left(P_{\pi_x}(z-x) - P_{\pi_j}(z-x)\right) \right| \leq \frac{1}{16}$$

that is,

$$\left| P_{\pi_x}(z-x) - P_{\pi_j}(z-x) \right| \leq \frac{|z-x|}{16} \tag{3.0.46}$$

To bound the first summand of (3.0.45), notice that since $y \in X_{\frac{3}{4}}(\pi_j, x)$ and $|y-x| = \rho$, then

$$|P_{\pi_j}(y-x)| \geq \frac{3}{4}|y-x| = \frac{3}{4}\rho \tag{3.0.47}$$

Moreover, by (3.0.42), we have

$$|P_{\pi_j}(y-z)| \le |y-z| \le \frac{\rho}{16}. \tag{3.0.48}$$

Thus, combining (3.0.47), (3.0.48) and using linearity of the projection function, we get

$$\begin{aligned} |P_{\pi_j}(z-x)| &= |P_{\pi_j}(z-y+y-x)| \\ &= |P_{\pi_j}(z-y) + P_{\pi_j}(y-x)| \\ &= |-P_{\pi_j}(y-z) + P_{\pi_j}(y-x)| \\ &\ge |P_{\pi_j}(y-x)| - |P_{\pi_j}(y-z)| \\ &\ge \frac{3}{4}\rho - \frac{\rho}{16}. \end{aligned} \tag{3.0.49}$$

Substituting (3.0.49) and (3.0.46) in (3.0.45) and recalling (3.0.43), we get

$$\begin{aligned} |P_{\pi_x}(z-x)| &\ge \frac{3\rho}{4} - \frac{\rho}{16} - \frac{|z-x|}{16} \\ &= \frac{11\rho}{16} - \frac{|z-x|}{16}. \end{aligned} \tag{3.0.50}$$

However, by (3.0.43),

$$|z-x| \le \frac{17\rho}{16}$$

that is,

$$\rho \geq \frac{16}{17}|z-x|. \tag{3.0.51}$$

Replacing (3.0.51) in (3.0.50), we get

$$\begin{aligned}
|P_{\pi_x}(z-x)| &\geq \frac{11}{16}\left(\frac{16}{17}|z-x|\right) - \frac{|z-x|}{16} \\
&= \left(\frac{11}{17} - \frac{1}{16}\right)|z-x| \\
&\geq \frac{1}{2}|z-x|.
\end{aligned}$$

Thus, we proved that

$$B(y, \frac{\rho}{16}) \subset X_{\frac{1}{2}}(\pi_x, x) \cap B(x, 2\rho).$$

So, by (3.0.40), we get

$$\begin{aligned}
\mu\left(X_{\frac{1}{2}}(\pi_x, x) \cap B(x, 2\rho)\right) &\geq \mu\left(B(y, \frac{\rho}{16})\right) \\
&\geq \theta_o(1-\epsilon)\frac{\alpha_n \rho^n}{16^n}
\end{aligned}$$

since $\epsilon = \dfrac{1}{2^n 16^n + 1}$, we get a contradiction with (3.0.41). This finishes the proof of Claim 3.

Now , we are ready to prove that Ej's are our required sets

Claim # 4: $\forall j = 1, 2 \cdots , N$, $\forall x_o \in E_j$ we have,

$$E_j \cap B\left(x_o, \frac{\delta}{4}\right) \subset f_{x_o,j}(\mathbb{R}^n)$$

where

$$f_{x_o,j} : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+k} \text{ is Lipschitz}$$

Proof of Claim # 4:We want to use the result of Lemma 3.0.2.

Let $x \in E_j \cap B\left(x_o, \frac{\delta}{2}\right)$ and suppose there exists $y \neq x \in E_j \cap B(x, \rho)$.

Then, by Claim 3 , we have $y \notin X_{\frac{3}{4}}(\pi_j, x)$

So,

$$|P_{\pi_j}(y - x)| < \frac{3}{4}|y - x| \text{ and } |y - x| < \rho$$

Translating by x, we get

$$|P_{\pi_j + x}(y) - x| < \frac{3}{4}\rho$$

Thus, by Lemma 3.0.2, used on $\epsilon = \frac{3}{4}$, we get

$$E_j \cap B(x_o, \frac{\delta}{4}) \subset f_{x_o,j}(\mathbb{R}^n)$$

where $f_{x_o,j} : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+k}$ is Lipschitz.This finihses the proof of Claim 4.

Now, we have to construct countable sets that are eligible to be our Lipschitz images.

Let $\left\{B(x_i, \frac{\delta}{4})\right\}_{i=1}^{\infty}$ denote a countable cover for E with $|x_i - x_i^{'}| \geq \frac{\delta}{4}$.

$\forall i, i^{'} \in \mathbb{N}$, we have

$$E_j \subset E \subset \bigcup_{i=1}^{\infty} B\left(x_i, \frac{\delta}{4}\right)$$

that is,

$$E_j \cap \left(\bigcup_{i=1}^{\infty} B\left(x_i, \frac{\delta}{4}\right)\right) = E_j$$

so,

$$\bigcup_{i=1}^{\infty}\left(E_j \cap B\left(x_i, \frac{\delta}{2}\right)\right) = E_j$$

Taking the union over j on both sides and recalling that $E = \bigcup_{j=1}^{N} E_j$, we get

$$\bigcup_{j=1}^{N}\bigcup_{i=1}^{\infty}\left(E_j \cap B(x_i, \frac{\delta}{4})\right) = \bigcup_{j=1}^{N} E_j = E$$

so, by Claim 4, we get

$$E = \bigcup_{i,j=1}^{\infty,N}\left(E_j \cap B(x_i, \frac{\delta}{4})\right) \subset \bigcup_{i,j=1}^{\infty,N} f_{x_i,j}(\mathbb{R}^n)$$

Renaming, we get

$$E \subset \bigcup_{l=1}^{\infty} f_l(\mathbb{R}^n) \tag{3.0.52}$$

where $f_l : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+k}$ is Lipschitz for all $l = 1, 2 \cdots$. Thus, we got that E is inside countably many Lipschitz images.

Recall from (3.0.37), we chose E such that $\mu(F \setminus E) < \frac{1}{2}\mu(\mathbb{R}^{n+k})$. Set $E^1 := E$. Thus, on E^1 (3.0.38) holds and we have (3.0.52). Since we renamed E to E^1, let us rename (3.0.52) to say that

$$E^1 \subset \bigcup_{l=1}^{\infty} f_l^1(\mathbb{R}^n) \tag{3.0.53}$$

where $f_l : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+k}$ is Lipschitz for all $l = 1, 2 \cdots$. Now, by Egoroff Theorem (see Theorem 2.0.15), construct a set E^2 with $E^2 \subset F \setminus E^1$ (that is $E^1 \cap E^2 = \phi$)

$$\mu\left((F \setminus E^1) \setminus E^2\right) < \frac{1}{2^2}\mu(\mathbb{R}^{n+k}),$$

and (3.0.38) holds on E^2. Repeating the same work we did in Theorem from (3.0.38) to (3.0.53), we get

$$E^2 \subset \bigcup_{l=1}^{\infty} f_l^2(\mathbb{R}^n)$$

where $f_l : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+k}$ is Lipschitz for all $l = 1, 2 \cdots$.

In general, for $p \in \mathbb{N}$,by by Egoroff Theorem (see Theorem 2.0.15), construct a set E^p with $E^p \subset F \setminus \bigcup_{i=1}^{p-1} E^i$ (that is $E^p \cap \bigcup_{i=1}^{p-1} E^i = \phi$),

$$\mu\left((F \setminus \bigcup_{i=1}^{p-1} E^i) \setminus E^p\right) < \frac{1}{2^p}\mu(\mathbb{R}^{n+k}) \tag{3.0.54}$$

and (3.0.38) holds on E^p. Repeating the same work we did in Theorem from (3.0.38) to (3.0.52), we get

$$E^p \subset \bigcup_{l=1}^{\infty} f_l^p(\mathbb{R}^n) \tag{3.0.55}$$

where $f_l : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+k}$ is Lipschitz for all $l = 1, 2 \cdots$. Thus, we have now constructed countably many disjoint μ - measurable sets $E^i \subset F$ such that, (3.0.55) holds $\forall i \in \mathbb{N}$. Moreover, using (3.0.54), we have

$$\mu\left(F \setminus \bigcup_{i=1}^{\infty} E^i\right) = 0 \tag{3.0.56}$$

Thus, by (3.0.56) and (3.0.55), we get

$$F \subset \bigcup_{i,l=1}^{\infty} f_l^i(\mathbb{R}^n) \bigcup F_o^1 \tag{3.0.57}$$

with $\mu(F_o^1) = 0$ and $f_l^i : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+k}$ is Lipschitz for all $l = 1, 2, \cdots$. Renaming, we get

$$F \subset \bigcup_{r=1}^{\infty} f_r(\mathbb{R}^n) \bigcup F_o^1 \tag{3.0.58}$$

Notice that by (3.0.58), we have written F inside a countable union of Lipschitz images.

Now, go back to the construction of F in (3.0.26) and call it F_1. Thus,(3.0.58) becomes

$$F_1 \subset \bigcup_{r=1}^{\infty} f_r^1(\mathbb{R}^n) \bigcup F_o^1. \tag{3.0.59}$$

Recall that

$$\begin{aligned}
\mathbb{R}^{n+k} \setminus F_1 &= \Big\{\theta \leq \theta_o\Big\} \\
&= \Big\{0 < \theta \leq \theta_o\Big\} \bigcup \Big\{\theta = 0\Big\} \\
&= \Big\{0 < \theta \leq \frac{1}{m_o}\Big\} \bigcup \Big\{\theta = 0\Big\} \\
&= \bigcup_{m > m_o} \Big\{\frac{1}{m} < \theta \leq \frac{1}{m_o}\Big\} \bigcup \Big\{\theta = 0\Big\}
\end{aligned}$$

Then,

$$\mu\Big(\mathbb{R}^{n+k} \setminus F_1\Big) = \mu\left(\bigcup_{m > m_o} \Big\{\frac{1}{m} \leq \theta \leq \frac{1}{m_o}\Big\}\right)$$

By (2.0.13), we get

$$\mu\Big(\mathbb{R}^{n+k} \setminus F_1\Big) = \lim_{m \to \infty, m > m_o} \mu\Big(\Big\{\frac{1}{m} < \theta \leq \frac{1}{m_o}\Big\}\Big)$$

Hence, there exists $m_1 > m_o$ such that

$$\mu\Big(\mathbb{R}^{n+k} \setminus F_1\Big) - \mu\left(\Big\{\frac{1}{m_1} < \theta \leq \frac{1}{m_o}\Big\}\right) \leq \frac{1}{2^2}\mu(\mathbb{R}^{n+k}) \tag{3.0.60}$$

Now, let $F_2 = \{\frac{1}{m_1} < \theta \leq \frac{1}{m_o}\}$ and $\theta_1 = \frac{1}{m_1}$. Thus, (3.0.60) becomes

$$\mu\Big(\mathbb{R}^{n+k} \setminus (F_1 \bigcup F_2)\Big) \leq \frac{1}{2^2}\mu(\mathbb{R}^{n+k})$$

As in the construction of F, Notice that $F_1 \cap F_2 = \phi$ and

$$\lim_{\rho \to 0} \frac{\mu\Big(B(x,\rho)\Big)}{\alpha_n \rho^n} \geq \theta_1 \quad \forall x \in F_2$$

Repeating the same process we did on F (Claim 1 until (3.0.59)) but on F_2 instead, we get

$$F_2 \subset \bigcup_{r=1}^{\infty} f_r^2(\mathbb{R}^n) \bigcup F_o^2$$

with $\mu(F_o^2) = 0$ and $f_r^2 : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+k}$ is Lipschitz for all $r = 1, 2, \cdots$
In general, construct $F_s = \{\frac{1}{m_{s-1}} < \theta \leq \frac{1}{m_{s-2}}\}$ with $\theta_{s-1} = \frac{1}{m_{s-1}}$ such that

$$\mu\Big(\mathbb{R}^{n+k} \setminus \bigcup_{i=1}^{s} F_i\Big) \leq \frac{1}{2^s}\mu(\mathbb{R}^{n+k}). \tag{3.0.61}$$

Repeating the same process we did on F_2 (Claim 1 until (3.0.59)) but on F_s instead, we get

$$F_s \subset \bigcup_{l=1}^{\infty} f_l^s(\mathbb{R}^n) \bigcup F_o^s \tag{3.0.62}$$

with $\mu(F_o^s) = 0$ and $f_l^s : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+k}$ is Lipschitz for all $l = 1, 2, \cdots$

Notice that from (3.0.62), we get

$$\mu\left(\mathbb{R}^{n+k}\setminus\bigcup_{i=1}^{\infty}F_i\right)=0 \tag{3.0.63}$$

We now prove that

$$\mu\left(\mathbb{R}^{n+k}\setminus\bigcup_{i=1}^{\infty}F_i\right)=\mu\left(M\setminus\bigcup_{i=1}^{\infty}F_i\right)=0. \tag{3.0.64}$$

But using (3.0.63) and the fact that $\mu(\mathbb{R}^{n+k}\setminus M)=0$, we get

$$\begin{aligned}\mu\left(\mathbb{R}^{n+k}\setminus\bigcup_{i=1}^{\infty}F_i\right) &= \mu\left(\left(\mathbb{R}^{n+k}\setminus M)\bigcup M\right)\setminus\bigcup_{i=1}^{\infty}F_i\right)\\ &= \mu\left(\left((\mathbb{R}^{n+k}\setminus M)\setminus\bigcup_{i1}^{\infty}F_i\right)\bigcup\left(M\setminus\bigcup_{i=1}^{\infty}F_i\right)\right)\\ &= \mu\left(\left(\mathbb{R}^{n+k}\setminus M\right)\setminus\bigcup_{i=1}^{\infty}F_i\right)+\mu\left(M\setminus\bigcup_{i=1}^{\infty}F_i\right)\\ &= \mu\left(M\setminus\bigcup_{i=1}^{\infty}F_i\right) \end{aligned} \tag{3.0.65}$$

Combinig (3.0.63) and (3.0.65), we get (3.0.64). Now, let

$$M\setminus\bigcup_{i=1}^{\infty}F_i=M_o^1 \text{ with } \mu(M_o^1)=0 \tag{3.0.66}$$

then,

$$M=\bigcup_{i=1}^{\infty}F_i\bigcup M_o^1$$

so, by (3.0.62) used on F_i, we get

$$M \subset \bigcup_{i=1}^{\infty}\left(\bigcup_{l=1}^{\infty} f_l^i(\mathbb{R}^n)\bigcup F_o^i\right)\bigcup M_o^1 = \bigcup_{i,l=1}^{\infty} f_l^i(\mathbb{R}^n)\bigcup\Big(\bigcup_{i=1}^{\infty} F_o^i\Big)\bigcup M_o^1$$

let

$$M_o = \bigcup_{i=1}^{\infty} F_o^i \bigcup M_o^1 \text{ such that } \mu(M_o^1) = 0$$

Notice that by the sentence below (3.0.62) and by (3.0.66), we have $\mu(M_o) = 0$.

Thus,

$$M \subset \bigcup_{i,l=1}^{\infty} f_l^i(\mathbb{R}^n)\bigcup M_o \text{ with } \mu(M_o) = 0$$

Renaming, we get

$$M \subset \bigcup_{j=1}^{\infty} f_j(\mathbb{R}^n)\bigcup M_o$$

where$f_j : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+k}$ is Lipschitz and $\mu(M_o) = 0$

To finish the proof of M being n-rectfiable, we still need to show that

$$\mathcal{H}^n(M_o) = 0$$

We know that

$$M_o \subset \Big\{\theta > 0\Big\} = \bigcup_{i=1}^{\infty}\Big\{\theta > \frac{1}{i}\Big\}$$

Let

$$M_o^i = M_o \cap \left\{\theta > \frac{1}{i}\right\}$$

then

$$M_o = \bigcup_{i=1}^{\infty} M_o^i$$

Now, fix i, so $\forall x \in M_o^i$, we have

$$\lim_{r\to 0} \frac{\mu\Big(B(x,r)\Big)}{\alpha_n r^n} > \frac{1}{i}$$

By Lemma 2.0.21, we get

$$\frac{c}{i}\mathcal{H}^n(M_o^i) < \mu(M_o^i)$$

where c is any constant depending only on n. Thus,

$$\begin{aligned} \mathcal{H}^n(M_o^i) &< \frac{i}{c}\mu(M_o^i) \\ &< \frac{i}{c}\mu(M_o) = 0 \end{aligned} \tag{3.0.67}$$

Then,

$$\mathcal{H}^n(M_o^i) = 0$$

Thus,

$$\begin{aligned} \mathcal{H}^n(M_o) &= \mathcal{H}^n(\bigcup_{i=1}^{\infty} M_o^i) \\ &\leq \sum_{i=1}^{\infty} \mathcal{H}^n(M_o^i) = 0 \end{aligned} \tag{3.0.68}$$

And hence the proof is done. □

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